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Research Article

Identities of Symmetry for Higher-Order Generalized q-Euler Polynomials

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We investigate the properties of symmetry in two variables related to multiple Euler q-l-function which interpolates higher-order q-Euler polynomials at negative integers. From our investigation, we can derive many interesting identities of symmetry in two variables related to generalized higher-order q-Euler polynomials and alternating generalized q-power sums.

1. Introduction

Throughout this paper, the notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$. Let χ be a Dirichlet character with $d \in \mathbb{N}$ with conductor $d \equiv 1 (\bmod 2)$. Then the generalized Euler polynomials attached to χ are defined by the following generating function (see [1–3]):

$$2\sum_{a=0}^{d-1} \frac{\chi(a)(-1)^a e^{(a+x)t}}{e^{dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi}(x)\frac{t^n}{n!}. \tag{1}$$

The generalized Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{a=0}^{d-1} \frac{\chi(a)(-1)^a e^{(a+x)t}}{e^{dt}+1}\right)^r = \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x)\frac{t^n}{n!}. \tag{2}$$

When $x = 0$, $E_{n,\chi}^{(r)} = E_{n\chi}^{(r)}(0)$ are called the generalized Euler numbers attached to χ (see [2, 4]).

Assume that $q \in \mathbb{C}$ with $|q| < 1$ and define q-numbers by (see [2–15])

$$[x]_q = \frac{1-q^x}{1-q}. \tag{3}$$

Note that $\lim_{q\to 1}[x]_q = x$.

In [4, 8], Kim initiated to consider various q-extensions (or (h,q)-extensions) of Euler numbers and polynomials and constructed analytic continuations which interpolate his q-numbers and polynomials. Until recently, many authors have studied q-Euler or (h,q)-Euler polynomials due to him (see [1–21]). In [4], Kim defined the (h,q)-extension of generalized higher-order Euler polynomials attached to χ which is given by the generating function:

$$F_{q,\chi}^{(h,r)}(t,x) = [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^{r}(h-j+1)m_j}(-1)^{\sum_{j=1}^{r} m_j} \times \left(\prod_{j=1}^{r}\chi(m_j)\right) e^{[x+\sum_{l=1}^{r} m_l]_q t}$$

$$= \sum_{n=0}^{\infty} E_{n,\chi,q}^{(h,r)}(x) \frac{t^n}{n!}, \tag{4}$$

where $h \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Note that

$$\begin{aligned}\lim_{q\to 1} F_q^{(h,r)}(t,x) &= \left(2\sum_{a=0}^{d-1} \frac{\chi(a)(-1)^a e^{(a+x)t}}{e^{dt}+1}\right)^r \\ &= \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x) \frac{t^n}{n!}.\end{aligned} \tag{5}$$

When $x = 0$, $E_{n,\chi,q}^{(h,r)} = E_{n,\chi,q}^{(h,r)}(0)$ are called the (h,q)-extension of generalized higher-order Euler numbers attached to χ.

We find from (4) that

$$\begin{aligned}E_{n,\chi,q}^{(h,r)}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,\chi,q}^{(h,r)} [x]_q^{n-l} \\ &= \left(q^x E_{\chi,q}^{(h,r)} + [x]_q\right)^n,\end{aligned} \tag{6}$$

with the usual convention about replacing $(E_{\chi,q}^{(h\cdot r)})^n$ by $E_{n,\chi,q}^{(h\cdot r)}$.

In [4], Dirichlet-type multiple (h,q)-l-function is defined by Kim to be

$$\begin{aligned}&l_{q,r}^{(h)}(s,x \mid \chi) \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} F_{q,\chi}^{(h,r)}(-t,x) t^{s-1} dt \\ &= [2]_q^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{q^{\sum_{l=1}^r (h-l+1)m_l} \left(\prod_{l=1}^r \chi(m_l)\right)(-1)^{\sum_{l=1}^r m_l}}{[m_1+\cdots+m_r+x]_q^s},\end{aligned} \tag{7}$$

where $s, h \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

By using Cauchy residue theorem, we get

$$l_{q,r}^{(h)}(-n,x \mid \chi) = E_{n,\chi,q}^{(h,r)}(x), \quad n \in \mathbb{Z}_{\geq 0}. \tag{8}$$

In this paper, we investigate certain properties of symmetry in two variables related to Dirichlet-type multiple (h,q)-function which interpolates the (h,q)-extension of generalized higher-order Euler polynomials attached to χ at negative integers. From our investigation, we can derive many interesting identities of symmetry in two variables related to (h,q)-extension of generalized higher-order Euler polynomials and alternating generalized q-power sums.

2. Identities for the (h,q)-Extension of Generalized Higher-Order Euler Polynomials

In this section, we assume that χ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\mathrm{mod}2)$.

Let $w_1, w_2, r \in \mathbb{N}$ with $w_1 \equiv 1(\mathrm{mod}2)$ and $w_2 \equiv 1(\mathrm{mod}2)$ and $h \in \mathbb{Z}$. First, we observe that

$$\begin{aligned}&\frac{1}{[2]_{q^{w_1}}^r} l_{q^{w_1},r}^{(h)}\left(s, w_2 x + \frac{w_2}{w_1}\sum_{l=1}^r j_l \mid \chi\right) \\ &= \sum_{m_1,\ldots,m_r=0}^{\infty} \left((-1)^{m_1+\cdots+m_r} q^{w_1 \sum_{l=1}^r (h-l+1)m_l} \left(\prod_{l=1}^r \chi(m_l)\right)\right. \\ &\qquad \times \left(\left[m_1+\cdots+m_r+w_2 x\right.\right. \\ &\qquad\qquad \left.\left.\left. + \frac{w_2}{w_1}(j_1+\cdots+j_r)\right]_{q^{w_1}}^s\right)^{-1}\right) \\ &= \sum_{m_1,\ldots,m_r=0}^{\infty} \left(q^{w_1 \sum_{l=1}^r (h-l+1)m_l} (-1)^{\sum_{l=1}^r m_l} \left(\prod_{l=1}^r \chi(m_l)\right) [w_1]_q^s\right. \\ &\qquad \times \left(\left[w_2(j_1+\cdots+j_r) + w_1 w_2 x\right.\right. \\ &\qquad\qquad \left.\left.\left. + w_1(m_1+\cdots+m_r)\right]_q^s\right)^{-1}\right) \\ &= [w_1]_q^s \\ &\quad \times \sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{i_1,\ldots,i_r=0}^{dw_2-1} \left((-1)^{\sum_{l=1}^r (i_l+n_l)} q^{w_1 \sum_{l=1}^r (h-l+1)(i_l+n_l db)}\right. \\ &\qquad \times \left(\prod_{l=1}^r \chi(i_l)\right) \\ &\qquad \times \left(\left[w_1 w_2\left(x + d\sum_{l=1}^r n_l\right)\right.\right. \\ &\qquad\qquad \left.\left.\left. + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r i_l\right]_q^s\right)^{-1}\right) \\ &= [w_1]_q^s \\ &\quad \times \sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{i_1,\ldots,i_r=0}^{w_2 d-1} \left((-1)^{\sum_{l=1}^r (i_l+n_l)} q^{w_1 \sum_{l=1}^r (h-l+1)(i_l+n_l w_2 d)}\right. \\ &\qquad \times \left(\prod_{l=1}^r \chi(i_l)\right) \\ &\qquad \times \left(\left[w_1 w_2\left(x + d\sum_{l=1}^r n_l\right)\right.\right. \\ &\qquad\qquad \left.\left.\left. + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r i_l\right]_q^s\right)^{-1}\right).\end{aligned} \tag{9}$$

Thus, by (9), we get

$$\begin{aligned}
&\frac{[w_2]_q^s}{[2]_{q^{w_1}}^r}\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\qquad\times q^{b\sum_{l=1}^r(h-l+1)j_l}l_{q^{w_1},r}^{(h)}\left(s,w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\mid\chi\right)\\
&=[w_1]_q^s[w_2]_q^s\\
&\quad\times\sum_{i_1,\dots,i_r=0}^{dw_2-1}\sum_{j_1,\dots,j_r=0}^{dw_1-1}\sum_{n_1,\dots,n_r=0}^{\infty}\Bigg(\Bigg((-1)^{\sum_{l=1}^r(i_l+j_l+n_l)}\\
&\qquad\times\left(\prod_{l=1}^r\chi(j_l)\right)\left(\prod_{l=1}^r\chi(i_l)\right)\\
&\qquad\times q^{w_2\sum_{l=1}^r(h-l+1)j_l+w_1\sum_{l=1}^r(h-l+1)i_l}\Bigg)\\
&\qquad\times\Bigg(\Bigg[w_1w_2\left(x+d\sum_{l=1}^r n_l\right)\\
&\qquad\quad+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r i_l\Bigg]_q^s\Bigg)^{-1}\Bigg)\\
&\qquad\times q^{w_1w_2d\sum_{l=1}^r(h-l+1)n_l}.
\end{aligned}\tag{10}$$

By using the same method as (10), we get

$$\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^s}\sum_{j_1,\dots,j_r=0}^{dw_2-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\qquad\times q^{w_1\sum_{l=1}^r(h-l+1)j_l}l_{q^{w_2},r}^{(h)}\left(s,w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l\mid\chi\right)\\
&=[w_1]_q^s[w_2]_q^s\\
&\quad\times\sum_{j_1,\dots,j_r=0}^{dw_2-1}\sum_{i_1,\dots,i_r=0}^{dw_1-1}\sum_{n_1,\dots,n_r=0}^{\infty}\Bigg(\Bigg((-1)^{\sum_{l=1}^r(i_l+j_l+n_l)}\\
&\qquad\times\left(\prod_{l=1}^r\chi(j_l)\right)\left(\prod_{l=1}^r\chi(i_l)\right)\\
&\qquad\times q^{w_1\sum_{l=1}^r(h-l+1)j_l+w_2\sum_{l=1}^r(h-l+1)i_l}\Bigg)\\
&\qquad\times\Bigg(\Bigg[w_1w_2\left(x+d\sum_{l=1}^r n_l\right)\\
&\qquad\quad+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r i_l\Bigg]_q^s\Bigg)^{-1}\Bigg)\\
&\qquad\times q^{w_1w_2d\sum_{l=1}^r(h-l+1)n_l}.
\end{aligned}\tag{11}$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 1. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\mathrm{mod}2)$ and $w_2 \equiv 1(\mathrm{mod}2)$, one has*

$$\begin{aligned}
&[2]_{q^{w_2}}^r[w_2]_q^s\sum_{j_1,\dots,j_r=0}^{w_1d-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\qquad\times q^{w_2\sum_{l=1}^r(h-l+1)j_l}l_{q^{w_1},r}^{(h)}\\
&\qquad\times\left(s,w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\mid\chi\right)\\
&=[2]_{q^{w_1}}^r[w_1]_q^s\sum_{j_1,\dots,j_r=0}^{w_2d-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\qquad\times q^{w_1\sum_{l=1}^r(h-l+1)j_l}l_{q^{w_2},r}^{(h)}\\
&\qquad\times\left(s,w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l\mid\chi\right).
\end{aligned}\tag{12}$$

By (8) and Theorem 1, we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{Z}_{\geq 0}$ and $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1(\mathrm{mod}2)$ and $w_2 \equiv 1(\mathrm{mod}2)$, one has*

$$\begin{aligned}
&[2]_{q^{w_2}}^r[w_1]_q^n\sum_{j_1,\dots,j_r=0}^{w_1d-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\qquad\times q^{w_2\sum_{l=1}^r(h-l+1)j_l}E_{n,\chi,q^{w_1}}^{(h,r)}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\right)\\
&=[2]_{q^{w_1}}^r[w_2]_q^n\\
&\quad\times\sum_{j_1,\dots,j_r=0}^{w_2d-1}(-1)^{\sum_{l=1}^r j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\\
&\quad\times q^{w_1\sum_{l=1}^r(h-l+1)j_l}E_{n,\chi,q^{w_2}}^{(h,r)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l\right).
\end{aligned}\tag{13}$$

From (6), we note that

$$\begin{aligned}
E_{n,\chi,q}^{(h,r)}(x+y)&=\left(q^{x+y}E_{\chi,q}^{(h,r)}+[x+y]_q\right)^n\\
&=\left(q^{x+y}E_{\chi,q}^{(h,r)}+q^x[y]_q+[x]_q\right)^n\\
&=\sum_{i=0}^n\binom{n}{i}q^{xi}\left(q^yE_{\chi,q}^{(h,r)}+[y]_q\right)^i[x]_q^{n-i}\\
&=\sum_{i=0}^n\binom{n}{i}q^{xi}E_{i,\chi,q}^{(h,r)}(y)[x]_q^{n-i}.
\end{aligned}\tag{14}$$

By (14), we get

$$\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}q^{w_2\sum_{l=1}^r(h-l+1)j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\times E_{n,\chi,q^{w_1}}^{(h,r)}\left(w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l\right)$$

$$=\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}q^{w_2\sum_{l=1}^r(h-l+1)j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\times\sum_{i=0}^{n}\binom{n}{i}q^{iw_2(j_1+\cdots+j_r)}E_{i,\chi,q^{w_1}}^{(h,r)}(w_2x)\times\left[\frac{w_2(j_1+\cdots+j_r)}{w_1}\right]_{q^{w_1}}^{n-i}$$

$$=\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}q^{w_2\sum_{l=1}^r(h-l+1)j_l}\left(\prod_{l=1}^r\chi(j_l)\right)\times\sum_{i=0}^{n}\binom{n}{i}q^{(n-i)w_2\sum_{l=1}^r j_l}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)\left[\frac{w_2}{w_1}\sum_{l=1}^r j_l\right]_{q^{w_1}}^{i}$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^{i}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)\times\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}q^{w_2\sum_{l=1}^r(h-l+n-i+1)j_l}\times\left(\prod_{l=1}^r\chi(j_l)\right)[j_1+\cdots+j_r]_{q^{w_2}}^{i}$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^{i}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)\,S_{n,i,q^{w_2}}^{(h,r)}(w_1d\mid\chi),\tag{15}$$

where

$$S_{n,i,q}^{(h,r)}(w\mid\chi)=\sum_{j_1,\dots,j_r=0}^{w-1}(-1)^{\sum_{l=1}^r j_l}q^{\sum_{l=1}^r(h-l+n-i+1)j_l}\times[j_1+\cdots+j_r]_q^{i}\left(\prod_{l=1}^r\chi(j_l)\right).\tag{16}$$

From (15), we have

$$[2]_{q^{w_2}}^r[w_1]_q^n\times\sum_{j_1,\dots,j_r=0}^{dw_1-1}(-1)^{\sum_{l=1}^r j_l}q^{w_2\sum_{l=1}^r(h-l+1)j_l}\times\left(\prod_{l=1}^r\chi(j_l)\right)E_{n,\chi,q^{w_1}}^{(h,r)}\times\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)\right)$$

$$=[2]_{q^{w_2}}^r\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{n-i}[w_2]_q^{i}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)\,S_{n,i,q^{w_2}}^{(h,r)}\times(w_1d\mid\chi).\tag{17}$$

By using the same method as in (17), we get

$$[2]_{q^{w_1}}^r[w_2]_q^n\sum_{j_1,\dots,j_r=0}^{dw_2-1}(-1)^{\sum_{l=1}^r j_l}q^{w_1\sum_{l=1}^r(h-l+1)j_l}\times\left(\prod_{l=1}^r\chi(j_l)\right)E_{n,\chi,q^{w_2}}^{(h,r)}\left(w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l\right)$$

$$=[2]_{q^{w_1}}^r\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^{n-i}[w_1]_q^{i}E_{n-i,\chi,q^{w_2}}^{(h,r)}\times(w_1x)\,S_{n,i,q^{w_1}}^{(h,r)}(w_2d\mid\chi).\tag{18}$$

Therefore, by (17) and (18), we obtain the following theorem.

Theorem 3. *For $n\in\mathbb{Z}_{\ge0}$ and $w_1,w_2\in\mathbb{N}$, with $w_1\equiv1(\mathrm{mod}\,2)$ and $w_2\equiv1(\mathrm{mod}2)$, one has*

$$[2]_{q^{w_2}}^r\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{n-i}[w_2]_q^{i}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)\,S_{n,i,q^{w_2}}^{(h,r)}(w_1d\mid\chi)$$

$$=[2]_{q^{w_1}}^r\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^{n-i}[w_1]_q^{i}E_{n-i,\chi,q^{w_2}}^{(h,r)}(w_1x)\,S_{n,i,q^{w_1}}^{(h,r)}\times(w_2d\mid\chi).\tag{19}$$

Now, we observe that

$$e^{[x]_qu}\sum_{m_1,\dots,m_r=0}^{\infty}q^{\sum_{l=1}^r(h-l+1)m_l}(-1)^{\sum_{l=1}^r m_l}\times\left(\prod_{l=1}^r\chi(m_l)\right)e^{[y+\sum_{l=1}^r m_l]_qq^x(u+v)}$$
$$=e^{-[x]_qu}\sum_{m_1,\dots,m_r=0}^{\infty}q^{\sum_{l=1}^r(h-l+1)m_l}(-1)^{\sum_{l=1}^r m_l}\times\left(\prod_{l=1}^r\chi(m_l)\right)e^{[x+y+\sum_{l=1}^r m_l]_q(u+v)}.\tag{20}$$

The left hand side of (20) multiplied by $[2]_q^r$ is given by

$$[2]_q^re^{[x]_qu}\times\sum_{m_1,\dots,m_r=0}^{\infty}q^{\sum_{l=1}^r(h-l+1)m_l}(-1)^{\sum_{l=1}^r m_l}e^{[y+\sum_{l=1}^r m_l]_qq^x(u+v)}$$

$$
\begin{aligned}
&\times\left(\prod_{l=1}^{r}\chi\left(m_l\right)\right)\\
&= e^{[x]_q u}\sum_{n=0}^{\infty}q^{nx}E_{n,\chi,q}^{(h,r)}\left(y\right)\frac{(u+v)^n}{n!}\\
&= \left(\sum_{l=0}^{\infty}[x]_q^l\frac{u^l}{l!}\right)\\
&\quad\times\left(\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}q^{(k+n)x}E_{k+n,\chi,q}^{(h,r)}\left(y\right)\frac{u^k}{k!}\frac{v^n}{n!}\right)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}q^{(k+n)x}E_{k+n,\chi,q}^{(h,r)}\left(y\right)[x]_q^{m-k}\right)\\
&\quad\times\frac{u^m}{m!}\frac{v^n}{n!}.
\end{aligned}\tag{21}
$$

The right hand side of (20) multiplied by $[2]_q^r$ is given by

$$
\begin{aligned}
&[2]_q^r e^{-[x]_q v}\sum_{m_1,\ldots,m_r=0}^{\infty}(-1)^{\sum_{l=1}^{r}m_l}q^{\sum_{l=1}^{r}(h-l+1)m_l}\\
&\quad\times\left(\prod_{l=1}^{r}\chi\left(m_l\right)\right)e^{[x+\sum_{l=1}^{r}m_l]_q(u+v)}\\
&= e^{-[x]_q v}\sum_{n=0}^{\infty}E_{n,\chi,q}^{(h,r)}\left(x+y\right)\frac{(u+v)^n}{n!}\\
&= \left(\sum_{l=0}^{\infty}\frac{\left(-[x]_q\right)^l}{l!}v^l\right)\\
&\quad\times\left(\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}E_{m+k,\chi,q}^{(h,r)}\left(x+y\right)\frac{u^m}{m!}\frac{v^k}{k!}\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{m+k,\chi,q}^{(h,r)}\left(x+y\right)\left(-[x]_q\right)^{n-k}\right)\\
&\quad\times\frac{u^m}{m!}\frac{v^n}{n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{m+k,\chi,q}^{(h,r)}\left(x+y\right)q^{(n-k)x}[-x]_q^{n-k}\right)\\
&\quad\times\frac{u^m}{m!}\frac{v^n}{n!}.
\end{aligned}\tag{22}
$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 4. *For $m, n \in \mathbb{Z}_{\geq 0}$, one has*

$$
\begin{aligned}
&\sum_{k=0}^{m}\binom{m}{k}q^{kx}E_{n+k,\chi,q}^{(h,r)}\left(y\right)[x]_q^{m-k}\\
&= \sum_{k=0}^{n}\binom{n}{k}q^{-kx}E_{m+k,\chi,q}^{(h,r)}\left(x+y\right)[-x]_q^{n-k}.
\end{aligned}\tag{23}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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