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Research Article

Error Estimate of Eigenvalues of Perturbed Higher-Order Discrete Vector Boundary Value Problems

Haiyan Lv,[1] Yuming Shi,[2] and Guojing Ren[3]

[1] *School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China*
[2] *School of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[3] *School of Statistics and Mathematics, Shandong Provincial Key Laboratory of Digital Media Technology, Shandong University of Finance and Economics, Jinan, Shandong 250014, China*

Correspondence should be addressed to Haiyan Lv; lhy_776@163.com

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This paper is concerned with the eigenvalues of perturbed higher-order discrete vector boundary value problems. A suitable admissible function space is first introduced, a new variational formula of eigenvalues is then established under certain nonsingularity conditions, and error estimates of eigenvalues of problems with small perturbation are finally given by using the variational formula. As a direct consequence, continuous dependence of eigenvalues on boundary value problems is obtained under the nonsingularity conditions. In addition, two special perturbed cases are discussed.

1. Introduction

Consider the following $2n$-order vector difference equation:

$$\sum_{i=0}^{n}\Delta^{i}\left[r_i(t)\,\Delta^{i}y(t-i)\right]=\lambda w(t)\,y(t),\qquad t\in[n,N+n],\tag{1}$$

with the boundary condition

$$R\begin{pmatrix}-u(0,y)\\ u(N+1,y)\end{pmatrix}+S\begin{pmatrix}v(0,y)\\ v(N+1,y)\end{pmatrix}=0,\tag{2}$$

where Δ is the forward difference operator; that is, $\Delta y(t)=y(t+1)-y(t)$; $y(t)$ is a d-dimensional column vector-valued function on interval $[0,N+2n]$ of integer, $n\ge 1$ and $N\ge 2n-1$; $r_i(t)$ $(t\in[n,N+n+i],\,0\le i\le n)$ and $w(t)$ $(t\in[n,N+n])$ are $d\times d$ Hermitian matrices, $w(t)>0$ $(t\in[n,N+n])$;

$$r_n(t)\ \text{is nonsingular on}\ [n,2n-1]\cup[N+n+1,N+2n];\tag{3}$$

R and S are $2nd\times 2nd$ matrices satisfying the following self-adjoint condition [1, Lemma 2.1]:

$$\operatorname{rank}(R,S)=2nd,\qquad RS^{*}=SR^{*};\tag{4}$$

$u^T(t,y)=(u_1^T(t,y),\dots,u_n^T(t,y))$, $v^T(t,y)=(v_1^T(t,y),\dots,v_n^T(t,y))$ are nd-dimensional vectors;

$$\begin{aligned}u_i(t,y)&=\Delta^{i-1}y(t+n-i),\\ v_i(t,y)&=(-1)^{i-1}\sum_{k=i}^{n}\Delta^{k-i}\left[r_k(t+n)\,\Delta^{k}y(t+n-k)\right];\end{aligned}\tag{5}$$

y^T denotes the transpose of y; R^* denotes the complex conjugate transpose of R; and $\lambda\in\mathbb{C}$ is the spectral parameter.

Higher-order discrete linear problems also have been investigated by some scholars besides second-order discrete Sturm-Liouville problems and discrete linear Hamiltonian systems (cf. [2–14] and their references). Zhou [15] and Grzegorczyk and Werbowski [7] studied a higher-order linear difference equation in which the leading coefficient is equal to 1 and established some criteria for the oscillation of solutions. Shi and Chen [1] investigated higher-order discrete linear boundary value problems (1)-(2) and obtained some spectral results, including Rayleigh's principle, the minimax theorem, the dual orthogonality, and the number of eigenvalues. These results establish the theoretical foundation for our further research. Ren and Shi [16] discussed the defect index of singular symmetric linear difference equations of order $2n$ with real coefficients and one singular endpoint and showed that the defect index d satisfies the inequalities $n \leq d \leq 2n$ and that all values of d in this range are realized. However, because of the characteristics of higher-order difference equations, compared with the research of second-order difference equations and discrete Hamiltonian systems, it is more difficult to study higher-order difference equations. Thus, there are few references in higher-order difference equations. For more information about higher-order discrete linear problems, the reader is referred to [6, 12, 14].

Recently, we have studied second-order discrete Sturm-Liouville problems and obtained error estimates of eigenvalues of perturbed problem under some hypotheses in [17]. Motivated by the ideas and methods used in [17], we extend the results to $2n$-order discrete vector boundary value problems (1)-(2) by means of the results obtained in [1]. Although the method is similar, the problems we investigate in this paper are more complex, since they are not only of higher order but also of higher dimension.

If $r_n(t)$ is nonsingular on $[n, N+n]$, then the $2n$-order vector difference equation (1) can be converted into the discrete linear Hamiltonian system studied in [18]:

$$\begin{aligned} \Delta x(t) &= A(t)x(t+1) + B(t)u(t), \\ \Delta u(t) &= [C(t) - \lambda \mathscr{W}(t)]x(t+1) - A^*(t)u(t), \\ & \qquad t \in [0, N], \end{aligned} \tag{6}$$

where

$$\begin{aligned} A(t) &= \begin{pmatrix} 0 & I_{d(n-1)} \\ 0 & 0 \end{pmatrix}, \\ B(t) &= \operatorname{diag}\left\{0, \ldots, 0, (-1)^{n-1} r_n^{-1}(t+n)\right\}, \\ C(t) &= \operatorname{diag}\left\{-r_0(t+n), r_1(t+n), \ldots, (-1)^n r_{n-1}(t+n)\right\}, \\ \mathscr{W}(t) &= \operatorname{diag}\{w(t+n), 0, \ldots, 0\}. \end{aligned} \tag{7}$$

However, hypothesis (3) does not require the leading coefficient $r_n(t)$ to be always nonsingular in $[n, N+n]$. So, the coefficient $B(t)$ and the weight functions $\mathscr{W}(t)$ of the corresponding discrete linear Hamiltonian system do not satisfy assumption (2.1) and the positive definiteness of the weight function in [18]. Hence the Hamiltonian system considered in [18] does not include the equation we discuss in this paper.

In the present paper, we study error estimate of eigenvalues of (1)-(2) under small perturbation. By employing a variational property—the minimax theorem established in [1]—an error estimate of eigenvalues of all perturbed problems sufficiently close to problem (1)-(2) is given under certain nonsingularity conditions. The continuous dependence of eigenvalues on problems is consequently obtained from the error estimate under the nonsingularity conditions. The continuous dependence of eigenvalues on problems may not hold in general. It is under certain nonsingularity conditions that we get the related result. In addition, the minimax theorem [1, Theorem 3.5] was established in an admissible function space, which is dependent on boundary condition (2). Hence, it is difficult to apply to the case that some perturbation occurs in boundary condition (2). So we will first establish a minimax theorem in an admissible function space with a new weight function that includes the data of (1) and boundary condition (2) by [1, Theorem 3.5]. Then, employing the new minimax theorem, we study the error estimate of eigenvalues of perturbed problem. Another difficulty results from the complicated calculations since the problem is not only of higher order but also of higher dimension and needs to estimate the norms of inverses of some perturbed matrices.

The setup of this paper is as follows. In the next section, we recall some useful existing results, introduce a new suitable admissible function space, and establish a new minimax theorem in it. In Section 3, we give the main results that provide error estimates of eigenvalues of perturbed problems of (1)-(2) under certain nonsingularity conditions. Finally, We discuss two special perturbed problems in Section 4.

2. Preliminaries

In this section, we first introduce some notations and results for convenience in the following discussion, then give a suitable admissible function space, and establish a new variational property of eigenvalues for (1)-(2) in this space.

Consider the following linear space:

$$L[0, N+2n] := \left\{ y = \{y(t)\}_{t=0}^{N+2n} \subset \mathbb{C}^d \right\}. \tag{8}$$

Obviously, $\dim L[0, N+2n] = (N+2n+1)d$. Let $\mathscr{L}$ denote the following difference operator:

$$(\mathscr{L}y)(t) := w^{-1}(t) \sum_{i=0}^{n} \Delta^i \left[r_i(t) \Delta^i y(t-i) \right], \quad t \in [n, N+n]. \tag{9}$$

For convenience, for $y \in L[0, N+2n]$, we write $y \in \mathscr{B}$ if y satisfies boundary condition (2). Denote

$$\widehat{L}[0, N+2n] := \{ y \in L[0, N+2n] : y \in \mathscr{B} \}. \tag{10}$$

Lemma 1 (see, [1, Lemma 2.2]). *Assume that* (3) *and* (4) *hold. Then $y \in \mathscr{B}$ if and only if there exists a unique vector $\xi \in \mathbb{C}^{2nd}$ such that*

$$\begin{pmatrix} -u(0, y) \\ u(N+1, y) \end{pmatrix} = -S^* \xi, \qquad \begin{pmatrix} v(0, y) \\ v(N+1, y) \end{pmatrix} = R^* \xi. \tag{11}$$

Let

$$\begin{aligned} Y^T(t, k) &:= \left(y^T(t+k-1), \ldots, y^T(t) \right), \\ & t \in [0, N+n+1], \quad k \in [1, n], \end{aligned} \tag{12}$$

$$Y(t) := Y(t, n). \tag{13}$$

In particular,

$$\begin{aligned} Y^T(0) &= \left(y^T(n-1),\dots,y^T(0)\right),\\ Y^T(n) &= \left(y^T(2n-1),\dots,y^T(n)\right),\\ Y^T(N+1) &= \left(y^T(N+n),\dots,y^T(N+1)\right),\\ Y^T(N+n+1) &= \left(y^T(N+2n),\dots,y^T(N+n+1)\right). \end{aligned} \tag{14}$$

Express u and v in terms of Y:

$$\begin{aligned} u(0,y) &= LY(0),\\ v(0,y) &= AY(n)+BY(0),\\ u(N+1,y) &= LY(N+1),\\ v(N+1,y) &= A_1Y(N+n+1)+B_1Y(N+1), \end{aligned} \tag{15}$$

where $L=(l_{ij})$, $A=(a_{ij})$, $B=(b_{ij})$, and A_1 and B_1 are $nd\times nd$ matrices; A_1 and B_1 are matrices about $r_i(t+N+1)$, which are the shifts of variable t of $r_i(t)$ in A and B to the right with $N+1$ units, respectively. More precisely, for $1\le i,j\le n$,

$$l_{ij}=\begin{cases}0_d, & i<j,\\ (-1)^{j-1}C_{i-1}^{j-1}I_d, & i\ge j,\end{cases} \tag{16}$$

$$a_{ij}=\begin{cases}(-1)^{i-1}\displaystyle\sum_{k=0}^{j-i}\sum_{h=0}^{k}(-1)^kC_{n-j+k}^{h}\\ \quad\times C_{i+n-j+k}^{k-h}r_{i+n-j+k}(2n-j+k-h), & i\le j,\\ 0_d, & i>j,\end{cases} \tag{17}$$

$$b_{ij}=\begin{cases}-\displaystyle\sum_{k=0}^{n-j}\sum_{h=0}^{k}(-1)^kC_{j-i+k}^{k-h}C_{j+k}^{h}r_{j+k}(n+k-h), & i\le j,\\ (-1)^{i-j+1}\displaystyle\sum_{k=0}^{n-i}\sum_{h=0}^{k}(-1)^kC_k^hC_{i+k}^{k-h+j}\\ \quad\times r_{i+k}(n+k-h), & i>j.\end{cases} \tag{18}$$

Obviously, L is nonsingular and $L^{-1}=L$. In upper triangular matrix A, $a_{ii}=(-1)^{i-1}r_n(2n-i)$. So if (3) holds, then A and A_1 are nonsingular. By Proposition 2.1 in [1], L^*B, L^*B_1, BL, and B_1L are Hermitian matrices.

Denote

$$R=(R_1,R_2),\qquad S=(S_1,S_2), \tag{19}$$

where R_j and S_j $(j=1,2)$ are $2nd\times nd$ matrices. Substitute (15) into (2) to get

$$\begin{aligned} &\Omega\operatorname{diag}\{L,-A_1\}\begin{pmatrix}Y(0)\\ Y(N+n+1)\end{pmatrix}\\ &\quad=(S_1A,R_2L+S_2B_1)\begin{pmatrix}Y(n)\\ Y(N+1)\end{pmatrix}, \end{aligned} \tag{20}$$

where

$$\Omega:=\left(R_1-S_1BL^{-1},S_2\right)=\left(R_1-S_1BL,S_2\right). \tag{21}$$

Next, we always assume that

$$\Omega\ \text{ is nonsingular.} \tag{22}$$

Let

$$\langle x,y\rangle:=\sum_{t=n}^{N+n}y^*(t)\,w(t)\,x(t),\qquad x,y\in L[0,N+2n]. \tag{23}$$

When (22) holds, $\widehat{L}[0,N+2n]$ is an $(N+1)d$-dimensional Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ by Theorem 2.3 in [1]. In this case, $\widehat{L}[0,N+2n]$ is the same as the admissible function space $L_\omega^2[0,N+2n]$ in [1].

A series of spectral results including the variational properties of eigenvalues for problem (1)-(2) have been established by Shi and Chen in [1]. We state some of these results for future use.

The following lemma is Theorem 3.1 in [1] in the special case that (22) holds.

Lemma 2. *Assume that* (3), (4), *and* (22) *hold. Then problem* (1)-(2) *has exactly* $(N+1)d$ *real eigenvalues (multiplicity included) arranged as*

$$\lambda_1\le\lambda_2\le\cdots\le\lambda_{(N+1)d}. \tag{24}$$

The Rayleigh quotient for the difference operator $\mathscr{L}$ on $\widehat{L}[0,N+2n]$ with $\langle\cdot,\cdot\rangle$ is defined by

$$R(y):=\frac{\langle\mathscr{L}y,y\rangle}{\langle y,y\rangle}, \tag{25}$$

where $y\in\widehat{L}[0,N+2n]$ and $y'=\{y(t)\}_{t=n}^{N+n}\ne0$.

The following lemma is the minimax theorem—Theorem 3.5 in [1] in the special case that (22) holds.

Lemma 3. *Assume that* (3), (4), *and* (22) *hold. Then, for each* k, $1\le k\le(N+1)d$,

$$\begin{aligned} \lambda_k=\max\Big\{f\left(z^{(1)},\dots,z^{(k-1)}\right):z^{(j)}\in\widehat{L}\\ \times[0,N+2n],1\le j\le k-1\Big\} \end{aligned} \tag{26}$$

with $f(z^{(1)},\dots,z^{(k-1)})=\min\{R(y):y\in\widehat{L}[0,N+2n],y\perp z^{(j)},1\le j\le k-1,y'\ne0\}$, *where* $y\perp z^{(j)}$ *denotes* $\langle y,z^{(j)}\rangle=0$.

Since the perturbation of (1) and (2) affects the space $\widehat{L}[0,N+2n]$, we need a new suitable admissible function space and a variational formula to apply (26).

For any $y\in\widehat{L}[0,N+2n]$, by Lemma 1 and (15) there exists a unique vector $\xi\in\mathbb{C}^{2nd}$ such that

$$\begin{aligned} &Y(0)=LS_1^*\xi,\qquad Y(n)=A^{-1}\left(R_1^*-BLS_1^*\right)\xi,\\ &Y(N+1)=-LS_2^*\xi,\quad Y(N+n+1)=A_1^{-1}\left(R_2^*+B_1LS_2^*\right)\xi; \end{aligned} \tag{27}$$

that is,

$$\begin{pmatrix} Y(0) \\ Y(n) \\ Y(N+1) \\ Y(N+n+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & L & 0 \\ A^{-1} & 0 & -A^{-1}BL & 0 \\ 0 & 0 & 0 & -L \\ 0 & A_1^{-1} & 0 & A_1^{-1}B_1L \end{pmatrix} (R,S)^*\xi. \tag{28}$$

From above we know that $\{Y(0), Y(n), Y(N+1), Y(N+n+1)\}$, and then $\{y(0), \dots, y(2n-1), y(N+1), \dots, y(N+2n)\}$ can be uniquely determined by ξ. Hence, we introduce the following new admissible function space:

$$X := \left\{z = \{\xi, y(2n), \dots, y(N)\} : \xi \in \mathbb{C}^{2nd}, \right. \\ \left. y(t) \in \mathbb{C}^d, 2n \le t \le N\right\}. \tag{29}$$

Since $w(t) > 0$ ($t \in [n, N+n]$), it follows from (3) and (22) that

$$W := \Omega \operatorname{diag}\left\{A^{-1^*}W_1A^{-1}, L^*W_2L\right\}\Omega^* > 0, \tag{30}$$

where

$$W_1 = \operatorname{diag}\{w(2n-1), \dots, w(n)\}, \\ W_2 = \operatorname{diag}\{w(N+n), \dots, w(N+1)\}. \tag{31}$$

Thus, we can define an inner product on X by

$$\langle z_1, z_2\rangle_1 := \eta^* W\xi + \sum_{t=2n}^{N} y^*(t)\, w(t)\, x(t), \tag{32}$$

where $z_1 = \{\xi, x(2n), \dots, x(N)\}$, $z_2 = \{\eta, y(2n), \dots, y(N)\} \in X$. Denote its induced norm by

$$\|z_1\|_1 := \left(\langle z_1, z_1\rangle_1\right)^{1/2}. \tag{33}$$

Obviously, $(X, \langle\cdot,\cdot\rangle_1)$ is also an $(N+1)d$-dimensional Hilbert space. Note that the elements of the space X are independent of (1) and boundary condition (2), which are partly put in the new weight function $\{W\} \cup \{w(t)\}_{t=2n}^{N}$.

In order to establish a connection between X and $\widehat{L}[0, N+2n]$, we define a linear map

$$T_1 : X \longrightarrow \widehat{L}[0, N+2n], \tag{34}$$

by $T_1(z) = y = \{y(t)\}_{t=0}^{N+2n} \in \widehat{L}[0, N+2n]$ with $\{Y(0), Y(n), Y(N+1), Y(N+n+1)\}$ determined by (28) for $z = \{\xi, y(2n), \dots, y(N)\} \in X$. Evidently, T_1 is an invertible linear map. Moreover, for any $z_1 = \{\xi, x(2n), \dots, x(N)\}$, $z_2 = \{\eta, y(2n), \dots, y(N)\} \in X$, set $T_1(z_1) = x$ and $T_1(z_2) = y$. Then, from (23), (27), and (30), we have

$$\begin{aligned} \langle T_1(z_1), T_1(z_2)\rangle &= \sum_{t=n}^{N+n} y^*(t)\, w(t)\, x(t) \\ &= Y^*(n) W_1 X(n) + Y^*(N+1) \\ &\quad \times W_2 X(N+1) + \sum_{t=2n}^{N} y^*(t)\, w(t)\, x(t) \\ &= \begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix}^* \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \begin{pmatrix} X(n) \\ X(N+1) \end{pmatrix} \\ &\quad + \sum_{t=2n}^{N} y^*(t)\, w(t)\, x(t) \\ &= \eta^* W\xi + \sum_{t=2n}^{N} y^*(t)\, w(t)\, x(t) \\ &= \langle z_1, z_2\rangle_1; \end{aligned} \tag{35}$$

that is, T_1 is a product-preserving map.

Next, we introduce the Rayleigh quotient corresponding to $\mathscr{L}$ on X with $\langle\cdot,\cdot\rangle_1$ as follows:

$$\mathscr{R}(z) := \frac{P(z)}{\langle z, z\rangle_1}, \\ z = \{\xi, y(2n), \dots, y(N)\} \in X, \quad z \ne 0, \tag{36}$$

where $P(z) = \langle \mathscr{L}(T_1(z)), T_1(z)\rangle$ and $T_1(z) = y$. By a direct calculation we have from (9) and (23) that

$$\begin{aligned} P(z) &= \langle \mathscr{L}(T_1(z)), T_1(z)\rangle = \langle \mathscr{L}y, y\rangle \\ &= \sum_{t=n}^{N+n} y^*(t) \left\{\sum_{i=0}^{n} \Delta^i \left[r_i(t)\, \Delta^i y(t-i)\right]\right\} \\ &= \sum_{t=n}^{N+n} y^*(t) \left\{\sum_{i=0}^{n} D_i(t)\, y(t+i) \right. \\ &\qquad \left. + \sum_{i=1}^{n} D_i(t-i)\, y(t-i)\right\} \\ &= Y^*(n) L_1 Y(n) + Y^*(N+1) L_2 Y(N+1) \\ &\quad + Y^*(n) L_3 Y(0) + Y^*(N+1) L_4 Y(N+n+1) \\ &\quad + 2\operatorname{Re}\{Y^*(N-n+1) L_5 Y(N+1)\} \\ &\quad + 2\operatorname{Re}\{Y^*(n) L_6 Y(2n)\} \\ &\quad + \sum_{t=2n}^{N} y^*(t)\, D_0(t)\, y(t) \\ &\quad + 2\operatorname{Re}\left\{\sum_{i=1}^{n} \sum_{t=2n}^{N-i} y^*(t)\, D_i(t)\, y(t+i)\right\}, \end{aligned} \tag{37}$$

where

$$D_i(t)=\sum_{k=0}^{n-i}\sum_{h=0}^{k}(-1)^k C_{i+k}^{k-h}C_{i+k}^{h} r_{i+k}(t+i+k-h),\quad n-i\le t\le N+n,\ 0\le i\le n. \tag{38}$$

For any $1\le i,j\le n$, and $1\le p\le 6$, $L_p=(l_{ij}^p)$ are $nd\times nd$ matrices, l_{ij}^p are $d\times d$ matrices, and

$$\begin{aligned}
l_{ij}^1&=\begin{cases}D_{j-i}(2n-j), & i\le j,\\ D_{i-j}(2n-i), & i>j,\end{cases}\\
l_{ij}^2&=\begin{cases}D_{j-i}(N+n-j+1), & i\le j,\\ D_{i-j}(N+n-i+1), & i>j,\end{cases}\\
l_{ij}^3&=\begin{cases}0, & i<j,\\ D_{n-i+j}(n-j), & i\ge j,\end{cases}\\
l_{ij}^4&=\begin{cases}D_{n+i-j}(N+n-i+1), & i\le j,\\ 0, & i>j,\end{cases}\\
l_{ij}^5&=\begin{cases}D_{n+i-j}(N-i+1), & i\le j,\\ 0, & i>j,\end{cases}\\
l_{ij}^6&=\begin{cases}D_{n+i-j}(2n-i), & i\le j,\\ 0, & i>j.\end{cases}
\end{aligned} \tag{39}$$

Further, it follows from (27) that

$$\begin{aligned}
P(z)={}&\xi^*\left(\Omega M\Omega^*+(R_1-S_1BL)A^{-1^*}L_3LS_1^*\right.\\
&\left.-S_2L^*L_4A_1^{-1}(R_2^*+B_1LS_2^*)\right)\xi\\
&-2\operatorname{Re}\left\{Y^*(N-n+1)L_5LS_2^*\xi\right\}\\
&+2\operatorname{Re}\left\{\xi^*(R_1-S_1BL)A^{-1^*}L_6Y(2n)\right\}\\
&+\sum_{t=2n}^{N}y^*(t)D_0(t)y(t)\\
&+2\operatorname{Re}\left\{\sum_{i=1}^{n}\sum_{t=2n}^{N-i}y^*(t)D_i(t)y(t+i)\right\},
\end{aligned} \tag{40}$$

where

$$M=\operatorname{diag}\left\{A^{-1^*}L_1A^{-1},L^*L_2L\right\}. \tag{41}$$

On the basis of the above discussion, we obtain the following variational formula of eigenvalues for (1)-(2) on X by Lemma 3, which plays an important role in the next section.

Theorem 4. *Assume that* (3), (4), *and* (22) *hold. Then, for each* k, $1\le k\le (N+1)d$,

$$\lambda_k=\max\left\{g\left(z^{(1)},\dots,z^{(k-1)}\right):z^{(j)}\in X,1\le j\le k-1\right\} \tag{42}$$

with $g(z^{(1)},\dots,z^{(k-1)})=\min\{\mathscr{R}(z):z\in X, z\perp_1 z^{(j)},1\le j\le k-1,z\ne 0\}$, *where* $z\perp_1 z^{(j)}$ *denotes* $\langle z,z^{(j)}\rangle_1=0$.

At the end of this section, we quote two lemmas about matrices and their perturbation. For convenience, we introduce the following notation for an invertible matrix $A=(a_{ij})\in\mathbb{C}^{d\times d}$:

$$h(A):=\frac{|\det A|}{2d\sqrt{d!}(\|A\|+1)^{d-1}}, \tag{43}$$

where the norm of matrix A is defined by

$$\|A\|:=\left(\sum_{i,j=1}^{d}|a_{ij}|^2\right)^{1/2}. \tag{44}$$

With the aid of [9, Corollary 7.8.2] we have immediately the following results.

Lemma 5. *For any matrix* $A=(a_{ij})\in\mathbb{C}^{d\times d}$, $|\det A|\le\|A\|^d$.

Lemma 6 ([17, Lemma 2.5]). *Let* $A\in\mathbb{C}^{d\times d}$ *be invertible. If a matrix* $\widetilde{A}\in\mathbb{C}^{d\times d}$ *satisfies*

$$\left\|\widetilde{A}-A\right\|\le\min\{h(A),1\}, \tag{45}$$

then $\widetilde{A}$ *is invertible, and*

$$\left\|\widetilde{A}^{-1}\right\|\le\frac{2d(\|A\|+1)^{d-1}}{|\det A|}. \tag{46}$$

3. Main Results

In this section, we discuss eigenvalues of perturbed problems sufficiently close to problem (1)-(2) and give error estimates of them.

For convenience, introduce the following notations and several constant matrices:

$$\begin{gathered}
w=\max_{n\le t\le N+n}\|w(t)\|,\qquad w_0=\min_{n\le t\le N+n}|\det w(t)|,\\
r=\|R\|,\qquad s=\|S\|,\qquad l=\|L\|,\\
\hat{r}=\max\left\{\|r_i(t)\|:t\in[n,N+n+i],0\le i\le n\right\},\\
r_0=\min\left\{|\det r_n(t)|:t\in[n,2n-1]\right.\\
\left.\cup[N+n+1,N+2n]\right\},\\
d_i=\sum_{k=0}^{n-i}\sum_{h=0}^{k}C_{i+k}^{k-h}C_{i+k}^{h},\quad 0\le i\le n,\ \hat{d}=d_0+2\sum_{i=1}^{n}d_i.
\end{gathered} \tag{47}$$

For any $0\le i,j\le n$,

$$e_1=\left(\sum_{i\le j}|d_{j-i}|^2+\sum_{i>j}|d_{i-j}|^2\right)^{1/2},$$

$$e_2=\left(\sum_{i\le j}|d_{n+i-j}|^2\right)^{1/2},$$

$$a_{ij}^0 = \begin{cases} \sum_{k=0}^{j-i}\sum_{h=0}^{k} C_{n-j+k}^{h} C_{i+n-j+k}^{k-h}, & i \le j, \\ 0, & i > j, \end{cases}$$

$$b_{ij}^0 = \begin{cases} \sum_{k=0}^{n-j}\sum_{h=0}^{k} C_{j-i+k}^{k-h} C_{j+k}^{h}, & i \le j, \\ \sum_{k=0}^{n-i}\sum_{h=0}^{k} C_{k}^{h} C_{i+k}^{k-h+j}, & i > j, \end{cases}$$

$$a = \left\| \left(a_{ij}^0\right)_{n\times n} \right\|, \qquad b = \left\| \left(b_{ij}^0\right)_{n\times n} \right\|, \qquad \widehat{s} = r + s + sb\widehat{r}l, \tag{48}$$

$$\beta = \min\left\{\min \sigma\left(w\left(t\right)\right) : t \in [n, N+n]\right\}, \qquad \gamma = \min \sigma\left(W\right), \tag{49}$$

where $\min \sigma(w(t))$ denotes the minimum value of all eigenvalues of $w(t)$ and W is the same as in (30). It is evident that $\beta > 0$ and $\gamma > 0$.

Based on the above discussion, we know

$$\begin{gathered} a < b, \qquad \|A\| \le a\widehat{r}, \qquad \|A_1\| \le a\widehat{r}, \qquad \|B\| \le b\widehat{r}, \\ \|B_1\| \le b\widehat{r}, \qquad \|L_1\| \le e_1\widehat{r}, \\ \|L_2\| \le e_1\widehat{r}, \qquad \|L_p\| \le e_2\widehat{r} \quad (3 \le p \le 6), \\ \|D_i(t)\| \le d_i\widehat{r} \quad (0 \le i \le n). \end{gathered} \tag{50}$$

Now, we consider the following perturbed problem of (1)-(2):

$$\sum_{i=0}^{n} \Delta^i \left[\widetilde{r}_i\left(t\right) \Delta^i y\left(t-i\right)\right] = \lambda \widetilde{w}\left(t\right) y\left(t\right), \quad t \in [n, N+n], \tag{1$'$}$$

$$\widetilde{R}\begin{pmatrix} -u\left(0,y\right) \\ u\left(N+1,y\right) \end{pmatrix} + \widetilde{S}\begin{pmatrix} v\left(0,y\right) \\ v\left(N+1,y\right) \end{pmatrix} = 0, \tag{2$'$}$$

where $\widetilde{r}_i(t)$ $(t \in [n, N+n+i], 0 \le i \le n)$ and $\widetilde{w}(t)$ $(t \in [n, N+n])$ are $d \times d$ Hermitian matrices, $\widetilde{w}(t) > 0$ $(t \in [n, N+n])$, and $\widetilde{R}$ and $\widetilde{S}$ are $2nd \times 2nd$ matrices and satisfy

$$\widetilde{R}\widetilde{S}^* = \widetilde{S}\widetilde{R}^*. \tag{51}$$

In the following, we will prove that if the perturbation is sufficiently small in norm, then

$$\widetilde{r}_n\left(t\right) \text{ is nonsingular on } [n, 2n-1] \cup [N+n+1, N+2n], \tag{52}$$

$$\operatorname{rank}\left(\widetilde{R}, \widetilde{S}\right) = 2nd, \quad \widetilde{\Omega} = \left(\widetilde{R}_1 - \widetilde{S}_1\widetilde{B}L, \widetilde{S}_2\right) \text{ is invertible}, \tag{53}$$

where $\widetilde{B}$ has the same form of B with $r_i(t)$ in (18) replaced by $\widetilde{r}_i(t)$. The matrices $\widetilde{B}_1$, $\widetilde{A}$, $\widetilde{A}_1$, $\widetilde{D}_i(t)$ $(0 \le i \le n)$, $\widetilde{L}_p$ $(1 \le p \le 6)$ are the perturbations of the matrices B, A, A_1, $D_i(t)$ $(0 \le i \le n)$, L_p $(1 \le p \le 6)$, respectively.

Proposition 7. *Let*

$$\begin{gathered} \varepsilon_1 := \min\left\{\frac{\sqrt{2}}{2}h\left(D\right), \frac{h\left(\Omega\right)}{\left(\widehat{r}+s+1\right)bl+2}, \frac{1}{\left(\widehat{r}+s+1\right)bl+2}, \right. \\ \left. \frac{r_0^n}{2nd\sqrt{(nd)!}(a\widehat{r}+1)^{nd-1}a}\right\}, \end{gathered} \tag{54}$$

where D is a $2nd \times 2nd$ nonsingular submatrix of (R,S). For any $0 < \varepsilon \le \varepsilon_1$, if

$$\left\|\widetilde{R} - R\right\| \le \varepsilon, \qquad \left\|\widetilde{S} - S\right\| \le \varepsilon, \tag{55}$$

$$\left\|\widetilde{r}_i\left(t\right) - r_i\left(t\right)\right\| \le \varepsilon, \quad t \in [n, N+n+i], \ 0 \le i \le n, \tag{56}$$

then

(i) (52) *holds, $\widetilde{A}$ and $\widetilde{A}_1$ are nonsingular, and*

$$\left\|\widetilde{A}^{-1}\right\| \le m, \qquad \left\|\widetilde{A}_1^{-1}\right\| \le m, \tag{57}$$

where

$$m = \frac{2nd(a\widehat{r}+1)^{nd-1}}{r_0^n}; \tag{58}$$

(ii) (53) *holds, and*

$$\left\|\widetilde{\Omega}^{-1}\right\| \le 4nd(\widehat{s}+1)^{2nd-1}|\det \Omega|^{-1}. \tag{59}$$

Proof. (i) We only prove that $\widetilde{A}$ is invertible. The invertibility of $\widetilde{A}_1$ can be similarly proved. Since

$$\begin{aligned} \varepsilon &\le \frac{r_0^n}{2nd\sqrt{(nd)!}(a\widehat{r}+1)^{nd-1}a} \\ &\le \frac{|\det A|}{2nd\sqrt{(nd)!}(\|A\|+1)^{nd-1}a} = \frac{h\left(A\right)}{a}, \end{aligned} \tag{60}$$

we have

$$\left\|\widetilde{A} - A\right\| \le a\varepsilon \le \min\left\{h\left(A\right), 1\right\}. \tag{61}$$

Thus, $\widetilde{A}$ is invertible by Lemma 6, and

$$\left\|\widetilde{A}^{-1}\right\| \le \frac{2nd(\|A\|+1)^{nd-1}}{|\det A|} \le m. \tag{62}$$

In addition, since

$$\det \widetilde{A} = (-1)^{(n(n-1)d)/2} \det \widetilde{r}_n\left(n\right) \cdots \det \widetilde{r}_n\left(2n-1\right), \tag{63}$$

then $\widetilde{r}_n(t)$ is nonsingular on $[n, 2n-1]$, which, together with the invertibility of $\widetilde{A}_1$, yields that (52) holds. So (i) is proved.

(ii) Let $\widetilde{D}$ be a $2nd \times 2nd$ submatrix of $(\widetilde{R}, \widetilde{S})$ and let its position be the same as that of D in (R,S). Since

$$\left\|\widetilde{D} - D\right\|^2 \le \left\|\widetilde{R} - R\right\|^2 + \left\|\widetilde{S} - S\right\|^2 \le 2\varepsilon^2, \tag{64}$$

that is,

$$\left\|\widetilde{D}-D\right\| \leq \sqrt{2}\,\varepsilon \leq \sqrt{2}\,\varepsilon_1 \leq \min\{h(D),1\}, \quad (65)$$

$\widetilde{D}$ is invertible by Lemma 6 and, consequently, $\operatorname{rank}(\widetilde{R},\widetilde{S}) = 2nd$.

In addition,

$$\begin{aligned}
&\left\|\widetilde{S}_1\widetilde{B}L - S_1BL\right\| \\
&\quad \leq \left(\left\|\widetilde{S}_1\right\|\left\|\widetilde{B}-B\right\| + \left\|\widetilde{S}_1 - S_1\right\|\|B\|\right)\|L\| \\
&\quad \leq ((s+1)\,b\varepsilon + b\widehat{r}\varepsilon)\,l \\
&\quad = (\widehat{r}+s+1)\,bl\varepsilon.
\end{aligned} \quad (66)$$

Then we have

$$\begin{aligned}
&\left\|\widetilde{\Omega}-\Omega\right\|^2 \\
&\quad = \left\|\widetilde{R}_1 - \widetilde{S}_1\widetilde{B}L - R_1 + S_1BL\right\|^2 + \left\|\widetilde{S}_2 - S_2\right\|^2 \\
&\quad \leq \left(\left\|\widetilde{R}_1 - R_1\right\| + \left\|\widetilde{S}_1\widetilde{B}L - S_1BL\right\|\right)^2 + \left\|\widetilde{S}_2 - S_2\right\|^2 \\
&\quad \leq (\varepsilon + (\widehat{r}+s+1)\,bl\varepsilon)^2 + \varepsilon^2 \\
&\quad \leq ((\widehat{r}+s+1)\,bl+2)^2\varepsilon^2.
\end{aligned} \quad (67)$$

Thus,

$$\begin{aligned}
\left\|\widetilde{\Omega}-\Omega\right\| &\leq ((\widehat{r}+s+1)\,bl+2)\,\varepsilon \\
&\leq ((\widehat{r}+s+1)\,bl+2)\,\varepsilon_1 \\
&\leq \min\{h(\Omega),1\}.
\end{aligned} \quad (68)$$

Hence, $\widetilde{\Omega}$ is invertible and $\|\widetilde{\Omega}^{-1}\| \leq 4nd(\|\Omega\|+1)^{2nd-1}|\det\Omega|^{-1}$ by Lemma 6. Further,

$$\begin{aligned}
\|\Omega\|^2 &= \left\|R_1 - S_1BL\right\|^2 + \left\|S_2\right\|^2 \\
&\leq (r + sb\widehat{r}l)^2 + s^2 \\
&\leq (r + s + sb\widehat{r}l)^2 = \widehat{s}^2,
\end{aligned} \quad (69)$$

which yields that (59) holds. The proof is complete. □

Under the assumptions of Proposition 7, $\widetilde{A}$ and $\widetilde{\Omega}$ are invertible. So, we can define the following inner product on X corresponding to problem $(1)'$-$(2)'$:

$$\langle z_1, z_2\rangle_2 := \eta^*\widetilde{W}\xi + \sum_{t=2n}^{N} y^*(t)\,\widetilde{w}(t)\,x(t) \quad (70)$$

for any $z_1 = \{\xi, x(2n),\dots,x(N)\}$, $z_2 = \{\eta, y(2n),\dots,y(N)\} \in X$, where

$$\begin{aligned}
\widetilde{W} &= \widetilde{\Omega}\operatorname{diag}\left\{\left(\widetilde{A}^{-1}\right)^*\widetilde{W}_1\widetilde{A}^{-1}, L^*\widetilde{W}_2L\right\}\widetilde{\Omega}^* > 0, \\
\widetilde{W}_1 &= \operatorname{diag}\{\widetilde{w}(2n-1),\dots,\widetilde{w}(n)\}, \\
\widetilde{W}_2 &= \operatorname{diag}\{\widetilde{w}(N+n),\dots,\widetilde{w}(N+1)\}.
\end{aligned} \quad (71)$$

The corresponding induced norm is denoted by

$$\|z\|_2 := \left(\langle z,z\rangle_2\right)^{1/2}, \quad z = \{\xi, y(2n),\dots,y(N)\} \in X. \quad (72)$$

Similarly, $(X, \langle\cdot,\cdot\rangle_2)$ is also an $(N+1)d$-dimensional Hilbert space.

Under the hypotheses of Proposition 7, if further (51) holds, then, by Lemma 2, the perturbed problem $(1)'$-$(2)'$ has also $(N+1)d$ real eigenvalues (multiplicity included) arranged as

$$\widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \cdots \leq \widetilde{\lambda}_{(N+1)d}. \quad (73)$$

Notice that the multiplicity of $\widetilde{\lambda}_k$, the kth eigenvalue of $(1)'$-$(2)'$, may be different from that of the kth eigenvalue λ_k of (1)-(2) in general.

Similarly, The Rayleigh quotient corresponding to the difference operator for $(1)'$-$(2)'$

$$\left(\widetilde{\mathscr{L}}y\right)(t) = \widetilde{w}^{-1}(t)\sum_{i=0}^{n}\Delta^i\left[\widetilde{r}_i(t)\,\Delta^i y(t-i)\right], \quad t\in[n,N+n], \quad (74)$$

on X with $\langle\cdot,\cdot\rangle_2$ can be defined by

$$\begin{aligned}
&\widetilde{\mathscr{R}}(z) := \frac{\widetilde{P}(z)}{\langle z,z\rangle_2}, \\
&z = \{\xi, y(2n),\dots,y(N)\} \in X \quad \text{with } z \neq 0,
\end{aligned} \quad (75)$$

where

$$\begin{aligned}
\widetilde{P}(z) = {}& \xi^*\left(\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^* + \left(\widetilde{R}_1 - \widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\right. \\
&\left.-\widetilde{S}_2L^*\widetilde{L}_4\widetilde{A}_1^{-1}\left(\widetilde{R}_2^* + \widetilde{B}_1L\widetilde{S}_2^*\right)\right)\xi \\
&- 2\operatorname{Re}\left\{Y^*(N-n+1)\,\widetilde{L}_5L\widetilde{S}_2^*\xi\right\} \\
&+ 2\operatorname{Re}\left\{\xi^*\left(\widetilde{R}_1 - \widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_6Y(2n)\right\} \\
&+ \sum_{t=2n}^{N} y^*(t)\,\widetilde{D}_0(t)\,y(t) \\
&+ 2\operatorname{Re}\left\{\sum_{i=1}^{n}\sum_{t=2n}^{N-i} y^*(t)\,\widetilde{D}_i(t)\,y(t+i)\right\},
\end{aligned} \quad (76)$$

in which

$$\widetilde{M} = \operatorname{diag}\left\{\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_1\widetilde{A}^{-1}, L^*\widetilde{L}_2L\right\}. \quad (77)$$

According to the above discussion, if (51), (55), and (56) hold, then we can get the following variational formula of eigenvalues for $(1)'$-$(2)'$ on X in a similar way to Theorem 4: for each k, $1 \leq k \leq (N+1)d$,

$$\widetilde{\lambda}_k = \max\left\{\widetilde{g}\left(z^{(1)},\dots,z^{(k-1)}\right) : z^{(j)} \in X, 1 \leq j \leq k-1\right\} \quad (78)$$

with $\widetilde{g}(z^{(1)},\dots,z^{(k-1)}) = \min\{\widetilde{\mathscr{R}}(z) : z \in X, z\perp_2 z^{(j)}, 1 \leq j \leq k-1, z \neq 0\}$, where $z\perp_2 z^{(j)}$ denotes $\langle z, z^{(j)}\rangle_2 = 0$.

In order to obtain an error estimate of eigenvalues for the perturbed problem by applying the above variational formula of eigenvalues, we will discuss the relationship between $\perp_2$ and $\perp_1$ and then give another form of variational formula of eigenvalues for $(1)'$-$(2)'$ on X. Now we introduce the following linear transformation:

$$T_2 : X \longrightarrow X, \tag{79}$$

where, for any $z = \{\xi, y(2n), \ldots, y(N)\} \in X$,

$$\begin{aligned} T_2(z) = \Big\{&\widetilde{W}^{-1}W\xi, \widetilde{w}^{-1}(2n)\, w(2n)\, y(2n), \\ &\ldots, \widetilde{w}^{-1}(N)\, w(N)\, y(N)\Big\}. \end{aligned} \tag{80}$$

Obviously, T_2 is invertible, and

$$\langle z_1, z_2\rangle_2 = \left\langle T_2^{-1}(z_1), z_2\right\rangle_1, \quad \forall\, z_1, z_2 \in X. \tag{81}$$

So, for any $z^{(1)}, \ldots, z^{(k-1)} \in X$, we get

$$\begin{aligned} \widetilde{g}\left(z^{(1)}, \ldots, z^{(k-1)}\right) &= \min\Big\{\widetilde{\mathscr{R}}(z) : z \in X, z\perp_2 z^{(j)}, \\ &\qquad 1 \le j \le k-1, z \ne 0\Big\} \\ &= \min\Big\{\widetilde{\mathscr{R}}(z) : z \in X, T_2^{-1}(z)\perp_1 z^{(j)}, \\ &\qquad 1 \le j \le k-1, z \ne 0\Big\} \\ &= \min\Big\{\widetilde{\mathscr{R}}(T_2(z)) : z \in X, z\perp_1 z^{(j)}, \\ &\qquad 1 \le j \le k-1, z \ne 0\Big\}. \end{aligned} \tag{82}$$

Thus, the variational formula of eigenvalues for $(1)'$-$(2)'$ on X can be restated as follows: if (51), (55), and (56) hold, then, for each k, $1 \le k \le (N+1)d$,

$$\widetilde{\lambda}_k = \max\left\{\widetilde{g}\left(z^{(1)}, \ldots, z^{(k-1)}\right) : z^{(j)} \in X, 1 \le j \le k-1\right\} \tag{83}$$

with $\widetilde{g}(z^{(1)}, \ldots, z^{(k-1)}) = \min\{\widetilde{\mathscr{R}}(T_2(z)) : z \in X, z\perp_1 z^{(j)}, 1 \le j \le k-1, z \ne 0\}$.

Before giving the main results, we prepare some estimates.

Lemma 8. *For any* $0 < \varepsilon \le \varepsilon_1$, *if* (3) *and* (56) *hold, then*

$$\left\|\widetilde{M} - M\right\| \le \left(m^2 e_1(2ma\widehat{r} + 1) + l^2 e_1\right)\varepsilon, \tag{84}$$

$$\|M\| \le e_1 \widehat{r}\left(m^4 + l^4\right)^{1/2}, \tag{85}$$

$$\left\|\widetilde{M}\right\| \le e_1(\widehat{r} + 1)\left(m^4 + l^4\right)^{1/2}, \tag{86}$$

where ε_1, m, M, *and* $\widetilde{M}$ *are the same as in* (54), (58), (41), *and* (77), *respectively.*

Proof. $A^a = (A_{ij})_{nd\times nd}$ denotes the adjoint matrix of A. Then, by Lemma 5, we get

$$\left|A_{ij}\right| \le \|A\|^{nd-1}, \tag{87}$$

which yields

$$\left\|A^a\right\| = \left(\sum_{i,j=1}^{nd}\left|A_{ij}\right|^2\right)^{1/2} \le nd\, \|A\|^{nd-1} \le nd(a\widehat{r})^{nd-1}. \tag{88}$$

So,

$$\left\|A^{-1}\right\| = \frac{\left\|A^a\right\|}{|\det A|} \le \frac{nd(a\widehat{r})^{nd-1}}{r_0^n} \le m. \tag{89}$$

Similarly, one gets

$$\left\|A_1^{-1}\right\| \le m. \tag{90}$$

Hence, we have from (57) and (89) that

$$\begin{aligned} \left\|\widetilde{A}^{-1} - A^{-1}\right\| &= \left\|A^{-1}A\widetilde{A}^{-1} - A^{-1}\widetilde{A}\widetilde{A}^{-1}\right\| \\ &\le \left\|A^{-1}\right\|\left\|A - \widetilde{A}\right\|\left\|\widetilde{A}^{-1}\right\| \\ &\le m^2 a\varepsilon. \end{aligned} \tag{91}$$

Similarly, we obtain

$$\left\|\widetilde{A}_1^{-1} - A_1^{-1}\right\| \le m^2 a\varepsilon. \tag{92}$$

It follows from (41) and (77) that

$$\begin{aligned} \left\|\widetilde{M} - M\right\|^2 &= \left\|\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_1\widetilde{A}^{-1} - A^{-1*}L_1A^{-1}\right\|^2 \\ &\quad + \left\|L^*\widetilde{L}_2L - L^*L_2L\right\|^2. \end{aligned} \tag{93}$$

From (91) one can get

$$\begin{aligned} &\left\|\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_1\widetilde{A}^{-1} - A^{-1*}L_1A^{-1}\right\| \\ &\quad\le \left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{L}_1\widetilde{A}^{-1} - L_1A^{-1}\right\| \\ &\qquad + \left\|\widetilde{A}^{-1} - A^{-1}\right\|\left\|L_1\right\|\left\|A^{-1}\right\| \\ &\quad\le \left\|\widetilde{A}^{-1}\right\|\Big(\left\|\widetilde{L}_1 - L_1\right\|\left\|\widetilde{A}^{-1}\right\| \\ &\qquad + \left\|L_1\right\|\left\|\widetilde{A}^{-1} - A^{-1}\right\|\Big) \\ &\qquad + \left\|\widetilde{A}^{-1} - A^{-1}\right\|\left\|L_1\right\|\left\|A^{-1}\right\| \\ &\quad\le m\left(me_1\varepsilon + m^2ae_1\widehat{r}\varepsilon\right) + m^3ae_1\widehat{r}\varepsilon \\ &\quad\le m^2e_1(2ma\widehat{r} + 1)\varepsilon. \end{aligned} \tag{94}$$

In addition,

$$\left\|L^*\widetilde{L}_2L - L^*L_2L\right\| \le \|L\|^2\left\|\widetilde{L}_2 - L_2\right\| \le l^2e_1\varepsilon. \tag{95}$$

Therefore, we have

$$\left\|\widetilde{M} - M\right\| \le \left(m^2e_1(2ma\widehat{r} + 1) + l^2e_1\right)\varepsilon. \tag{96}$$

It is easy to get from (41) that

$$\begin{aligned}\|M\| &= \left[\left\|A^{-1^*}L_1A^{-1}\right\|^2 + \left\|L^*L_2L\right\|^2\right]^{1/2} \\ &\le \left[\left\|A^{-1}\right\|^4\left\|L_1\right\|^2 + \|L\|^4\left\|L_2\right\|^2\right]^{1/2} \\ &\le \left(m^4e_1^2\widehat{r}^2 + l^4e_1^2\widehat{r}^2\right)^{1/2} \\ &= e_1\widehat{r}\left(m^4 + l^4\right)^{1/2}.\end{aligned} \tag{97}$$

Inequality (86) can be obtained by a similar argument. The proof is complete. □

Now, we study $|P(z)|$ for any $z \in X$.

Proposition 9. *For any $z \in X$, if (3) holds, then*

$$|P(z)| \le \left(G_1\gamma^{-1} + G_2\beta^{-1}\right)\|z\|_1^2, \tag{98}$$

where β and γ are defined as in (49),

$$G_1 := \widehat{s}^2e_1\widehat{r}\left(m^4 + l^4\right)^{1/2} + me_2\widehat{r}\,(r + sb\widehat{r}l)\,(2sl + 1) + e_2\widehat{r}ls, \tag{99}$$

$$G_2 := (r + sb\widehat{r}l)\,me_2\widehat{r} + e_2\widehat{r}ls + \widehat{d}\widehat{r}. \tag{100}$$

Proof. For any given $z = \{\xi, y(2n), \ldots, y(N)\} \in X$, it follows from (40) that

$$\begin{aligned}|P(z)| \le{}& \|\xi\|^2\left(\|\Omega\|^2\|M\| + \left\|R_1 - S_1BL\right\|\right. \\ &\times \left\|A^{-1}\right\|\left\|L_3\right\|\|L\|\left\|S_1\right\| \\ &+ \left\|S_2\right\|\|L\|\left\|L_4\right\| \\ &\left.\times\left\|A_1^{-1}\right\|\left\|R_2^* + B_1LS_2^*\right\|\right) \\ &+ 2\|\xi\|\,\|Y(N-n+1)\| \\ &\times \left\|L_5\right\|\|L\|\left\|S_2\right\| + 2\|\xi\|\,\|Y(2n)\| \\ &\times \left\|R_1 - S_1BL\right\|\left\|A^{-1}\right\|\left\|L_6\right\| \\ &+ \sum_{t=2n}^{N}\|y(t)\|^2\left\|D_0(t)\right\| + 2\sum_{i=1}^{n}\sum_{t=2n}^{N-i} \\ &\times \|y(t)\|\,\|y(t+i)\|\,\left\|D_i(t)\right\|,\end{aligned} \tag{101}$$

where $\|x\|$ is the Euclidean norm of $x \in \mathbb{C}^d$; that is,

$$\|x\| = \left(\sum_{i=1}^{d}\left|x_i\right|^2\right)^{1/2}. \tag{102}$$

Further, from (50), (69), (85), (89), and (90) we have

$$\begin{aligned}|P(z)| \le{}& \|\xi\|^2\left(\widehat{s}^2e_1\widehat{r}\left(m^4 + l^4\right)^{1/2}\right. \\ &\left.+2slme_2\widehat{r}\,(r + sb\widehat{r}l)\right) \\ &+ \left(\|\xi\|^2 + \|Y(N-n+1)\|^2\right)e_2\widehat{r}ls \\ &+ \left(\|\xi\|^2 + \|Y(2n)\|\right)(r + sb\widehat{r}l)\,me_2\widehat{r} + \widehat{d}\widehat{r}\sum_{t=2n}^{N}\|y(t)\|^2 \\ \le{}& \|\xi\|^2\left(\widehat{s}^2e_1\widehat{r}\left(m^4 + l^4\right)^{1/2}\right. \\ &\left.+me_2\widehat{r}\,(r + sb\widehat{r}l)\,(2sl + 1) + e_2\widehat{r}ls\right) \\ &+ \left(\|Y(2n)\|^2 + \|Y(N-n+1)\|^2\right) \\ &\times\left((r + sb\widehat{r}l)\,me_2\widehat{r} + e_2\widehat{r}ls\right) + \widehat{d}\widehat{r}\sum_{t=2n}^{N}\|y(t)\|^2 \\ \le{}& G_1\|\xi\|^2 + G_2\sum_{t=2n}^{N}\|y(t)\|^2 \\ \le{}& G_1\gamma^{-1}\xi^*W\xi + G_2\beta^{-1}\sum_{t=2n}^{N}y^*(t)\,w(t)\,y(t) \\ \le{}& \left(G_1\gamma^{-1} + G_2\beta^{-1}\right)\|z\|_1^2.\end{aligned} \tag{103}$$

This completes the proof. □

Next, we study the difference between $\|T_2(z)\|_2^2$ and $\|z\|_1^2$ for any $z \in X$.

Proposition 10. *Let*

$$\varepsilon_2 := \min\left\{\varepsilon_1, \frac{w_0}{2d\sqrt{d!}(w+1)^{d-1}}\right\}, \tag{104}$$

where ε_1 is defined as in (54). For any $0 < \varepsilon \le \varepsilon_2$, if (55) and (56) hold and

$$\|\widetilde{w}(t) - w(t)\| \le \varepsilon, \quad t \in [n, N+n], \tag{105}$$

then

$$\left|\left\|T_2(z)\right\|_2^2 - \|z\|_1^2\right| \le \left(G_3\gamma^{-1} + G_4\beta^{-1}\right)\|z\|_1^2\varepsilon, \quad \forall z \in X, \tag{106}$$

where

$$\begin{aligned}G_3 :={}& \left(32g_1n^3d^3(\widehat{s}+1)^{4nd-2}(w+1)^{d-1}w\widehat{s}^2\right. \\ &\left.\times\left[\left(m^4 + l^4\right)\left(a^4(\widehat{r}+1)^4 + l^4\right)\right]^{1/2}\right) \\ &\times\left(|\det\Omega|^2w_0\right)^{-1},\end{aligned} \tag{107}$$

$$g_1 := \sqrt{n}(\hat{s}+1)^2\left(m^2(2mwa+1)+l^2\right) + \sqrt{n}w(2\hat{s}+1)((\hat{r}+s+1)bl+2) \times \left(m^4+l^4\right)^{1/2}, \tag{108}$$

$$G_4 := 2d(w+1)^{d-1}ww_0^{-1}. \tag{109}$$

Proof. It follows from (80) that, for any given $z = \{\xi, y(2n), \ldots, y(N)\} \in X$,

$$\|T_2(z)\|_2^2 = \xi^* W\widetilde{W}^{-1}W\xi + \sum_{t=2n}^{N} y^*(t)\,w(t)\,\widetilde{w}^{-1}(t)\,w(t)\,y(t), \tag{110}$$

which, together with (33), yields that

$$\begin{aligned}\left|\|T_2(z)\|_2^2 - \|z\|_1^2\right| &= \left|\xi^* W\left(\widetilde{W}^{-1}W - I_{2nd}\right)\xi + \sum_{t=2n}^{N} y^*(t)\,w(t)\left(\widetilde{w}^{-1}(t)\,w(t) - I_d\right)y(t)\right| \\ &\le \|\xi\|^2\|W\|\left\|\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\| + \sum_{t=2n}^{N}\|y(t)\|^2\|w(t)\| \times \left\|\widetilde{w}^{-1}(t)\right\|\|w(t)-\widetilde{w}(t)\|.\end{aligned} \tag{111}$$

Since

$$\varepsilon \le \frac{w_0}{2d\sqrt{d!}(w+1)^{d-1}} \le h(w(t)), \quad t \in [n, N+n], \tag{112}$$

it follows from Lemma 6 that

$$\left\|\widetilde{w}^{-1}(t)\right\| \le \frac{2d(\|w(t)\|+1)^{d-1}}{|\det w(t)|} \le 2d(w+1)^{d-1}w_0^{-1}, \quad t \in [n, N+n]. \tag{113}$$

Thus,

$$\left\|\widetilde{W}_1^{-1}\right\| = \left\|\operatorname{diag}\left\{\widetilde{w}^{-1}(2n-1), \ldots, \widetilde{w}^{-1}(n)\right\}\right\| \le 2\sqrt{n}d(w+1)^{d-1}w_0^{-1}. \tag{114}$$

Similarly, we have

$$\left\|\widetilde{W}_2^{-1}\right\| \le 2\sqrt{n}d(w+1)^{d-1}w_0^{-1}. \tag{115}$$

In addition, from (59), (71), and (113) we get

$$\begin{aligned}\left\|\widetilde{W}^{-1}\right\| &= \left\|\widetilde{\Omega}^{*^{-1}}\operatorname{diag}\left\{\widetilde{A}\widetilde{W}_1^{-1}\widetilde{A}^*, L\widetilde{W}_2^{-1}L^*\right\}\widetilde{\Omega}^{-1}\right\| \\ &\le \left\|\widetilde{\Omega}^{-1}\right\|^2\left[\left(\left\|\widetilde{A}\right\|^2\left\|\widetilde{W}_1^{-1}\right\|\right)^2 + \left(\|L\|^2\left\|\widetilde{W}_2^{-1}\right\|\right)^2\right]^{1/2} \\ &\le \left(32n^{5/2}d^3(\hat{s}+1)^{4nd-2}(w+1)^{d-1} \times \left(a^4(\hat{r}+1)^4+l^4\right)^{1/2}\right) \times \left(|\det\Omega|^2 w_0\right)^{-1}.\end{aligned} \tag{116}$$

Now, we are in a position to estimate $\|\widetilde{W}-W\|$. Let

$$\begin{aligned}\widetilde{K} &= \operatorname{diag}\left\{\left(\widetilde{A}^{-1}\right)^*\widetilde{W}_1\widetilde{A}^{-1}, L^*\widetilde{W}_2L\right\}, \\ K &= \operatorname{diag}\left\{A^{-1^*}W_1A^{-1}, L^*W_2L\right\}.\end{aligned} \tag{117}$$

Then, from (89) we obtain

$$\|K\| = \left[\left\|A^{-1^*}W_1A^{-1}\right\|^2 + \left\|L^*W_2L\right\|^2\right]^{1/2} \le \sqrt{n}w\left(m^4+l^4\right)^{1/2}. \tag{118}$$

With a similar argument to that for (93), we get

$$\left\|\widetilde{K}-K\right\| \le \sqrt{n}\left(m^2(2mwa+1)+l^2\right)\varepsilon. \tag{119}$$

From (67) one has

$$\left\|\widetilde{\Omega}-\Omega\right\| \le ((\hat{r}+s+1)bl+2)\varepsilon \le ((\hat{r}+s+1)bl+2)\varepsilon_1 \le 1, \tag{120}$$

which, together with (69), implies that

$$\left\|\widetilde{\Omega}\right\| \le \|\Omega\|+1 \le \hat{s}+1. \tag{121}$$

Hence, it follows from (30), (71), and (121) that

$$\begin{aligned}\left\|\widetilde{W}-W\right\| &= \left\|\widetilde{\Omega}\widetilde{K}\widetilde{\Omega}^* - \Omega K\Omega^*\right\| \\ &\le \left\|\widetilde{\Omega}\right\|\left\|\widetilde{K}\widetilde{\Omega}^* - K\Omega^*\right\| + \left\|\widetilde{\Omega}-\Omega\right\|\left\|K\Omega^*\right\| \\ &\le \left\|\widetilde{\Omega}\right\|\left(\left\|\widetilde{K}-K\right\|\left\|\widetilde{\Omega}^*\right\| + \|K\|\left\|\widetilde{\Omega}^*-\Omega^*\right\|\right) + \left\|\widetilde{\Omega}-\Omega\right\|\|K\|\left\|\Omega^*\right\| \\ &\le (\hat{s}+1)^2\left\|\widetilde{K}-K\right\| + \|K\|(2\hat{s}+1) \times ((\hat{r}+s+1)bl+2)\varepsilon \\ &\le g_1\varepsilon,\end{aligned} \tag{122}$$

where g_1 is the same as in (108). It can be easily concluded from (69) that

$$\begin{aligned}\|W\| &= \|\Omega K \Omega^*\| \le \|\Omega\|^2 \|K\| \\ &\le \sqrt{n} w \hat{s}^2 \left(m^4 + l^4\right)^{1/2}.\end{aligned} \tag{123}$$

Therefore, from (113), (116), (122), and (123) we have

$$\begin{aligned}&\left| \|T_2(z)\|_2^2 - \|z\|_1^2 \right| \\ &\le \left(\left(32 g_1 n^3 d^3 (\hat{s}+1)^{4nd-2} (w+1)^{d-1} w \hat{s}^2 \right.\right. \\ &\quad \left.\times \left[\left(m^4 + l^4\right) \left(a^4 (\hat{r}+1)^4 + l^4\right) \right]^{1/2} \right) \\ &\quad \times \left(|\det \Omega|^2 w_0 \right)^{-1} \Big) \varepsilon \|\xi\|^2 \\ &\quad + 2d (w+1)^{d-1} w w_0^{-1} \varepsilon \sum_{t=2n}^{N} \|y(t)\|^2 \\ &= \left(G_3 \|\xi\|^2 + G_4 \sum_{t=2n}^{N} \|y(t)\|^2 \right) \varepsilon \\ &\le \left(G_3 \gamma^{-1} + G_4 \beta^{-1} \right) \|z\|_1^2 \varepsilon.\end{aligned} \tag{124}$$

Consequently, (106) holds and the proof is complete. □

The following result is about the estimate of difference between $\widetilde{P}(T_2(z))$ and $P(z)$.

Proposition 11. *For any $0 < \varepsilon \le \varepsilon_2$, in which ε_2 is defined as in (104), if (55), (56), and (105) hold, then*

$$\left| \widetilde{P}(T_2(z)) - P(z) \right| \le \left(G_5 \gamma^{-1} + G_6 \beta^{-1} \right) \|z\|_1^2 \varepsilon, \quad \forall z \in X, \tag{125}$$

where

$$\begin{aligned}G_5 := g_2 &+ \frac{G_3 (G_3 + g_1)}{g_1 \sqrt{n} w \hat{s}^2 (m^4 + l^4)^{1/2}} \\ &\times (\hat{r}+1) \Big(e_1 (\hat{s}+1)^2 \left(m^4 + l^4\right)^{1/2} \\ &\qquad + 2 m l e_2 (s+1)(r + sb\hat{r}l + 1) \Big) \\ &+ (\hat{s}+1)^2 \left(m^2 e_1 (2ma\hat{r} + 1) + l^2 e_1 \right) \\ &+ (2\hat{s}+1) ((\hat{r}+s+1) bl + 2) \\ &\times e_1 \hat{r} \left(m^4 + l^4\right)^{1/2} + 2 m l e_2 \\ &\times ((\hat{r}+s+1)(r + 2sb\hat{r}l + 1) \\ &\qquad + sa\hat{r}m (r + sb\hat{r}l + 1) + s\hat{r}) \\ &+ (\hat{r}+s+1) l e_2 + [(\hat{r}+1)((\hat{r}+s+1) bl + 1) \\ &\qquad + (r + sb\hat{r}l)(ma\hat{r}+1)] m e_2,\end{aligned} \tag{126}$$

$$\begin{aligned}G_6 := G_4 &(G_4+1) \hat{d} (\hat{r}+1) w^{-1} + \hat{d} \\ &+ g_2 + (\hat{r}+s+1) l e_2 \\ &+ [(\hat{r}+1)((\hat{r}+s+1) bl + 1) \\ &\quad + (r + sb\hat{r}l)(ma\hat{r}+1)] m e_2,\end{aligned} \tag{127}$$

$$\begin{aligned}g_2 := \Big\{ &\Big(\Big(64 g_1 n^3 d^4 (\hat{s}+1)^{4nd-2} (w+1)^{2d-2} \\ &\quad \times w \left(a^4 (\hat{r}+1)^4 + l^4 \right)^{1/2} \Big) \\ &\quad \times \left(|\det \Omega|^2 w_0^2 \right)^{-1} \Big) \\ &+ 2\sqrt{n} d (w+1)^{d-1} w_0^{-1} \Big\} \\ &\times (\hat{r}+1) ((s+1) l + (r + sb\hat{r}l + 1) m) e_2,\end{aligned} \tag{128}$$

and G_3, g_1, and G_4 are the same as in (107), (108), and (109), respectively.

Proof. It follows from (76) and (80) that, for any $z = \{\xi, y(2n), \dots, y(N)\} \in X$,

$$\begin{aligned}\widetilde{P}(T_2(z)) = \xi^* W \widetilde{W}^{-1} &\Big(\widetilde{\Omega} \widetilde{M} \widetilde{\Omega}^* + \left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L \right) \left(\widetilde{A}^{-1} \right)^* \widetilde{L}_3 L \widetilde{S}_1^* \\ &\quad - \widetilde{S}_2 L^* \widetilde{L}_4 \widetilde{A}_1^{-1} \left(\widetilde{R}_2^* + \widetilde{B}_1 L \widetilde{S}_2^* \right) \Big) \widetilde{W}^{-1} W \xi \\ &- 2 \operatorname{Re} \left\{ Y^* (N-n+1) W_3 \widetilde{L}_5 L \widetilde{S}_2^* \widetilde{W}^{-1} W \xi \right\} \\ &+ 2 \operatorname{Re} \Big\{ \xi^* W \widetilde{W}^{-1} \left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L \right) \\ &\qquad \times \left(\widetilde{A}^{-1} \right)^* \widetilde{L}_6 W_4 Y(2n) \Big\} \\ &+ \sum_{t=2n}^{N} y^*(t) w(t) \widetilde{w}^{-1}(t) \widetilde{D}_0(t) \widetilde{w}^{-1}(t) w(t) y(t) \\ &+ 2 \operatorname{Re} \Bigg\{ \sum_{i=1}^{n} \sum_{t=2n}^{N-i} y^*(t) w(t) \widetilde{w}^{-1}(t) \widetilde{D}_i(t) \widetilde{w}^{-1} \\ &\qquad \times (t+i) w(t+i) y(t+i) \Bigg\},\end{aligned} \tag{129}$$

where

$$\begin{aligned}W_3 &= \operatorname{diag} \left\{ w(N) \widetilde{w}^{-1}(N), \dots, w(N-n+1) \widetilde{w}^{-1} \right. \\ &\qquad \left. \times (N-n+1) \right\}, \\ W_4 &= \operatorname{diag} \left\{ \widetilde{w}^{-1}(3n-1) w(3n-1), \dots, \widetilde{w}^{-1} \right. \\ &\qquad \left. \times (2n) w(2n) \right\}.\end{aligned} \tag{130}$$

So we get from (40) and (129) that

$$\begin{aligned}
&\left|\widetilde{P}\left(T_2(z)\right)-P(z)\right| \\
&\leq \|\xi\|^2\Delta_1+2\|\xi\|\,\|Y(N-n+1)\|\,\Delta_2 \\
&\quad+2\|\xi\|\,\|Y(2n)\|\,\Delta_3 \\
&\quad+\sum_{t=2n}^{N}\|y(t)\|^2\Delta_4(t) \\
&\quad+2\sum_{i=1}^{n}\sum_{t=2n}^{N-i}\|y(t)\|\,\|y(t+i)\|\,\Delta_5(t) \\
&\leq \|\xi\|^2\left(\Delta_1+\Delta_2+\Delta_3\right) \\
&\quad+\sum_{t=2n}^{N}\|y(t)\|^2\left(\Delta_2+\Delta_3+\Delta_4(t)\right) \\
&\quad+2\sum_{i=1}^{n}\sum_{t=2n}^{N-i}\|y(t)\|\,\|y(t+i)\|\,\Delta_5(t),
\end{aligned} \tag{131}$$

where

$$\begin{aligned}
\Delta_1&=\Big\|W\widetilde{W}^{-1}\Big(\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*+\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^* \\
&\qquad-\widetilde{S}_2L^*\widetilde{L}_4\widetilde{A}_1^{-1}\left(\widetilde{R}_2^*+\widetilde{B}_1L\widetilde{S}_2^*\right)\Big)\widetilde{W}^{-1}W \\
&\quad-\Big(\Omega M\Omega^*+\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^* \\
&\qquad-S_2L^*L_4A_1^{-1}\left(R_2^*+B_1LS_2^*\right)\Big)\Big\|, \\
\Delta_2&=\left\|W_3\widetilde{L}_5L\widetilde{S}_2^*\widetilde{W}^{-1}W-L_5LS_2^*\right\|, \\
\Delta_3&=\Big\|W\widetilde{W}^{-1}\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_6W_4 \\
&\quad-\left(R_1-S_1BL\right)A^{-1^*}L_6\Big\|, \\
\Delta_4(t)&=\left\|w(t)\,\widetilde{w}^{-1}(t)\,\widetilde{D}_0(t)\,\widetilde{w}^{-1}(t)\,w(t)-D_0(t)\right\|, \\
\Delta_5(t)&=\left\|w(t)\,\widetilde{w}^{-1}(t)\,\widetilde{D}_i(t)\,\widetilde{w}^{-1}(t+i)\,w(t+i)-D_i(t)\right\|.
\end{aligned} \tag{132}$$

In the following we discuss Δ_j, $1\leq j\leq 5$, term by term. It follows from the first relation in (132) that

$$\begin{aligned}
\Delta_1&\leq\left\|W\widetilde{W}^{-1}\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\widetilde{W}^{-1}W-\Omega M\Omega^*\right\| \\
&\quad+\Big\|W\widetilde{W}^{-1}\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\widetilde{W}^{-1}W \\
&\qquad-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\Big\| \\
&\quad+\Big\|W\widetilde{W}^{-1}\widetilde{S}_2L^*\widetilde{L}_4\widetilde{A}_1^{-1}\left(\widetilde{R}_2^*+\widetilde{B}_1L\widetilde{S}_2^*\right)\widetilde{W}^{-1}W \\
&\qquad-S_2L^*L_4A_1^{-1}\left(R_2^*+B_1LS_2^*\right)\Big\|,
\end{aligned} \tag{133}$$

in which the right-hand side is a sum of three terms. Now, we calculate the first term.

$$\begin{aligned}
&\left\|W\widetilde{W}^{-1}\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\widetilde{W}^{-1}W-\Omega M\Omega^*\right\| \\
&\quad\leq\left\|W-\widetilde{W}\right\|\left\|\widetilde{W}^{-1}\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\widetilde{W}^{-1}W\right\| \\
&\qquad+\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|+\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*-\Omega M\Omega^*\right\| \\
&\quad\leq\left\|\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|\times\left(\left\|\widetilde{W}^{-1}W\right\|+1\right)\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\right\| \\
&\qquad+\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*-\Omega M\Omega^*\right\|.
\end{aligned} \tag{134}$$

From (67), (69), and (121) we have

$$\begin{aligned}
\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*-\Omega M\Omega^*\right\|&\leq\left\|\widetilde{\Omega}\right\|\left\|\widetilde{M}\widetilde{\Omega}^*-M\Omega^*\right\| \\
&\quad+\left\|\widetilde{\Omega}-\Omega\right\|\left\|M\Omega^*\right\| \\
&\leq\left\|\widetilde{\Omega}\right\|\Big(\left\|\widetilde{M}-M\right\|\left\|\widetilde{\Omega}^*\right\| \\
&\qquad+\|M\|\left\|\widetilde{\Omega}^*-\Omega^*\right\|\Big) \\
&\quad+\left\|\widetilde{\Omega}-\Omega\right\|\|M\|\left\|\Omega^*\right\| \\
&\leq(\widehat{s}+1)^2\left\|\widetilde{M}-M\right\|+\|M\| \\
&\quad\times(2\widehat{s}+1)\left(\left(\widehat{r}+s+1\right)bl+2\right)\varepsilon,
\end{aligned} \tag{135}$$

which, together with (84) and (85) in Lemma 8, implies that

$$\begin{aligned}
&\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*-\Omega M\Omega^*\right\| \\
&\quad\leq\Big\{(\widehat{s}+1)^2\left(m^2e_1\left(2ma\widehat{r}+1\right)+l^2e_1\right) \\
&\qquad+(2\widehat{s}+1)\left(\left(\widehat{r}+s+1\right)bl+2\right) \\
&\qquad\times e_1\widehat{r}\left(m^4+l^4\right)^{1/2}\Big\}\,\varepsilon.
\end{aligned} \tag{136}$$

In addition, from (86) and (121) we get

$$\begin{aligned}
\left\|\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\right\|&\leq\left\|\widetilde{\Omega}\right\|^2\left\|\widetilde{M}\right\| \\
&\leq(\widehat{s}+1)^2e_1\left(\widehat{r}+1\right)\left(m^4+l^4\right)^{1/2}.
\end{aligned} \tag{137}$$

Hence, it follows from (134)–(137) that

$$\begin{aligned}
&\left\|W\widetilde{W}^{-1}\widetilde{\Omega}\widetilde{M}\widetilde{\Omega}^*\widetilde{W}^{-1}W-\Omega M\Omega^*\right\| \\
&\quad\leq\left\|\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|\left(\left\|\widetilde{W}^{-1}W\right\|+1\right) \\
&\qquad\times(\widehat{s}+1)^2e_1\left(\widehat{r}+1\right)\left(m^4+l^4\right)^{1/2} \\
&\qquad+\Big\{(\widehat{s}+1)^2\left(m^2e_1\left(2ma\widehat{r}+1\right)+l^2e_1\right) \\
&\qquad+(2\widehat{s}+1)\left(\left(\widehat{r}+s+1\right)bl+2\right) \\
&\qquad\times e_1\widehat{r}\left(m^4+l^4\right)^{1/2}\Big\}\,\varepsilon.
\end{aligned} \tag{138}$$

Next, we study the second term in the right-hand side of (133):

$$\begin{aligned}
&\left\|W\widetilde{W}^{-1}\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\widetilde{W}^{-1}W\right.\\
&\quad\left.-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|W-\widetilde{W}\right\|\left\|\widetilde{W}^{-1}\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\right.\\
&\qquad\left.\times\widetilde{L}_3L\widetilde{S}_1^*\widetilde{W}^{-1}W\right\|\\
&\qquad+\left\|\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\widetilde{W}^{-1}\right\|\\
&\qquad\times\left\|W-\widetilde{W}\right\|+\left\|\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\right.\\
&\qquad\left.-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|\left(\left\|\widetilde{W}^{-1}W\right\|+1\right)\\
&\qquad\times\left\|\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\right\|\\
&\qquad+\left\|\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\right.\\
&\qquad\left.-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\right\|.
\end{aligned}\tag{139}$$

Since $\varepsilon\le\varepsilon_1\le 1/((\widehat{r}+s+1)bl+1)$, from (66) we have

$$\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L-R_1+S_1BL\right\|\le\left(\left(\widehat{r}+s+1\right)bl+1\right)\varepsilon\le 1.\tag{140}$$

So,

$$\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right\|\le r+sb\widehat{r}l+1,\tag{141}$$

which, together with (57), (89), and (91), yields that

$$\begin{aligned}
&\left\|\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right\|\left\|\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*-A^{-1^*}L_3LS_1^*\right\|\\
&\qquad+\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L-R_1+S_1BL\right\|\left\|A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right\|\left(\left\|\widetilde{A}^{-1}\right\|\left\|\widetilde{L}_3L\widetilde{S}_1^*-L_3LS_1^*\right\|\right.\\
&\qquad\left.+\left\|\widetilde{A}^{-1}-A^{-1}\right\|\left\|L_3LS_1^*\right\|\right)\\
&\qquad+\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L-R_1+S_1BL\right\|\left\|A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right\|\left[\left\|\widetilde{A}^{-1}\right\|\left(\left\|\widetilde{L}_3L\right\|\left\|\widetilde{S}_1^*-S_1^*\right\|\right.\right.\\
&\qquad\left.+\left\|\widetilde{L}_3-L_3\right\|\left\|LS_1^*\right\|\right)\\
&\qquad\left.+\left\|\widetilde{A}^{-1}-A^{-1}\right\|\left\|L_3LS_1^*\right\|\right]\\
&\qquad+\left\|\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L-R_1+S_1BL\right\|\left\|A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le mle_2\left(\left(\widehat{r}+s+1\right)\left(r+2sb\widehat{r}l+1\right)\right.\\
&\qquad\left.+sa\widehat{r}m\left(r+sb\widehat{r}l+1\right)+s\widehat{r}\right)\varepsilon.
\end{aligned}\tag{142}$$

Hence, it follows from (139)–(142) that

$$\begin{aligned}
&\left\|W\widetilde{W}^{-1}\left(\widetilde{R}_1-\widetilde{S}_1\widetilde{B}L\right)\left(\widetilde{A}^{-1}\right)^*\widetilde{L}_3L\widetilde{S}_1^*\widetilde{W}^{-1}W\right.\\
&\quad\left.-\left(R_1-S_1BL\right)A^{-1^*}L_3LS_1^*\right\|\\
&\quad\le\left\|\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|\left(\left\|\widetilde{W}^{-1}W\right\|+1\right)\\
&\qquad\times mle_2\left(\widehat{r}+1\right)\left(s+1\right)\left(r+sb\widehat{r}l+1\right)\\
&\qquad+mle_2\left(\left(\widehat{r}+s+1\right)\left(r+2sb\widehat{r}l+1\right)\right.\\
&\qquad\left.+sa\widehat{r}m\left(r+sb\widehat{r}l+1\right)+s\widehat{r}\right)\varepsilon.
\end{aligned}\tag{143}$$

With a similar argument, one can obtain an estimate of the third term in the right-hand side of (133), which is the same as (143). Then, from (116), (122), (123), (133), (138), and (143), one can get

$$\begin{aligned}
\Delta_1\le&\left\{\left(\left(32g_1n^{5/2}d^3\left(\widehat{s}+1\right)^{4nd-2}\right.\right.\right.\\
&\left.\times\left(w+1\right)^{d-1}\left(a^4\left(\widehat{r}+1\right)^4+l^4\right)^{1/2}\right)\\
&\times\left(\left|\det\Omega\right|^2w_0\right)^{-1}\Big)\\
&\times\left(\left(32n^3d^3\left(\widehat{s}+1\right)^{4nd-2}\left(w+1\right)^{d-1}w\widehat{s}^2\right.\right.\\
&\left.\times\left[\left(m^4+l^4\right)\left(a^4\left(\widehat{r}+1\right)^4+l^4\right)\right]^{1/2}\right)\\
&\times\left(\left|\det\Omega\right|^2w_0\right)^{-1}+1\Big)\\
&\times\left(\widehat{r}+1\right)\left(e_1\left(\widehat{s}+1\right)^2\left(m^4+l^4\right)^{1/2}\right.\\
&\left.+2mle_2\left(s+1\right)\left(r+sb\widehat{r}l+1\right)\right)\\
&+\left(\widehat{s}+1\right)^2\left(m^2e_1\left(2ma\widehat{r}+1\right)+l^2e_1\right)\\
&+\left(2\widehat{s}+1\right)\left(\left(\widehat{r}+s+1\right)bl+2\right)\\
&\times e_1\widehat{r}\left(m^4+l^4\right)^{1/2}\\
&+2mle_2\left(\left(\widehat{r}+s+1\right)\left(r+2sb\widehat{r}l+1\right)\right.\\
&\left.\left.+sa\widehat{r}m\left(r+sb\widehat{r}l+1\right)+s\widehat{r}\right)\right\}\varepsilon.
\end{aligned}\tag{144}$$

Next, we consider the second relation in (132). It is evident that

$$\begin{aligned}
\Delta_2\le&\left\|W_3\widetilde{L}_5L\widetilde{S}_2^*\widetilde{W}^{-1}\right\|\left\|W-\widetilde{W}\right\|\\
&+\left\|W_3-I_{nd}\right\|\left\|\widetilde{L}_5L\widetilde{S}_2^*\right\|\\
&+\left\|\widetilde{L}_5L\widetilde{S}_2^*-L_5LS_2^*\right\|.
\end{aligned}\tag{145}$$

From (50) we have

$$\begin{aligned}\left\|\widetilde{L}_5 L\widetilde{S}_2^* - L_5 L S_2^*\right\| &\leq \left\|\widetilde{L}_5 - L_5\right\| \left\|L\widetilde{S}_2^*\right\| \\ &\quad + \left\|L_5 L\right\| \left\|\widetilde{S}_2 - S_2\right\| \\ &\leq (\widehat{r} + s + 1)\, l e_2 \varepsilon.\end{aligned} \tag{146}$$

It follows from the expression of W_3 that

$$\begin{aligned}\left\|W_3\right\| &\leq 2\sqrt{n} d (w+1)^{d-1} w w_0^{-1}, \\ \left\|W_3 - I_{nd}\right\| &\leq 2\sqrt{n} d (w+1)^{d-1} w_0^{-1} \varepsilon.\end{aligned} \tag{147}$$

Additionally,

$$\left\|\widetilde{L}_5 L\widetilde{S}_2^*\right\| \leq (\widehat{r} + 1)(s+1)\, l e_2. \tag{148}$$

Further, from (116) and (122) one has

$$\begin{aligned}\Delta_2 \leq &\left\{ \left(64 g_1 n^3 d^4 (\widehat{s}+1)^{4nd-2} \right.\right. \\ &\left.\times (w+1)^{2d-2} w \left(a^4 (\widehat{r}+1)^4 + l^4\right)^{1/2}\right) \\ &\times \left(|\det\Omega|^2\ w_0^2\right)^{-1} \\ &\left. + 2\sqrt{n} d (w+1)^{d-1} w_0^{-1} \right\} \\ &\times (\widehat{r}+1)(s+1)\, l e_2 \varepsilon + (\widehat{r} + s + 1)\, l e_2 \varepsilon.\end{aligned} \tag{149}$$

From the third relation in (132) we get

$$\begin{aligned}\Delta_3 \leq &\left\|W - \widetilde{W}\right\| \left\|\widetilde{W}^{-1} \left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L\right) \left(\widetilde{A}^{-1}\right)^* \widetilde{L}_6 W_4\right\| \\ &+ \left\|\left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L\right) \left(\widetilde{A}^{-1}\right)^* \widetilde{L}_6\right\| \left\|W_4 - I_{nd}\right\| \\ &+ \left\|\left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L\right) \left(\widetilde{A}^{-1}\right)^* \widetilde{L}_6 - (R_1 - S_1 B L) A^{-1*} L_6\right\|.\end{aligned} \tag{150}$$

From (57), (66), (89), and (91) we obtain

$$\begin{aligned}&\left\|\left(\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L\right) \left(\widetilde{A}^{-1}\right)^* \widetilde{L}_6 - (R_1 - S_1 B L) A^{-1*} L_6\right\| \\ &\quad \leq \left\|\widetilde{R}_1 - \widetilde{S}_1 \widetilde{B} L - R_1 + S_1 B L\right\| \left\|\left(\widetilde{A}^{-1}\right)^* \widetilde{L}_6\right\| \\ &\qquad + \left\|R_1 - S_1 B L\right\| \left(\left\|\left(\widetilde{A}^{-1}\right)^*\right\| \left\|\widetilde{L}_6 - L_6\right\| \right. \\ &\qquad \left. + \left\|\left(\widetilde{A}^{-1}\right)^* - A^{-1*}\right\| \left\|L_6\right\| \right) \\ &\quad \leq [(\widehat{r}+1)((\widehat{r}+s+1) b l + 1) \\ &\qquad + (r + s b \widehat{r} l)(m a \widehat{r} + 1)]\, m e_2 \varepsilon.\end{aligned} \tag{151}$$

According to the expression of W_4, we know that it has the same estimate as W_3 in (147). Thus, we have

$$\begin{aligned}\Delta_3 \leq &\left\{ \left(64 g_1 n^3 d^4 (\widehat{s}+1)^{4nd-2} \right.\right. \\ &\left.\times (w+1)^{2d-2} w \left(a^4 (\widehat{r}+1)^4 + l^4\right)^{1/2}\right) \\ &\times \left(|\det\Omega|^2 w_0^2\right)^{-1} \\ &\left. + 2\sqrt{n} d (w+1)^{d-1} w_0^{-1} \right\} \\ &\times (\widehat{r}+1)(r + s b \widehat{r} l + 1)\, m e_2 \varepsilon \\ &+ [(\widehat{r}+1)((\widehat{r}+s+1) b l + 1) \\ &+ (r + s b \widehat{r} l)(m a \widehat{r} + 1)]\, m e_2 \varepsilon.\end{aligned} \tag{152}$$

It follows from (113) and (132) that, for any $t \in [2n, N]$,

$$\begin{aligned}\Delta_4(t) &\leq \|w(t) - \widetilde{w}(t)\| \\ &\quad \times \left\|\widetilde{w}^{-1}(t) \widetilde{D}_0(t) \widetilde{w}^{-1}(t) w(t)\right\| \\ &\quad + \left\|\widetilde{D}_0(t) \widetilde{w}^{-1}(t)\right\| \|w(t) - \widetilde{w}(t)\| \\ &\quad + \left\|\widetilde{D}_0(t) - D_0(t)\right\| \\ &\leq d_0 (\widehat{r}+1) \left\|\widetilde{w}^{-1}(t)\right\| \\ &\quad \times \left(\left\|\widetilde{w}^{-1}(t)\right\| w + 1\right) \varepsilon + d_0 \varepsilon \\ &\leq \left(G_4 (G_4 + 1) d_0 (\widehat{r}+1) w^{-1} + d_0\right) \varepsilon.\end{aligned} \tag{153}$$

Similarly, it can be concluded that

$$\Delta_5(t) \leq \left(G_4 (G_4 + 1) d_i (\widehat{r}+1) w^{-1} + d_i\right) \varepsilon. \tag{154}$$

So, by the assumptions and the Hölder inequality, we have

$$\begin{aligned}&\sum_{i=1}^{n} \sum_{t=2n}^{N-i} \|y(t)\| \|y(t+i)\| \Delta_5(t) \\ &\quad \leq \sum_{i=1}^{n} \left(G_4 (G_4 + 1) d_i (\widehat{r}+1) w^{-1} + d_i\right) \varepsilon \\ &\quad \times \sum_{t=2n}^{N} \|y(t)\|^2.\end{aligned} \tag{155}$$

Therefore, from (144) and (149)–(155) we obtain

$$\left|\widetilde{P}(T_2(z)) - P(z)\right| \leq \left(G_5 \|\xi\|^2 + G_6 \sum_{t=2n}^{N} \|y(t)\|^2\right) \varepsilon, \tag{156}$$

which, together with (49), implies that (125) holds. The proof is complete. □

Now we give the main result of the present paper—an error estimate of eigenvalues of the perturbed problem $(1)'$-$(2)'$.

Theorem 12. *Assume that* (3), (4), (22), *and* (51) *hold. Let*

$$\varepsilon_0 := \min\left\{\varepsilon_2, \frac{\beta\gamma}{2(G_3\beta + G_4\gamma)}\right\}, \quad (157)$$

where ε_2, β, γ, G_3, *and* G_4 *are the same as in* (104), (49), (107), *and* (109), *respectively. For any* $0 < \varepsilon \le \varepsilon_0$, *if* (55), (56), *and* (105) *hold, then the kth eigenvalue* $\widetilde{\lambda}_k$ *of* $(1)'$-$(2)'$ *and the kth eigenvalue* λ_k *of* (1)-(2) *(in the increasing order as in* (73) *and* (24), *resp.) satisfy*

$$\left|\widetilde{\lambda}_k - \lambda_k\right| \le 2\Gamma\varepsilon, \quad 1 \le k \le (N+1)d, \quad (158)$$

where

$$\Gamma = G_5\gamma^{-1} + G_6\beta^{-1} + \left(G_1\gamma^{-1} + G_2\beta^{-1}\right)\left(G_3\gamma^{-1} + G_4\beta^{-1}\right) \quad (159)$$

and G_1, G_2, G_5, *and* G_6 *are the same as in* (99), (100), (126), *and* (127), *respectively.*

Proof. By Propositions 9–11, we have that, for any $z \in X$ with $z \neq 0$,

$$\begin{aligned}\left|\widetilde{\mathscr{R}}(T_2(z)) - \mathscr{R}(z)\right| &= \left|\frac{\widetilde{P}(T_2(z))}{\|T_2(z)\|_2^2} - \frac{P(z)}{\|z\|_1^2}\right| \\ &\le \left(\left|\widetilde{P}(T_2(z)) - P(z)\right| + \frac{|P(z)|\left|\|T_2(z)\|_2^2 - \|z\|_1^2\right|}{\|z\|_1^2}\right) \\ &\quad \times \frac{1}{\|T_2(z)\|_2^2} \le \frac{\Gamma \| z\|_1^2}{\|T_2(z)\|_2^2}\varepsilon.\end{aligned} \quad (160)$$

Since

$$\varepsilon \le \varepsilon_0 \le \frac{\beta\gamma}{2(G_3\beta + G_4\gamma)}, \quad (161)$$

we have from (106) that

$$\begin{aligned}\left|\frac{\|T_2(z)\|_2^2}{\|z\|_1^2} - 1\right| &= \frac{\left|\|T_2(z)\|_2^2 - \|z\|_1^2\right|}{\|z\|_1^2} \\ &\le \left(G_3\gamma^{-1} + G_4\beta^{-1}\right)\varepsilon \le \frac{1}{2},\end{aligned} \quad (162)$$

which implies that $\|T_2(z)\|_2^2/\|z\|_1^2 \ge 1/2$; that is, $\|z\|_1^2/\|T_2(z)\|_2^2 \le 2$. Hence, it follows from (160) that

$$\left|\widetilde{\mathscr{R}}(T_2(z)) - \mathscr{R}(z)\right| \le 2\Gamma\varepsilon. \quad (163)$$

Therefore, for each k, $1 \le k \le (N+1)d$ and for any $z^{(1)}, \ldots, z^{(k-1)} \in X$, we get from Theorem 4 and (82) that

$$\begin{aligned}&\left|\widetilde{g}\left(z^{(1)}, \ldots, z^{(k-1)}\right) - g\left(z^{(1)}, \ldots, z^{(k-1)}\right)\right| \\ &= \Big|\min\left\{\widetilde{\mathscr{R}}(T_2(z)) : z \in X, z\perp_1 z^{(j)}, 1 \le j \le k-1, z \neq 0\right\} \\ &\quad - \min\left\{\mathscr{R}(z) : z \in X, z\perp_1 z^{(j)}, 1 \le j \le k-1, z \neq 0\right\}\Big| \\ &\le 2\Gamma\varepsilon,\end{aligned} \quad (164)$$

which, together with (83), yields that (158) holds. The proof is complete. □

The following result is a direct consequence of Theorem 12.

Corollary 13. *Assume that all the assumptions in Theorem 12 hold. Then each eigenvalue of problem* (1)-(2) *is continuously dependent on the coefficients and weight function of* (1) *and the coefficients of the boundary condition* (2).

Remark 14. The nonsingularity assumption (22) for Ω can be illustrated by giving examples. Since $2n$-order discrete vector boundary value problems include second-order discrete boundary value problems and the necessity of the nonsingularity assumption for Ω has been clarified through an example in [17]. Here we will not discuss it.

4. Two Special Cases

In this section, we consider two special perturbed problems. The error estimates will be simpler for these two special cases.

Case 1. The perturbed problem consists of (1)-$(2)'$; that is, only the coefficients of boundary condition (2) are perturbed, and the coefficients and weight function of (1) are invariant. Since the method of proof is similar to that of Theorem 12, only the related result is given.

Theorem 15. *Assume that* (3), (4), (22), *and* (51) *hold. Let*

$$\widehat{\varepsilon}_0 := \min\left\{\frac{\sqrt{2}}{2}h(D), \frac{h(\Omega)}{b\widehat{r}l + 2}, \frac{1}{b\widehat{r}l + 2}, \frac{\gamma}{2\widehat{G}_3}\right\}, \quad (165)$$

where D is a $2nd \times 2nd$ *nonsingular submatrix of* (R, S),

$$\begin{aligned}\widehat{G}_3 &= \Big(16\widehat{g}_1 n^3 d^3 (\widehat{s} + 1)^{4nd-2} \\ &\quad \times w^d \widehat{s}^2 \left(\left(m^4 + l^4\right)\left(a^4\widehat{r}^4 + l^4\right)\right)^{1/2}\Big) \\ &\quad \times \left(|\det\Omega|^2 \, w_0\right)^{-1},\end{aligned} \quad (166)$$

$$\widehat{g}_1 = \sqrt{n}w(2\widehat{s} + 1)(b\widehat{r}l + 2)\left(m^4 + l^4\right)^{1/2}.$$

For any $0 < \varepsilon \leq \widehat{\varepsilon}_0$, *if* (55) *holds, then the kth eigenvalue* $\widehat{\lambda}_k$ *of* (1)-(2)$'$ *and the kth eigenvalue* λ_k *of* (1)-(2) *satisfy*

$$\left|\widehat{\lambda}_k - \lambda_k\right| \leq 2\left(\widehat{G}_5\gamma^{-1} + \widehat{G}_6\beta^{-1} + \left(G_1\gamma^{-1} + G_2\beta^{-1}\right)\widehat{G}_3\gamma^{-1}\right)\varepsilon, \tag{167}$$

where $1 \leq k \leq (N+1)d$, β, γ, G_1, *and* G_2 *are the same as in* (49), (99), *and* (100), *respectively,*

$$\begin{aligned}
\widehat{G}_5 &= \widehat{g}_2 + \frac{\widehat{G}_3\left(\widehat{G}_3 + \widehat{g}_1\right)}{\widehat{g}_1\sqrt{n}w\widehat{s}^2\left(m^4 + l^4\right)^{1/2}} \\
&\quad \times \widehat{r}\left(e_1(\widehat{s}+1)^2\left(m^4 + l^4\right)^{1/2}\right. \\
&\qquad \left. + 2mle_2(s+1)(r + sb\widehat{r}l + 1)\right) \\
&\quad + (2\widehat{s}+1)(b\widehat{r}l + 2)e_1\widehat{r}\left(m^4 + l^4\right)^{1/2} \\
&\quad + 2mle_2\widehat{r}(2sb\widehat{r}l + r + s + 1) \\
&\quad + e_2\widehat{r}(l + (b\widehat{r}l + 1)m), \\
\widehat{G}_6 &= \widehat{g}_2 + e_2\widehat{r}(l + (b\widehat{r}l + 1)m), \\
\widehat{g}_2 &= \left(16\widehat{g}_1 n^{5/2} d^3 (\widehat{s}+1)^{4nd-2} w^{d-1} e_2 \widehat{r}\right. \\
&\quad \left. \times ((s+1)l + (r + sb\widehat{r}l + 1)m)\left(a^4\widehat{r}^4 + l^4\right)^{1/2}\right) \\
&\quad \times \left(|\det\Omega|^2 w_0\right)^{-1}.
\end{aligned} \tag{168}$$

Case 2. The perturbed problem consists of (1)$'$-(2); that is, only the coefficients and weight function of (1) are perturbed, and the coefficients of boundary condition (2) are invariant.

Since boundary condition contains the coefficients $r_i(t+n-1)$ and $r_i(t+N+n)$ ($1 \leq i \leq n$ and $1 \leq t \leq i$) of equation, the coefficients are invariant in this case; then A, A_1, B, B_1, L_3, and L_4 are invariant.

In addition, since in this case the admissible function space $\widehat{L}[0, N+2n]$ of perturbed problem is the same as that for the original problem, it can be directly applied instead of the space X. However, since the weight function is perturbed, the inner product on $\widehat{L}[0, N+2n]$ for the perturbed problem changes with it. Define an inner product on $\widehat{L}[0, N+2n]$ for the perturbed problem by

$$\langle x, y\rangle_0 := \sum_{t=n}^{N+n} y^*(t)\,\widetilde{w}(t)\,x(t), \quad x, y \in \widehat{L}[0, N+2n], \tag{169}$$

and the following induced norm

$$\|y\|_0 := \left(\langle y, y\rangle_0\right)^{1/2}, \quad y \in \widehat{L}[0, N+2n]. \tag{170}$$

Obviously, $\widehat{L}[0, N+2n]$ is still an $(N+1)d$-dimensional Hilbert space with the inner product $\langle\cdot,\cdot\rangle_0$ by [1, Theorem 2.3].

For convenience, we now introduce the Rayleigh quotient corresponding to the difference operator $\widetilde{\mathscr{L}}$ on $\widehat{L}[0, N+2n]$ with $\langle\cdot,\cdot\rangle_0$ as follows:

$$\begin{aligned}
\overline{R}(x) &:= \frac{\left\langle \widetilde{\mathscr{L}}y, y\right\rangle_0}{\langle y, y\rangle_0}, \\
y &\in \widehat{L}[0, N+2n] \quad \text{with } y' = \{y(t)\}_{t=n}^{N+n} \neq 0,
\end{aligned} \tag{171}$$

where $\widetilde{\mathscr{L}}$ is the same as in (74).

By Lemma 2, problem (1)$'$-(2) has also $(N+1)d$ real eigenvalues (multiplicity included) arranged as

$$\overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots \leq \overline{\lambda}_{(N+1)d}. \tag{172}$$

The variational property (26) of eigenvalues $\overline{\lambda}_k$ for perturbed problem (1)$'$-(2) on $\widehat{L}[0, N+2n]$ still holds, where λ_k, f, $R(y)$, $\perp$ and $\langle\cdot,\cdot\rangle$ are replaced by $\overline{\lambda}_k$, $\overline{g}$, $\overline{R}(y)$, $\perp_0$, and $\langle\cdot,\cdot\rangle_0$, respectively.

In a similar way to the discussion in Section 3, we first discuss the relation between $\perp_0$ and $\perp$ and then give another form of variational formula of eigenvalues for problem (1)$'$-(2) on $\widehat{L}[0, N+2n]$. Now we introduce the following linear transformation:

$$T_3 : \widehat{L}[0, N+2n] \longrightarrow \widehat{L}[0, N+2n]; \tag{173}$$

for any $y = \{y(t)\}_{t=0}^{N+2n} \in \widehat{L}[0, N+2n]$, we have

$$\begin{aligned}
T_3(y)(t) &= \widetilde{w}^{-1}(t)\,w(t)\,y(t), \quad t \in [n, N+n], \\
Y_{T_3(y)}(0) &= LS_1^*\Omega^{*-1}\operatorname{diag}\{A, -L\}\begin{pmatrix}\widetilde{W}_1^{-1}W_1Y(n) \\ \widetilde{W}_2^{-1}W_2Y(N+1)\end{pmatrix}, \\
Y_{T_3(y)}(N+n+1) &= A_1^{-1}\left(R_2^* + B_1LS_2^*\right)\Omega^{*-1}\operatorname{diag}\{A, -L\} \\
&\quad \times \begin{pmatrix}\widetilde{W}_1^{-1}W_1Y(n) \\ \widetilde{W}_2^{-1}W_2Y(N+1)\end{pmatrix},
\end{aligned} \tag{174}$$

where W_1, W_2, $\widetilde{W}_1$, and $\widetilde{W}_2$ are the same as in (31) and (71), respectively, and $Y_{T_3(y)}(t)$ has the same definition as $Y(t)$ in (13) only with $y(t)$ replaced by $T_3(y)(t)$.

Evidently, T_3 is invertible and

$$\langle y_1, y_2\rangle_0 = \left\langle T_3^{-1}(y_1), y_2\right\rangle, \quad \forall y_1, y_2 \in \widehat{L}[0, N+2n]. \tag{175}$$

Hence, for any $z^{(1)}, \ldots, z^{(k-1)} \in \widehat{L}[0, N+2n]$, we get

$$\begin{aligned}
&\overline{g}\left(z^{(1)}, \ldots, z^{(k-1)}\right) \\
&= \min\left\{\overline{R}(y) : y \in \widehat{L}[0, N+2n], y \perp_0 z^{(j)},\right. \\
&\qquad \left. 1 \leq j \leq k-1, y' \neq 0\right\} \\
&= \min\left\{\overline{R}(y) : y \in \widehat{L}[0, N+2n], T_3^{-1}(y) \perp z^{(j)},\right. \\
&\qquad \left. 1 \leq j \leq k-1, y' \neq 0\right\} \\
&= \min\left\{\overline{R}(T_3(y)) : y \in \widehat{L}[0, N+2n], y \perp z^{(j)},\right. \\
&\qquad \left. 1 \leq j \leq k-1, y' \neq 0\right\}.
\end{aligned} \tag{176}$$

Therefore, the variational property (26) of eigenvalues $\overline{\lambda}_k$ for problem $(1)'$-(2) on $\widehat{L}[0, N+2n]$ still holds, where λ_k, f, and $R(y)$ are replaced by $\overline{\lambda}_k$, $\overline{g}$, and $\overline{R}(T_3(y))$, respectively.

Now, we give an error estimate of eigenvalues of the perturbed problem $(1)'$-(2).

Theorem 16. *Assume that* (3), (4), *and* (22) *hold. Let*

$$\varepsilon_* = \min\left\{\frac{\beta w_0}{4d(w+1)^{d-1}w}, \frac{w_0}{2d\sqrt{d!}(w+1)^{d-1}}, 1\right\}, \quad (177)$$

where β is the same as in (49). *For any $0 < \varepsilon \le \varepsilon_*$, if* (56) *and* (105) *hold, then the kth eigenvalue $\overline{\lambda}_k$ of $(1)'$-(2) and the kth eigenvalue λ_k of* (1)-(2) *(in the increasing order as in* (172) *and* (24), *resp.) satisfy*

$$\left|\overline{\lambda}_k - \lambda_k\right| \le 2\left(M_1 + M_2\right)\varepsilon, \quad 1 \le k \le (N+1)d, \quad (178)$$

where

$$\begin{aligned} M_1 &:= \left[\sqrt{2n}G_4\left(G_4 n + 1\right)\right. \\ &\quad \times e_2\widehat{r}c\left(ls + m\left(r + sb\widehat{r}l\right)\right)\left(a^2\widehat{r}^2 + l^2\right)^{1/2} \\ &\quad \left. + G_4\left(G_4+1\right)\widehat{d}\left(\widehat{r}+1\right) + \widehat{d}w\right]\beta^{-1}w^{-1}, \\ M_2 &:= \left[e_2\widehat{r}c\left(ls + m\left(r + sb\widehat{r}l\right)\right)\left(a^2\widehat{r}^2 + l^2\right)^{1/2} + \widehat{d}\widehat{r}\right]G_4\beta^{-2}, \\ c &:= 2nd\widehat{s}^{2nd-1}\left|\det\Omega\right|^{-1}, \end{aligned} \quad (179)$$

and G_4 is the same as in (109).

Proof. It follows from (25) and (171) that, for any $y \in \widehat{L}[0, N+2n]$ with $y' \ne 0$,

$$\begin{aligned} &\left|\overline{R}\left(T_3(y)\right) - R(y)\right| \\ &\quad = \left|\frac{\left\langle\widetilde{\mathscr{L}}\left(T_3(y)\right), T_3(y)\right\rangle_0}{\left\langle T_3(y), T_3(y)\right\rangle_0} - \frac{\langle\mathscr{L}y, y\rangle}{\langle y, y\rangle}\right| \\ &\quad \le \left(\left\langle\widetilde{\mathscr{L}}\left(T_3(y)\right), T_3(y)\right\rangle_0 - \langle\mathscr{L}y, y\rangle\right. \\ &\qquad \left. + \frac{\left|\langle\mathscr{L}y, y\rangle\right|\left|\left\|T_3(y)\right\|_0^2 - \|y\|_*^2\right|}{\|y\|_*^2}\right)\frac{1}{\left\|T_3(y)\right\|_0^2}, \end{aligned} \quad (180)$$

where $\|y\|_* = (\langle y, y\rangle)^{1/2}$, and

$$\begin{aligned} \langle\mathscr{L}y, y\rangle &= \sum_{t=n}^{N+n} y^*(t)\left\{\sum_{i=0}^{n}\Delta^i\left[r_i(t)\Delta^i y(t-i)\right]\right\} \\ &= \sum_{t=n}^{N+n} y^*(t)\left\{\sum_{i=0}^{n} D_i(t)\, y(t+i)\right. \\ &\qquad \left. + \sum_{i=1}^{n} D_i(t-i)\, y(t-i)\right\} \\ &= Y^*(n) L_3 Y(0) + Y^*(N+1) \\ &\quad \times L_4 Y(N+n+1) + \sum_{t=n}^{N+n} y^*(t) D_0(t)\, y(t) \\ &\quad + 2\operatorname{Re}\left\{\sum_{i=1}^{n}\sum_{t=n}^{N+n-i} y^*(t) D_i(t)\, y(t+i)\right\}. \end{aligned} \quad (181)$$

It follows from (27) that

$$\begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix} = \operatorname{diag}\left\{A^{-1}, -L\right\}\Omega^*\xi; \quad (182)$$

that is,

$$\xi = \Omega^{*-1}\operatorname{diag}\{A, -L\}\begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix}. \quad (183)$$

So,

$$\begin{aligned} Y(0) &= LS_1^*\Omega^{*-1}\operatorname{diag}\{A, -L\}\begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix}, \\ Y(N+n+1) &= A_1^{-1}\left(R_2^* + B_1 LS_2^*\right)\Omega^{*-1} \\ &\qquad \times \operatorname{diag}\{A, -L\}\begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix}. \end{aligned} \quad (184)$$

Hence,

$$\begin{aligned} &\left|Y^*(n) L_3 Y(0)\right| \\ &\quad = \left|Y^*(n) L_3 LS_1^*\Omega^{*-1}\operatorname{diag}\{A, -L\}\begin{pmatrix} Y(n) \\ Y(N+1) \end{pmatrix}\right| \\ &\quad \le \left\|L_3 LS_1^*\Omega^{*-1}\operatorname{diag}\{A, -L\}\right\|\|Y(n)\| \\ &\qquad \times \sqrt{\|Y(n)\|^2 + \|Y(N+1)\|^2} \\ &\quad \le e_2\widehat{r}ls\left(a^2\widehat{r}^2 + l^2\right)^{1/2}\left\|\Omega^{-1}\right\| \\ &\qquad \times \frac{2\|Y(n)\|^2 + \|Y(N+1)\|^2}{2} \\ &\quad \le e_2\widehat{r}ls\left(a^2\widehat{r}^2 + l^2\right)^{1/2}\left\|\Omega^{-1}\right\|\sum_{t=n}^{N+n}\|y(t)\|^2. \end{aligned} \quad (185)$$

Similarly,

$$\begin{aligned} &\left|Y^*(N+1) L_4 Y(N+n+1)\right| \\ &\quad \le me_2\widehat{r}\left(r + sb\widehat{r}l\right)\left(a^2\widehat{r}^2 + l^2\right)^{1/2} \\ &\qquad \times \left\|\Omega^{-1}\right\|\sum_{t=n}^{N+n}\|y(t)\|^2. \end{aligned} \quad (186)$$

From (69) and (88), we have

$$\left\|\Omega^{-1}\right\| = \frac{\left\|\Omega^a\right\|}{|\det\Omega|} \le \frac{2nd\;\|\Omega\|^{2nd-1}}{|\det\Omega|} \le \frac{2nd\widehat{s}^{2nd-1}}{|\det\Omega|} = c. \quad (187)$$

Thus, it follows from (185)–(187) that

$$\begin{aligned}|\langle \mathscr{L} y, y\rangle| &\leq e_2\hat{r}c\,(ls + m\,(r + sb\hat{r}l))\left(a^2\hat{r}^2 + l^2\right)^{1/2} \\ &\quad\times \sum_{t=n}^{N+n} \|y(t)\|^2 + \hat{d}\hat{r} \sum_{t=n}^{N+n} \|y(t)\|^2 \\ &\leq \Big[e_2\hat{r}c\,(ls + m\,(r + sb\hat{r}l))\left(a^2\hat{r}^2 + l^2\right)^{1/2} \\ &\quad + \hat{d}\hat{r}\Big]\,\beta^{-1}\|y\|_*^2.\end{aligned} \tag{188}$$

In addition, from (174), we get

$$\begin{aligned}&\left|\|T_3(y)\|_0^2 - \|y\|_*^2\right| \\ &= \Bigg|\sum_{t=n}^{N+n}\left(\widetilde{w}^{-1}(t)\,w(t)\,y(t)\right)^*\widetilde{w}(t) \\ &\qquad\times\left(\widetilde{w}^{-1}(t)\,w(t)\,y(t)\right) - \sum_{t=n}^{N+n} y^*(t)\,w(t)\,y(t)\Bigg| \\ &= \left|\sum_{t=n}^{N+n} y^*(t)\left(w(t)\,\widetilde{w}^{-1}(t)\,w(t) - w(t)\right)y(t)\right| \\ &\leq \sum_{t=n}^{N+n}\|y(t)\|^2\,\|w(t)\|\,\left\|\widetilde{w}^{-1}(t)\right\|\,\|w(t) - \widetilde{w}(t)\|,\end{aligned} \tag{189}$$

which, together with (113), yields that

$$\begin{aligned}&\left|\|T_3(y)\|_0^2 - \|y\|_*^2\right| \\ &\quad\leq 2d(w+1)^{d-1}ww_0^{-1}\varepsilon\sum_{t=n}^{N+n}\|y(t)\|^2 \\ &\quad= G_4\varepsilon\sum_{t=n}^{N+n}\|y(t)\|^2 \leq G_4\beta^{-1}\varepsilon\|y\|_*^2.\end{aligned} \tag{190}$$

By the assumption $\varepsilon \leq \beta w_0/(4d(w+1)^{d-1}w)$, one can easily obtain

$$\|y\|_*^2 \leq 2\,\|T_3(y)\|_0^2. \tag{191}$$

With a similar argument to that used in the proof of Proposition 11, from (174) and (184) one can get that

$$\begin{aligned}&\left|\left\langle \widetilde{\mathscr{L}}(T_3(y)), T_3(y)\right\rangle_0 - \langle \mathscr{L} y, y\rangle\right| \\ &= \Bigg|\Big(Y^*(n)\,W_1\widetilde{W}_1^{-1}L_3LS_1^* + Y^*(N+1) \\ &\qquad\times W_2\widetilde{W}_2^{-1}L_4A_1^{-1}\left(R_2^* + B_1LS_2^*\right)\Big) \\ &\qquad\times \Omega^{*-1}\operatorname{diag}\{A, -L\}\begin{pmatrix}\widetilde{W}_1^{-1}W_1Y(n) \\ \widetilde{W}_2^{-1}W_2Y(N+1)\end{pmatrix} \\ &\qquad- Y^*(n)\,L_3Y(0) - Y^*(N+1)\,L_4Y(N+n+1) \\ &\qquad+ \sum_{t=n}^{N+n} y^*(t)\Big(w(t)\,\widetilde{w}^{-1}(t)\,\widetilde{D}_0(t)\,\widetilde{w}^{-1}(t)\,w(t) \\ &\qquad\qquad -D_0(t)\Big)\,y(t) \\ &\qquad+ 2\operatorname{Re}\Bigg\{\sum_{i=1}^{n}\sum_{t=n}^{N+n-i} y^*(t)\Big(w(t)\,\widetilde{w}^{-1}(t)\,\widetilde{D}_i(t)\,\widetilde{w}^{-1} \\ &\qquad\qquad\times(t+i)\,w(t+i) \\ &\qquad\qquad - D_i(t)\Big)\,y(t+i)\Bigg\}\Bigg| \\ &\leq M_1\|y\|_*^2\,\varepsilon,\end{aligned} \tag{192}$$

which, together with (180) and (188)–(191), implies that

$$\left|\overline{R}(T_3(y)) - R(y)\right| \leq 2\,(M_1 + M_2)\,\varepsilon. \tag{193}$$

By Theorem 4, we have

$$\left|\overline{\lambda}_k - \lambda_k\right| \leq 2\,(M_1 + M_2)\,\varepsilon, \quad 1 \leq k \leq (N+1)\,d. \tag{194}$$

This completes the proof. □

Remark 17. Since ε_* in Theorem 16 is greater than ε_0 in Theorem 12, the perturbed amplitude in Theorem 16 is even bigger.

Remark 18. The error estimate of eigenvalues of the special perturbed problem $(1)'$-(2) can be deduced from the proof of Theorem 12. Here, we give the proof instead of using the method of the space transformation T_1 from $\widehat{L}[0, N+2n]$ into X. The proof here is simpler and more direct.

Remark 19. The estimate obtained in Theorem 16 does not involve γ of (49), so we do not need to calculate the eigenvalues of matrix W when Theorem 16 is applied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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