Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 307974, 8 pages
http://dx.doi.org/10.1155/2013/307974

Research Article

$(1,1)$-Coherent Pairs on the Unit Circle

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Received 23 July 2013; Accepted 19 September 2013

Academic Editor: Jinde Cao

A pair $(\mathscr{U},\mathscr{V})$ of Hermitian regular linear functionals on the unit circle is said to be a $(1,1)$-coherent pair if their corresponding sequences of monic orthogonal polynomials $\{\phi_n(x)\}_{n\geq 0}$ and $\{\psi_n(x)\}_{n\geq 0}$ satisfy $\phi_n^{[1]}(z)+a_n\phi_{n-1}^{[1]}(z)=\psi_n(z)+b_n\psi_{n-1}(z)$, $a_n\neq 0$, $n\geq 1$, where $\phi_n^{[1]}(z)=\phi'_{n+1}(z)/(n+1)$. In this contribution, we consider the cases when $\mathscr{U}$ is the linear functional associated with the Lebesgue and Bernstein-Szegő measures, respectively, and we obtain a classification of the situations where $\mathscr{V}$ is associated with either a positive nontrivial measure or its rational spectral transformation.

1. Introduction

A pair $(\mathscr{U},\mathscr{V})$ of regular linear functionals on the linear space of polynomials with real coefficients $\mathbb{P}$ is a $(1,1)$-*coherent pair* if and only if their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\frac{P'_{n+1}(x)}{n+1}+a_n\frac{P'_n(x)}{n} = Q_n(x)+b_nQ_{n-1}(x), \quad a_n\neq 0,\ n\geq 1. \tag{1}$$

This concept is a generalization of the notion of *coherent pair*, for us $(1,0)$-*coherent pair*, introduced by Iserles et al. in [1], where $b_n=0$, for every $n\geq 1$.

In the work by Delgado and Marcellán [2], the notion of a *generalized coherent pair* of measures, in short, $(1,1)$-coherent pair of measures, arose as a necessary and sufficient condition for the existence of an algebraic relation between the SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$ associated with the Sobolev inner product

$$\begin{aligned}\langle p(x),r(x)\rangle_\lambda &= \int_{\mathbb{R}} p(x)\,r(x)\,d\mu_0\\ &\quad+\lambda\int_{\mathbb{R}} p'(x)\,r'(x)\,d\mu_1, \quad \lambda>0,\ p,r\in\mathbb{P},\end{aligned} \tag{2}$$

and the SMOP $\{P_n(x)\}_{n\geq 0}$ associated with the positive Borel measure μ_0 in the real line as follows:

$$\begin{aligned}&S_{n+1}(x;\lambda)+c_n(\lambda)\,S_n(x;\lambda)\\ &\quad= P_{n+1}(x)+\frac{n+1}{n}\,a_nP_n(x), \quad n\geq 1,\end{aligned} \tag{3}$$

where $\{c_n(\lambda)\}_{n\geq 1}$ are rational functions in $\lambda>0$. Besides, they obtained the classification of all $(1,1)$-coherent pairs of regular functionals $(\mathscr{U},\mathscr{V})$ and proved that at least one of them must be semiclassical of class at most 1, and $\mathscr{U}$ and $\mathscr{V}$ are related by a rational type expression. This is a generalization of the results of Meijer [3] for the $(1,0)$-coherence case (when $b_n=0$, $n\geq 1$), where either $\mathscr{U}$ or $\mathscr{V}$ must be a classical linear functional.

The most general case of the notion of coherent pair was studied by de Jesus et al. in [4] (see also [5]), the so-called (M,N)-*coherent pairs of order* (m,k), where the derivatives of order m and k of two SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ with respect to the regular linear functionals $\mathscr{U}$ and $\mathscr{V}$ are related by

$$\sum_{i=0}^{M}a_{n-i,n,m}P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N}b_{n-i,n,k}Q_{n+k-i}^{(k)}(x), \quad n\geq 0, \tag{4}$$

where M, N, m, $k \in \mathbb{Z}^+ \cup \{0\}$ and the real numbers $a_{n-i,n,m}, b_{n-i,n,k}$ satisfy some natural conditions. They showed that the regular linear functionals $\mathscr{U}$ and $\mathscr{V}$ are related by a rational factor, and, when $m \neq k$, those linear functionals are semiclassical. Besides, they proved that if (μ_0, μ_1) is a (M, N)-coherent pair of order $(m, 0)$ of positive Borel measures on the real line, then

$$\sum_{j=0}^{\max\{M,N\}} c_{n-j,n,m}(\lambda) S_{n-j+m,m}(x;\lambda) = \sum_{j=0}^{M} a_{n-j,n,m} P_{n-j+m}(x), \quad n \geq 0, \tag{5}$$

holds, where $c_{n-j,n,m}(\lambda)$, $0 < j \leq \max\{M, N\}$, $n \geq 0$, are rational functions in λ such that $c_{n-j,n,m}(\lambda) = 0$ for $n < j \leq \max\{M, N\}$, and $\{S_{n,m}(x;\lambda)\}_{n\geq 0}$ is the Sobolev SMOP with respect to the inner product

$$\langle p(x), r(x)\rangle_{\lambda,m} = \int_{\mathbb{R}} p(x) r(x)\, d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x) r^{(m)}(x)\, d\mu_1, \quad \lambda > 0,\ m \in \mathbb{Z}^+, \tag{6}$$

$p, r \in \mathbb{P}$. Also, they showed that $(M, \max\{M, N\})$-coherence of order $(m, 0)$ is a necessary condition for the algebraic relation (5). For a historical summary about coherent pairs on the real line, see, for example, the introductory sections in the recent papers of de Jesus et al. [6] and of Marcellán and Pinzón-Cortés [7].

On the other hand, the notion of coherent pair was extended to the theory of orthogonal polynomials in a discrete variable by Area et al. in [8–10]. They used the difference operator D_ω as well as the q-derivative operator D_q defined by

$$\begin{aligned}(D_\omega p)(x) &= \frac{p(x+\omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\}, \\ (D_q p)(x) &= \frac{p(qx) - p(x)}{(q-1)x} \quad \text{for } x \neq 0, \\ (D_q p)(0) &= p'(0), \quad q \in \mathbb{C} \setminus \{0, 1\},\end{aligned} \tag{7}$$

instead of the usual derivative operator D. In this way, they obtained similar results to those by Meijer and similar classification as a limit case when either $\omega \to 0$ or $q \to 1$, respectively. Likewise, Marcellán and Pinzón-Cortés in [11, 12] studied the analogue of the generalized coherent pairs introduced by Delgado and Marcellán, that is, $(1,1)$-D_ω-*coherent pairs* and $(1,1)$-D_q-*coherent pairs*. Finally, Álvarez-Nodarse et al. [13] analyzed the more general case, (M,N)-D_ω-*coherent pairs of order* (m, k) and (M,N)-D_q-*coherent pairs of order* (m, k), proving the analogue results to those in [4].

Furthermore, Branquinho et al. in [14] extended the concept of coherent pair to Hermitian linear functionals associated with nontrivial probability measures supported on the unit circle. They studied (3) in the framework of orthogonal polynomials on the unit circle (OPUC). Also, they concluded that if $(\mathscr{U}, \mathscr{V})$ is a $(1, 0)$-coherent pair of Hermitian regular linear functionals, then $\{P_n(z)\}_{n\geq 0}$ is semiclassical and $\{Q_n(z)\}_{n\geq 0}$ is quasiorthogonal of order at most 6 with respect to the functional $[zA(z) + (1/z)\overline{A}(1/z)]\mathscr{U}$, $A \in \mathbb{P}$. Besides, they analyzed the cases when either $\mathscr{U}$ or $\mathscr{V}$ is the Lebesgue measure or $\mathscr{U}$ is the Bernstein-Szegő measure.

Later on, Branquinho and Rebocho in [15] obtained that if the sequences $\{P_n(z)\}_{n\geq 0}$ and $\{Q_n(z)\}_{n\geq 0}$ satisfy, for $n \geq 0$,

$$\sum_{j=0}^{M_1} \alpha_{n,j} \frac{P'_{n+1+M_1-j}(z)}{n+1+M_1-j} + \sum_{j=0}^{M_2} \eta_{n,j} \left(P^*_{n+M_2-j}(z)\right)' = \sum_{j=0}^{N_1} \beta_{n,j} Q_{n+N_1-j}(z) + \sum_{j=0}^{N_2} \gamma_{n,j} Q^*_{n+N_2-j}(z), \tag{8}$$

with $N_1 = M_1$, $\max\{M_2, N_2\} < N_1$, and some extra conditions, then $\{P_n(z)\}_{n\geq 0}$ and $\{Q_n(z)\}_{n\geq 0}$ are semiclassical sequences of OPUC. Moreover, when $P_n(z) = Q_n(z)$ for all n and under some extra conditions, (8) is a necessary condition for the semiclassical character of $\{P_n(z)\}_{n\geq 0}$. Finally, they analyzed the $(0, 1)$-coherence case $(P'_{n+1}(z))/(n+1) = Q_n(z) + b_n Q_{n-1}(z)$, $b_n \neq 0$, $n \geq 1$, when $\mathscr{U}$ is the linear functional associated with either the Lebesgue measure or the Bernstein-Szegő measure.

The aim of our contribution is to describe the $(1, 1)$-coherence pair $(\mathscr{U}, \mathscr{V})$ when $\mathscr{U}$ and $\mathscr{V}$ are regular linear functionals, focusing our attention on the cases when $\mathscr{U}$ is either the Lebesgue or the Bernstein-Szegő linear functional. The structure of this work is as follows. In Section 2, we state some definitions and basic results which will be useful in the forthcoming sections. In Section 3, we introduce the concept of $(1, 1)$-coherent pair of Hermitian regular linear functionals, and we obtain some results that will be applied in the sequel. In Section 4, we analyze $(1, 1)$-coherent pairs when $\mathscr{U}$ is the linear functional associated with the Lebesgue measure on the unit circle. We determine the cases when the linear functional $\mathscr{V}$ is associated with a positive measure on the unit circle, or a rational spectral transformation of it. Finally, in Section 5, we deal with a similar analysis for the case when $\mathscr{U}$ is the linear functional associated with the Bernstein-Szegő measure.

2. Preliminaries

Let us consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the linear space of Laurent polynomials with complex coefficients $\Lambda = \operatorname{span}\{z^n : n \in \mathbb{Z}\}$, and a linear functional $\mathscr{U} : \Lambda \to \mathbb{C}$. We can associate with $\mathscr{U}$ a sequence of moments $\{c_n\}_{n\in\mathbb{Z}}$ defined by $c_n = \langle \mathscr{U}, z^n\rangle$, $n \in \mathbb{Z}$, and a bilinear form as follows:

$$\langle p(z), q(z)\rangle = \left\langle \mathscr{U}, p(z)\overline{q}\left(\frac{1}{z}\right)\right\rangle, \tag{9}$$

where $p, q \in \mathbb{P}$, the linear space of polynomials with complex coefficients. Its Gram matrix with respect to $\{z^n\}_{n\geq 0}$

is an infinite Toeplitz matrix $(c_{j-k})_{j,k\geq 0}$ with leading principal minors given by $\Delta_n = \det((c_{j-k})_{j,k=0}^{n})$, $n \in \mathbb{Z}^+ \cup \{0\}$.

The linear functional $\mathcal{U}$ is said to be *Hermitian* if $c_{-n} = \overline{c}_n$, *quasidefinite* or *regular* if $\Delta_n \neq 0$ for all $n \in \mathbb{Z}^+ \cup \{0\}$, and *positive definite* if $\Delta_n > 0$ for all $n \in \mathbb{Z}^+ \cup \{0\}$. We will denote by $\mathscr{H}$ the set of Hermitian linear functionals defined on Λ.

$\mathcal{U} \in \mathscr{H}$ is regular if and only if there exists a (unique) *sequence of monic orthogonal polynomials* on the unit circle (OPUC) $\{\phi_n(z)\}_{n\geq 0}$; this is, it satisfies that $\deg(\phi_n(z)) = n$ and $\langle \phi_m(z), \phi_n(z)\rangle = \kappa_n \delta_{m,n}$, with $\kappa_n \neq 0$, for $n, m \in \mathbb{Z}^+ \cup \{0\}$. Every monic OPUC $\phi_n(z)$ has an explicit representation, the so-called *Heine's formula*, as follows:

$$\phi_n(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ c_{-(n-1)} & c_{-(n-2)} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \qquad n \geq 1, \quad \phi_0(z) = 1. \tag{10}$$

Besides, they satisfy the *forward and backward Szegő recurrence relations*

$$\begin{gathered} \phi_n(z) = z\phi_{n-1}(z) + \alpha_n \phi_{n-1}^*(z), \\ \phi_n(z) = \left(1 - |\alpha_n|^2\right) z\phi_{n-1}(z) + \alpha_n \phi_n^*(z), \\ n \geq 1, \quad \phi_0(z) = 1, \end{gathered} \tag{11}$$

where $\alpha_n = \phi_n(0)$, $n \geq 1$, are said to be the *Verblunsky (reflection, Schur, Szegő, or Geronimus) coefficients* and $\phi_n^*(z) = z^n \overline{\phi}_n(1/z)$, $n \in \mathbb{Z}^+ \cup \{0\}$, is called the *reversed polynomial* of $\phi_n(z)$. Conversely, if $\{\phi_n(z)\}_{n\geq 0}$ is a sequence of monic polynomials which satisfies (11) and $|\alpha_n| \neq 1$ for $n \geq 1$, then $\{\phi_n(z)\}_{n\geq 0}$ is the sequence of monic OPUC with respect to some Hermitian regular linear functional.

If $\mathcal{U}$ is a Hermitian regular (resp., positive definite) linear functional, then (see [16–18]) $|\alpha_n| \neq 1$ (resp., $|\alpha_n| < 1$), for $n \geq 1$.

A positive definite Hermitian linear functional $\mathcal{U}$ has an integral representation (see [19])

$$\begin{aligned} \langle p(z), q(z)\rangle &= \left\langle \mathcal{U}, p(z)\overline{q}\left(\frac{1}{z}\right)\right\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} p(z)\overline{q}\left(\frac{1}{z}\right) d\mu(\theta), \\ &\qquad z = e^{i\theta}, \quad p, q \in \mathbb{P}, \end{aligned} \tag{12}$$

where μ is a nontrivial probability measure supported on an infinite subset of $\mathbb{T}$. A measure μ belongs to the *Nevai class* (see [20, 21]) if $\lim_{n\to\infty}|\phi_n(0)| = 0$.

On the other hand (see [19]), an analytic function $F(z)$, defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is said to be a *Carathéodory function* if and only if $F(0) = 1$ and $\operatorname{Re} F(z) > 0$ on $\mathbb{D}$. If μ is a probability measure on $\mathbb{T}$, then

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \tag{13}$$

is a Carathéodory function. Conversely, the *Herglotz representation theorem* claims that every Carathéodory function $F(z)$ has a representation given by (13) for a unique probability measure μ on $\mathbb{T}$.

Besides (see [22]), a Carathéodory function (13) admits the expansions

$$\begin{aligned} F(z) &= c_0 + 2\sum_{n=1}^{\infty} c_{-n} z^n, \quad |z| < 1, \\ F(z) &= -c_0 - 2\sum_{n=1}^{\infty} c_n z^{-n}, \quad |z| > 1, \end{aligned} \tag{14}$$

where $\{c_n\}_{n\geq 0}$ are the moments of the measure associated with $F(z)$.

To complete this section, we state the following definitions. Let $\{\phi_n(z)\}_{n\geq 0}$ be a sequence of monic OPUC with corresponding Verblunsky coefficients $\{\alpha_n\}_{n\geq 1}$, and let $N \in \mathbb{Z}^+ \cup \{0\}$. The polynomials defined by

$$\begin{gathered} \phi_n^{(N)}(z) = z\phi_{n-1}(z) + \alpha_{n+N}\phi_{n-1}^*(z), \\ n \geq 1,\ \phi_0^{(N)}(z) = 1, \end{gathered} \tag{15}$$

are called the *associated polynomials of* $\{\phi_n(z)\}_{n\geq 0}$ *of order* N. Similarly, given a finite set of complex numbers $\{\gamma_n\}_{n=1}^{N}$, with $|\gamma_n| \neq 1$, $n = 1, 2, \ldots, N$, let us define the new Verblunsky coefficients $\{\widetilde{\alpha}_n\}_{n\geq 1} = \{\gamma_1, \ldots, \gamma_N, \alpha_1, \alpha_2, \ldots\}$. Then the monic OPUC defined by the forward Szegő relation associated with $\{\widetilde{\alpha}_n\}_{n\geq 1}$ are said to be the *antiassociated polynomials of* $\{\phi_n(z)\}_{n\geq 0}$ *of order* N.

3. $(1, 1)$-Coherent Pairs on the Unit Circle

A pair of Hermitian regular linear functionals $(\mathcal{U}, \mathcal{V})$ defined on the linear space of Laurent polynomials is said to be a $(1, 1)$-*coherent pair* if their corresponding sequences of monic OPUC, $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$, are related by

$$\begin{aligned} &\phi_n^{[1]}(z) + a_n \phi_{n-1}^{[1]}(z) \\ &\quad = \psi_n(z) + b_n \psi_{n-1}(z), \quad a_n \neq 0,\ n \geq 1, \end{aligned} \tag{16}$$

where $\phi_n^{[1]}(z) = (\phi_{n+1}'(z))/(n+1)$, for $n \in \mathbb{N}$. In such a case, the pair $\{\phi_n(z)\}_{n\geq 0}$ and $\{\psi_n(z)\}_{n\geq 0}$ is also said to be a $(1, 1)$-coherent pair. If $b_n = 0$ for every $n \geq 1$, then $(\mathcal{U}, \mathcal{V})$ is called a $(1, 0)$-*coherent pair*.

Lemma 1. *If* $(\mathcal{U}, \mathcal{V})$ *satisfies* (16)*, then, one has the following.*

(i) $a_1 \neq b_1$ *if and only if* $\phi_n^{[1]}(z) \neq \psi_n(z)$*, for every* $n \geq 1$.

(ii) *For $n \geq 1$, one has*

$$\begin{aligned}\phi_n^{[1]}(z) &= \psi_n(z) + (b_n - a_n)\psi_{n-1}(z) \\ &+ \sum_{k=0}^{n-2}(-1)^{n-(k+1)} a_n a_{n-1}\cdots a_{k+2}(b_{k+1} - a_{k+1})\psi_k(z),\end{aligned} \tag{17}$$

$$\begin{aligned}\psi_n(z) &= \phi_n^{[1]}(z) + (a_n - b_n)\phi_{n-1}^{[1]}(z) \\ &+ \sum_{k=0}^{n-2}(-1)^{n-(k+1)} b_n b_{n-1}\cdots b_{k+2}(a_{k+1} - b_{k+1})\phi_k^{[1]}(z).\end{aligned} \tag{18}$$

Proof. From (16) it is easy to check that $a_1 = b_1$ if and only if there exists $N \in \mathbb{N}$, $N \geq 1$, such that $\phi_N^{[1]}(z) = \psi_N(z)$. Also, from (16) and using induction on n, it is immediate to prove (17) and (18). □

Corollary 2. *If $(\mathcal{U}, \mathcal{V})$ is a $(1,1)$-coherent pair given by* (16), *then*

$$\left\langle \mathcal{V}, \phi_n^{[1]}(z)\right\rangle = (-1)^n (a_1 - b_1)\prod_{j=2}^{n} a_j \langle \mathcal{V}, 1\rangle, \quad n \geq 1, \tag{19}$$

where $\prod_{j=k_1}^{k_2} a_j = 1$ whenever $k_2 < k_1$.

We will study the $(1,1)$-coherence relations when $\mathcal{U}$ is the linear functional associated with basic positive measures on the unit circle, namely, the Lebesgue and Bernstein-Szegő measures.

The Lebesgue linear functional is the linear functional associated with the Lebesgue measure $d\mu(\theta) = d\theta/2\pi$, and its corresponding sequence of monic OPUC is $\phi_n(z) = z^n$, for $n \in \mathbb{Z}^+ \cup \{0\}$. Besides, the reversed polynomials are $\phi_n^*(z) = 1$, $n \in \mathbb{Z}^+ \cup \{0\}$, and its Verblunsky coefficients are $\alpha_n = \phi_n(0) = 0$, for $n \geq 1$. Furthermore, its moments are $c_n = \delta_{n,0}$, for $n \in \mathbb{Z}^+ \cup \{0\}$, and its Carathéodory function is $F(z) = 1$.

The Bernstein-Szegő linear functional is associated with the measure $d\mu(\theta) = ((1 - |C|^2)/|1 + Ce^{i\theta}|^2)(d\theta/2\pi)$, with $C \in \mathbb{C}$ and $|C| < 1$. Its corresponding monic OPUC are $\phi_n(z) = z^{n-1}(z + C)$ for $n \geq 1$ and $\phi_0(z) = 1$. Its reversed polynomials are $\phi_n^*(z) = 1 + \overline{C}z$, for $n \geq 1$, and its Verblunsky coefficients are $\alpha_n = \phi_n(0) = 0$, for $n \geq 2$ and $\alpha_1 = C$. Besides, its moments are $c_n = (-C)^n$ for $n \in \mathbb{Z}^+ \cup \{0\}$, and its Carathéodory function is $F(z) = (1 - zC)/(1 + zC)$.

We begin by analyzing the first one.

4. The Lebesgue Linear Functional

Theorem 3. *Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle such that their corresponding monic OPUC satisfy* (16), *and let $\mathcal{U}$ be the Lebesgue linear functional.*

(i) *If $a_1 = b_1$, then $\mathcal{V}$ is also the linear functional associated with the Lebesgue measure, and $a_n = b_n$ for $n \geq 1$.*

(ii) *If $a_1 \neq b_1$ and $|\psi_n(0)| = |\beta_n| \neq 1$, $n \geq 1$, then*

$$v_n = (-a_2)^{n-1}(b_1 - a_1)v_0, \quad n \geq 1, \tag{20}$$

$$\begin{gathered} a_2 = b_2\left(1 - |\beta_1|^2\right) + \beta_1, \quad a_n = a_2, \\ b_n = \frac{b_{n-1}}{1 - |\beta_{n-1}|^2} = \frac{b_2}{\prod_{k=2}^{n-1}\left(1 - |\beta_k|^2\right)}, \quad n \geq 3, \end{gathered} \tag{21}$$

$$\begin{gathered} \beta_1 = a_1 - b_1, \\ \beta_n = (-1)^{n-1} b_n \cdots b_2 \beta_1 = -b_n\beta_{n-1}, \quad n \geq 2, \end{gathered} \tag{22}$$

$$\psi_1(z) = z + a_1 - b_1, \quad \textit{and for } n \geq 2,$$

$$\begin{aligned}\psi_n(z) &= z^n + (a_2 - b_n)z^{n-1} \\ &+ \sum_{k=1}^{n-2}(-1)^{n-k-1} b_n \cdots b_{k+2}(a_2 - b_{k+1})z^k + \beta_n,\end{aligned} \tag{23}$$

where $\{v_n\}_{n\geq 0}$ is the sequence of moments associated with $\mathcal{V}$.

Proof. Since $\phi_n^{[1]}(z) = z^n$ for $n \in \mathbb{Z}^+ \cup \{0\}$, then (16) becomes

$$z^n + a_n z^{n-1} = \psi_n(z) + b_n\psi_{n-1}(z), \quad a_n \neq 0,\ n \geq 1. \tag{24}$$

Thus, applying the linear functional $\mathcal{V}$ on the previous expression, we get

$$\begin{aligned} v_n &= -a_n v_{n-1} \\ &= (-1)^{n-1} a_n \cdots a_2 (b_1 - a_1) v_0, \quad n \geq 2, \\ v_1 &= (b_1 - a_1) v_0. \end{aligned} \tag{25}$$

(i) If $a_1 = b_1$, then from (25) we have $v_n = 0$ for $n \geq 1$. Thus, $\psi_n(z) = z^n$ for $n \geq 1$, and, as a consequence, from (24) we obtain $a_n = b_n$ for every $n \geq 1$.

(ii) From (18), we have

$$\begin{aligned}\psi_n(z) &= z^n + (a_n - b_n)z^{n-1} \\ &+ \sum_{k=0}^{n-2}(-1)^{n-(k+1)} b_n b_{n-1}\cdots b_{k+2}(a_{k+1} - b_{k+1})z^k.\end{aligned} \tag{26}$$

Multiplying (26) by z^{-1} and applying $\mathcal{V}$, we obtain

$$\begin{aligned} 0 &= v_{n-1} + (a_n - b_n)v_{n-2} \\ &+ \sum_{k=0}^{n-2}(-1)^{n-(k+1)} b_n b_{n-1}\cdots b_{k+2}(a_{k+1} - b_{k+1})v_{k-1}. \end{aligned} \tag{27}$$

Thus, multiplying this equation by b_{n+1} and adding it to the previous equation for $n+1$, we get

$$\begin{aligned} 0 &= v_n + a_{n+1}v_{n-1} \overset{(25)}{=} -a_n v_{n-1} + a_{n+1}v_{n-1} \\ &= (a_{n+1} - a_n)v_{n-1}, \quad n \geq 2. \end{aligned} \tag{28}$$

Since $a_n \neq 0$, $n \geq 1$, and $a_1 \neq b_1$, (25) yields $v_n \neq 0$ for $n \geq 1$. Thus, from (28), we conclude that $a_{n+1} = a_n$ for $n \geq 2$ or, equivalently, $a_{n+1} = a_2$ for $n \geq 2$. Therefore, (25) becomes (20).

On the other hand, from (26) we obtain (22) and (23). Besides, from the forward Szegő relation and (26), we can obtain another expression for $\psi_{n+1}(z)$, $n \geq 0$. By comparing the coefficients of z^n, we get $a_{n+1} - b_{n+1} = a_n - b_n - b_{n+1}|\beta_n|^2$, for $n \geq 1$. Hence, since $a_{n+1} = a_n$ and $|\beta_{n-1}| \neq 1$, for $n \geq 2$, (21) follows. □

We are interested in the cases where $\mathcal{V}$ is also a positive definite linear functional. Notice that, aside from the trivial case when $a_1 = b_1$, all of the coherence coefficients are determined from the values of a_1, b_1, and b_2 (or, equivalently, a_1, b_1, and a_2). Not every choice of these parameters will yield a positive definite linear functional $\mathcal{V}$. For instance, if $|b_2| = 1$ and $|a_1 - b_1| = |\beta_1| = \sqrt{2}$, then we can see from (22) that $|b_n| = 1$, $n \geqslant 3$, and $|\beta_n| = \sqrt{2}$, $n \geqslant 2$. However, it is possible to choose the values of a_1, b_1, and b_2 in order to get a positive definite linear functional $\mathcal{V}$, or at least its rational spectral transformation. We have the following cases.

Proposition 4. *Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair on the unit circle such that their corresponding monic OPUC satisfy* (16), *and let $\mathcal{U}$ be the linear functional associated with the Lebesgue measure. Assume that $\mathcal{V}$ is normalized (i.e., $v_0 = 1$). Then, one has the following.*

(i) *Let $|b_1 - a_1| < 1$. If $a_2 = a_1 - b_1$ (i.e., $b_2 = 0$), then $b_n = 0$ and $a_n = a_1 - b_1$ for every $n \geq 2$. Besides, $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure with parameter $b_1 - a_1$. Furthermore, if $b_N = 0$ for some $N \geq 2$, then $b_2 = 0$.*

(ii) *If a_1, b_1, $a_2 \in \mathbb{R}$ and either $0 < a_1 - b_1 < a_2 < 1$ or $-1 < a_2 < a_1 - b_1 < 0$ holds, then the Carathéodory function associated with $\mathcal{V}$ is*

$$F_{\mathcal{V}} = -\frac{b_1 - a_1}{a_2} F_B(z) + \frac{b_1 - a_1 + a_2}{a_2}, \tag{29}$$

where $F_B(z)$ is the Carathéodory function associated with the Bernstein-Szegő measure with parameter $-a_2$. As a consequence, the orthogonality measure associated with $\mathcal{V}$ is

$$\begin{aligned} d\mu_2 = & -\frac{b_1 - a_1}{a_2} \frac{1 - |a_2|^2}{|1 + a_2 e^{i\theta}|^2} \frac{d\theta}{2\pi} \\ & + \frac{b_1 - a_1 + a_2}{a_2} \frac{d\theta}{2\pi}. \end{aligned} \tag{30}$$

(iii) *For any values of a_1, b_1, the value of b_2 can be chosen in such a way that $\mathcal{V}$ is the linear functional associated with a rational spectral transformation of a Nevai class measure.*

Proof. (i) Notice that $a_1 \neq b_1$ because $a_2 \neq 0$. We first prove that if $b_N = 0$ for some $N \geq 2$, then $b_n = 0$ for $n \geq 2$. Assume that for some $N \geq 2$, $b_N = 0$. From (21), (22), and (23) it follows that $b_n = 0 = \beta_n$ and $\psi_n(z) = z^{n-1}(z + a_2)$ for $n \geq N$. Besides, another expression for $\psi_N(z)$ is $\psi_N(z) = z\psi_{N-1}(z) + \beta_N \psi^*_{N-1}(z) = z\psi_{N-1}(z)$, where $\psi_{N-1}(z)$ is given by (23). Thus, the comparison of the coefficients of z^{N-1} in both expressions of $\psi_N(z)$ yields $a_2 = a_2 - b_{N-1}$, and thus, $b_{N-1} = 0$. Following the same argument for $b_{N-1}, \ldots, b_2$, we conclude that $b_n = 0$ for $n = 2, \ldots, N-1$ and $a_2 = a_1 - b_1$. Therefore, $b_n = 0 = \beta_n$ for $n \geq 2$, $\beta_1 = a_1 - b_1 = a_2$, and $\psi_n(z) = z^{n-1}(z + a_1 - b_1)$ for $n \geq 1$. As a consequence, from (21) and (20), it follows that $a_{n+1} = a_1 - b_1$ and $v_n = (b_1 - a_1)^n$, $n \geq 0$. Finally, since $|\beta_1| = |b_1 - a_1| < 1$, then $\mathcal{V}$ is the linear functional associated with the Bernstein-Szegő measure.

(ii) From (20), the Carathéodory function associated with $\mathcal{V}$ is $F_{\mathcal{V}} = 1 + 2\sum_{k\geq 1}(b_1 - a_1)(-a_2)^{k-1}z^k$. Since $|a_2| < 1$, then (see [19]) the Bernstein-Szegő polynomials of parameter $-a_2$ have moments $c_n = (-a_2)^n$ and are orthogonal with respect to the measure $((1 - |a_2|^2)/|1 + a_2 e^{i\theta}|^2)(d\theta/2\pi)$, and their associated Carathéodory function is $F_B(z) = 1 - 2a_2\sum_{k\geq 1}(-a_2)^{k-1}z^k$. Therefore, (29) holds. In other words (see [23]), $F_{\mathcal{V}}$ can be obtained by applying a rescaling to the moments of $F_B(z)$, followed by a perturbation of its first moment (i.e., a diagonal perturbation of the corresponding Toeplitz matrix). Thus, the orthogonality measure associated with $\mathcal{V}$ is given by (30).

(iii) From (21), given $\beta_1 = a_1 - b_1$, we have $b_3 = b_2/(1 - |\beta_2|^2) = b_2/(1 - |b_2\beta_1|^2)$, so we can choose $|b_2|$ small enough so that β_2 is sufficiently close to 0. Thus, b_3 will also be close to 0, and since

$$\begin{aligned} \beta_n &= -b_n\beta_{n-1}, \quad n \geq 2, \\ b_n &= \frac{b_{n-1}}{1 - |\beta_{n-1}|^2}, \quad n \geq 3, \end{aligned} \tag{31}$$

$\{|b_n|\}_{n\geqslant 2}$ will be an increasing sequence and, as a consequence, $\{|\beta_n|\}_{n\geqslant 2}$ will be a decreasing sequence. Besides, b_2 can be chosen so that $|b_n|$ converges to a constant b, $0 < b < 1$, and therefore the product $\prod_{k=2}^{n-1}|1 - |\beta_k|^2|$ will also converge to $|b_2|/b$. This shows that $\beta_n \to 0$, and thus $\{\beta_n\}_{n\geqslant 2}$ defines a Nevai measure μ. As a consequence, since $\mathcal{V}$ has $\{\beta_n\}_{n\geqslant 1}$ as Verblunsky coefficients, $\mathcal{V}$ can be expressed as an antiassociated perturbation of order 1 (see [24]) applied to the measure μ. □

5. The Bernstein-Szegő Linear Functional

Now, we proceed to analyze the companion measure $\mathcal{V}$ when $\mathcal{U}$ is the Bernstein-Szegő linear functional defined as above.

Theorem 5. *Let $\mathscr{U}$ be the Bernstein-Szegő linear functional, and let $(\mathscr{U},\mathscr{V})$ be a $(1,1)$-coherent pair on the unit circle given by* (16). *Then, the moments of $\mathscr{V}$ are*

$$v_n = (-1)^n \times \left[(a_1 - b_1) \sum_{k=0}^{n-1} \frac{n+1-k}{n+1} C^k \prod_{j=2}^{n-k} a_j + \frac{1}{n+1} C^n \right] v_0, \quad n \geq 1, \tag{32}$$

where $\prod_{j=k_1}^{k_2} a_j = 1$ whenever $k_2 < k_1$, and the sequence of monic OPUC $\{\psi_n(z)\}_{n\geq 0}$ is given by $\psi_0(z) = 1$, $\psi_1(z) = z + (a_1 - b_1) + (1/2)C$, and, for $n \geq 2$,

$$\begin{aligned} \psi_n(z) &= z^n + \left[(a_n - b_n) + \frac{n}{n+1} C \right] z^{n-1} \\ &\quad - \left[b_n (a_{n-1} - b_{n-1}) - \frac{n-1}{n} C (a_n - b_n) \right] z^{n-2} \\ &\quad + \sum_{k=0}^{n-3} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+3} \\ &\quad \times \left[b_{k+2} (a_{k+1} - b_{k+1}) - \frac{k+1}{k+2} C (a_{k+2} - b_{k+2}) \right] z^k. \end{aligned} \tag{33}$$

Furthermore, $|\beta_n| = |\psi_n(0)| \neq 1$, $n \geq 1$, and

$$\begin{aligned} \beta_1 &= (a_1 - b_1) + \frac{1}{2} C, \qquad \beta_2 = -\left[b_2 \beta_1 - \frac{1}{2} C a_2 \right], \\ &\qquad \text{and for } n \geq 3, \\ \beta_n &= (-1)^{n-1} b_n b_{n-1} \cdots b_3 \left[b_2 (a_1 - b_1) - \frac{1}{2} C (a_2 - b_2) \right] \\ &= -b_n \beta_{n-1}, \end{aligned} \tag{34}$$

$$\begin{aligned} a_n + b_n \left[|\beta_{n-1}|^2 - 1 \right] &= -\frac{n}{n+1} C + \beta_1 + \frac{1}{2} C a_2 \overline{\beta}_1 - \sum_{k=2}^{n-1} b_k |\beta_{k-1}|^2, \\ &\qquad n \geq 2. \end{aligned} \tag{35}$$

Proof. Since $\phi_n^{[1]}(z) = z^n + (n/(n+1)) C z^{n-1}$, for $n \geq 0$, then, from (19), we get

$$v_n = -\frac{n}{n+1} C v_{n-1} + (-1)^n (a_1 - b_1) \times \prod_{j=2}^{n} a_j v_0, \quad n \geq 1, \tag{36}$$

where $\prod_{j=k_1}^{k_2} a_j = 1$ whenever $k_2 < k_1$. From (36) and using induction on n, it is easy to verify that the moments of $\mathscr{V}$ are given by (32). Besides, from (18) and (33), (34) holds. Furthermore, since $\{\psi_n(z)\}_{n\geq 0}$ is a sequence of monic OPUC, then it follows that $|\beta_n| \neq 1$, $n \geq 1$.

On the other hand, from the forward Szegő relation and (33), we can get another expression of $\psi_n(z)$, for $n \geq 2$. Hence, comparing the coefficients of z and using (34), (35) follows. □

As in the previous section, we are interested in the situations where $\mathscr{V}$ is also a positive definite linear functional. Notice now that the values of a_1, b_1, a_2, b_2, and b_3 determine all other coherence coefficients. We have the following cases.

Proposition 6. *Let $\mathscr{U}$ be the Bernstein-Szegő linear functional, and let $(\mathscr{U},\mathscr{V})$ be a $(1,1)$-coherent pair on the unit circle given by* (16). *Then, one has the following.*

(i) *If $a_1 = b_1$, then $C = 0$ and, therefore, $\mathscr{U}$ and $\mathscr{V}$ are Lebesgue linear functionals, and $a_n = b_n$ for $n \geq 1$.*

(ii) *Let $a_1 \neq b_1$.*

 (1) *If $\mathscr{V}$ is normalized (i.e., $v_0 = 1$) and $b_N = 0$ for some $N \geq 3$, then $C = 0$; this is, $\mathscr{U}$ is the Lebesgue linear functional. As a consequence, $b_{n+1} = 0$, $a_{n+1} = a_1 - b_1$, $\psi_n(z) = z^{n-1}(z + a_1 - b_1)$, and $v_n = (b_1 - a_1)^n$ for every $n \geq 1$. In other words, for $|b_1 - a_1| < 1$, $\mathscr{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1$.*

 (2) *If $(1/2)Ca_2 = b_2\beta_1$, then $\psi_n(z) = z^{n-1}(z + a_1 - b_1 + (1/2)C)$ for $n \geq 1$; this is, for $|b_1 - a_1 - (1/2)C| < 1$, $\mathscr{V}$ is the linear functional associated with the Bernstein-Szegő measure, with parameter $b_1 - a_1 - (1/2)C$.*

 (3) *If $(1/2)Ca_2 \neq b_2\beta_1$ and $b_n \neq 0$, for $n \geq 3$, then*

$$b_n = \frac{b_{n-1}}{1 - |\beta_{n-1}|^2} = \frac{b_3}{\prod_{k=3}^{n-1} \left(1 - |\beta_{k-1}|^2\right)}, \quad n \geq 4, \tag{37}$$

 and b_3 can be chosen so that $\mathscr{V}$ is the linear functional associated with an antiassociated perturbation of order 2 applied to a Nevai measure.

Proof. (i) If we multiply (33) by z^{-1} and apply $\mathscr{V}$, then we get, for $n \geq 2$,

$$\begin{aligned} 0 &= v_{n-1} + \left[(a_n - b_n) + \frac{n}{n+1} C \right] v_{n-2} \\ &\quad - \left[b_n (a_{n-1} - b_{n-1}) - \frac{n-1}{n} C (a_n - b_n) \right] v_{n-3} \\ &\quad + \sum_{k=0}^{n-3} (-1)^{n-(k+1)} b_n b_{n-1} \cdots b_{k+3} \\ &\quad \times \left[b_{k+2} (a_{k+1} - b_{k+1}) - \frac{k+1}{k+2} C (a_{k+2} - b_{k+2}) \right] v_{k-1}. \end{aligned} \tag{38}$$

If we multiply this equation by b_{n+1} and we add it to the previous equation for $n+1$, then we obtain

$$0 = v_n + \left[a_{n+1} + \frac{n+1}{n+2}C\right] v_{n-1} + \frac{n}{n+1}Ca_{n+1}v_{n-2}, \quad n \geq 2. \tag{39}$$

Hence, from (39) and (36), it follows that

$$\begin{aligned} 0 &= (-1)^{n+1}\left(a_1 - b_1\right) \\ &\quad \times \prod_{j=2}^{n+1} a_j v_0 \left[a_{n+3} - a_{n+2} + \frac{1}{(n+3)(n+4)}C\right] \\ &\quad + \left[\frac{1}{(n+2)(n+3)}a_{n+3} + \frac{n+1}{(n+2)(n+3)(n+4)}C\right] Cv_n, \\ &\qquad n \geq 0. \end{aligned} \tag{40}$$

On the other hand, if we apply the linear functional $\mathscr{V}$ to both sides of the (1, 1)-coherence relation (16), we get $v_1 + [a_1 + (C/2)]v_0 = b_1 v_0$ and

$$v_n + \left[a_n + \frac{Cn}{n+1}\right] v_{n-1} + a_n\frac{C(n-1)}{n}v_{n-2} = 0, \quad n \geq 2. \tag{41}$$

Thus, from (39) and (41), we obtain, for $n \geq 2$,

$$\begin{aligned} 0 = &\left[a_{n+1} - a_n + \frac{C}{(n+1)(n+2)}\right] v_{n-1} \\ &+ \left[\frac{na_{n+1}}{n+1} - \frac{(n-1)a_n}{n}\right] Cv_{n-2}. \end{aligned} \tag{42}$$

Therefore, if $a_1 = b_1$, then from (32), the moments of $\mathscr{V}$ are $v_n = (1/(n+1))(-C)^n v_0$ for $n \geq 0$, and, as a consequence, (40) becomes

$$\begin{aligned} 0 = &(-1)^n \frac{1}{(n+1)(n+2)(n+3)} \\ &\times C^{n+1}\left[a_{n+3} + \frac{n+1}{n+4}C\right] v_0, \quad n \geq 0, \end{aligned} \tag{43}$$

and (42) is, for $n \geq 2$,

$$\begin{aligned} 0 = &(-1)^{n-1}\frac{1}{n(n+1)}C^{n-1} \\ &\times \left[\frac{1}{n+2}C - \frac{1}{n-1}a_{n+1}\right] v_0, \quad n \geq 2. \end{aligned} \tag{44}$$

Then, if $C \neq 0$, from (43) and (44) it follows that $a_n = -((n-2)/(n+1))C$, for $n \geq 3$, and $a_n = ((n-2)/(n+1))C$, for $n \geq 3$, respectively, which is a contradiction. Thus, if $a_1 = b_1$, then $C = 0$; that is, $\mathscr{U}$ is the Lebesgue linear functional, and in case the part i of Theorem 3 holds.

Now, let us assume $a_1 \neq b_1$.

(ii)(1) From part (i) of Proposition 4, it suffices to show that $\mathscr{U}$ is the Lebesgue linear functional. Thus, let us prove that if $b_N = 0$ for some $N \geq 3$ (and therefore $\beta_N = 0$), then $C = 0$. Indeed, if $b_N = 0$ for some $N \geq 3$, then from (33) for $n = N+1$, $N \geq 2$, it follows that $\beta_{N+1} = 0$, for $N \geq 3$. Furthermore, from the forward Szegő relation and (33) for $n = N$, we obtain an expression of $\psi_{N+1}(z)$, for $N \geq 3$. Hence, comparing the coefficients of this expression and (33) for $n = N+1$, we obtain, for $N \geq 3$,

$$\left(a_{N+1} - b_{N+1}\right) + \frac{N+1}{N+2}C = a_N + \frac{N}{N+1}C, \tag{45}$$

$$-b_{N+1}a_N + \frac{N}{N+1}C\left(a_{N+1} - b_{N+1}\right) = \frac{N-1}{N}Ca_N, \tag{46}$$

$$b_{N+1}\frac{N-1}{N}Ca_N = 0. \tag{47}$$

Since $a_N \neq 0$, then from (47) it follows that either $C = 0$ or $b_{N+1} = 0$. If $C = 0$, then from (46) we get $b_{N+1} = 0$ and, as a consequence, from (45) we have $a_{N+1} = a_N$. If $b_{N+1} = 0$, then from (46) it follows that either $C = 0$ (and thus, from (45), $a_{N+1} = a_N$) or $a_{N+1} = ((N^2-1)/N^2)a_N$. If $b_{N+1} = 0$ and $a_{N+1} = ((N^2-1)/N^2)a_N$, from (45) it follows that $C = (((N+1)(N+2))/N^2)a_N$. But if $b_{N+1} = 0$, we can follow a similar argument and conclude that $C = ((N+2)(N+3)/(N+1)^2)a_{N+1}$, and since $a_{N+1} = ((N^2-1)/N^2)a_N$, then we also have $C = ((N+2)(N+3)(N-1)/(N+1)N^2)a_N$, which yields a contradiction. Therefore, $C = 0$.

(ii)(2) If $(1/2)Ca_2 = b_2\beta_1$, then from (34) it follows that $\beta_2 = 0$ and, as a consequence, $\beta_n = 0$ for every $n \geq 2$. Therefore, from the forward Szegő relation it follows that $\psi_n(z) = z^{n-1}(z+\beta_1)$ for $n \geq 1$.

(ii)(3) From the forward Szegő relation and (33) we obtain an expression of $\psi_n(z)$, for $n \geq 3$. If we compare the coefficients of z of this expression and (33), we get $\beta_2[b_4 - b_3] = b_4 b_3 \beta_2 \sum_{k=3}^{3} \overline{b}_k |\beta_{k-1}|^2$ and

$$\begin{aligned} &b_{n-1}\cdots b_4\beta_2\left[b_n - b_3\right] \\ &\quad = b_n b_{n-1}\cdots b_3\beta_2\sum_{k=3}^{n-1}\overline{b}_k\left|\beta_{k-1}\right|^2, \quad n \geq 5. \end{aligned} \tag{48}$$

Thus, if $(1/2)Ca_2 \neq b_2\beta_1$, then from (34) it follows that $\beta_2 \neq 0$, and, as a consequence, if $b_4, \ldots, b_{n-1}$, $n \geq 5$, are nonzero, then from (48) we get

$$b_n = \frac{b_3}{1 - b_3\sum_{k=3}^{n-1}\overline{b}_k\left|\beta_{k-1}\right|^2}, \quad n \geq 4. \tag{49}$$

Besides, from (34), $|\beta_n| = |b_n\beta_{n-1}|$ for $n \geq 3$, and if $b_3 \neq 0$, then by induction on n we can prove that $b_n = b_{n-1}/(1 - |\beta_{n-1}|^2)$, for $n \geq 4$, which is (37). Therefore, proceeding as in the proof of Proposition 4, we can choose $|b_3|$ small enough so that β_3 is sufficiently close to 0. As a consequence, $\{|b_n|\}_{n\geqslant 3}$ will be an increasing sequence, and hence $\{|\beta_n|\}_{n\geqslant 3}$ will be a decreasing sequence. Also, we can choose b_3 such that $|b_n|$ converges to a constant b, with $0 < b < 1$. The infinite product $\prod_{k=3}^{n-1}|1-|\beta_k|^2|$ will then converge to $|b_3|/b$. Therefore, since $\{\beta_n\}_{n\geqslant 1}$ are the Verblunsky coefficients of $\mathscr{V}$, this linear functional $\mathscr{V}$ is an antiassociated perturbation of order 2 (see [24]) applied to a Nevai measure μ. □

Acknowledgments

The authors thank the referee the valuable comments. They greatly contributed to improve the contents of the paper. The work of Luis Garza was supported by Conacyt Grant no. 156668 and Beca Santander Iberoamérica para Jóvenes Profesores e Investigadores (Mexico). The work of Francisco Marcellán and Natalia C. Pinzón-Cortés has been supported by Dirección General de Investigación, Desarrollo e Innovación, Ministerio de Economía y Competitividad of Spain, Grant MTM2012-36732-C03-01.

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