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*Research Article*

# On $\xi^{(s)}$-Quadratic Stochastic Operators on Two-Dimensional Simplex and Their Behavior

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A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of differing species in biology. Some quadratic stochastic operators have been studied by Lotka and Volterra. The general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for quadratic stochastic operators which are the simplest nonlinear operators. To study this problem, several classes of QSO were investigated. We study $\xi^{(s)}$-QSO defined on 2D simplex. We first classify $\xi^{(s)}$-QSO into 20 nonconjugate classes. Further, we investigate the dynamics of three classes of such operators.

## 1. Introduction

The history of quadratic stochastic operators can be traced back to Bernstein's work [1]. The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modelings in various fields such as biology [1–7], physics [8, 9], economics, and mathematics [3, 6, 10, 11].

One of such systems which relates to the population genetics is given by a quadratic stochastic operator [1]. A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of species in biology, which arises as follows. Consider a population consisting of $m$ species (or traits) $1, 2, \ldots, m$. We denote a set of all species (traits) by $I = \{1, 2, \ldots, m\}$. Let $x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)})$ be a probability distribution of species at an initial state and let $P_{ij,k}$ be a probability that individuals in the $i$th and $j$th species (traits) interbreed to produce an individual from $k$th species (trait). Then, a probability distribution $x^{(1)} = (x_1^{(1)}, \ldots, x_m^{(1)})$ of the species (traits) in the first generation can be found as a total probability; that is,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}. \tag{1}$$

This means that the association $x^{(0)} \to x^{(1)}$ defines a mapping $V$ called *the evolution operator*. The population evolves by starting from an arbitrary state $x^{(0)}$, then passing to the state $x^{(1)} = V(x^{(0)})$ (the first generation), then to the state $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})) = V^{(2)}(x^{(0)})$ (the second generation), and so on. Therefore, the evolution states of the population system are described by the following discrete dynamical system:

$$\begin{gathered} x^{(0)}, \qquad x^{(1)} = V\left(x^{(0)}\right), \\ x^{(2)} = V^{(2)}\left(x^{(0)}\right), \qquad x^{(3)} = V^{(3)}\left(x^{(0)}\right) \cdots . \end{gathered} \tag{2}$$

In other words, a QSO describes a distribution of the next generation if the distribution of the current generation was given. The fascinating applications of QSO to population genetics were given in [6].

In [12], it was given a long self-contained exposition of the recent achievements and open problems in the theory of the QSO. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully finished even in the class of QSO (the QSO is the simplest nonlinear operator). The difficulty of the problem depends on the given cubic matrix $(P_{ijk})_{i,j,k=1}^{m}$. An asymptotic behavior of the QSO even on the small dimensional simplex is complicated [11, 13–16]. In order to solve this problem, many researchers always introduced a certain class of QSO and studied their behavior, for example, Volterra-QSO [11, 17–20], permutated Volterra-QSO [21, 22], Quasi-Volterra-QSO [23], $\ell$-Volterra-QSO [24, 25], non-Volterra-QSO [13, 15], strictly non-Volterra-QSO [26], F-QSO [27], and non-Volterra operators generated by product measure [28–30]. However, all these classes together would not cover a set of all QSO. Therefore, there are many classes of QSO which were not studied yet. Recently, in the papers [31, 32], a new class of QSO was introduced. This class was called a $\xi^{(s)}$-QSO. In this paper, we are going to continue the study of $\xi^{(s)}$-QSO. This class of operators depends on a partition of the coupled index set (the coupled trait set) $\mathbf{P}_m = \{(i,j) : i < j\} \subset I \times I$. In case of two-dimensional simplex ($m = 3$), the coupled index set (the coupled trait set) $\mathbf{P}_3$ has five possible partitions. The dynamics of $\xi^{(s)}$-QSO corresponding to the point partition (the maximal partition) of $\mathbf{P}_3$ have been investigated in [31, 32]. In the present paper, we are going to describe and classify such operators generated by other three partitions. Further, we also investigate the dynamics of three classes of such operators.

The paper is organized as follows. In Section 2, we give some preliminary definitions. In Section 3, we discuss the classification of $\xi^{(s)}$-QSO related to $|\xi| = 2$. It turns out that some obtained operators are $\ell$-Volterra-QSO (see [24, 25]) and permuted $\ell$-Volterra-QSO. The dynamics of $\ell$-Volterra-QSO are not fully studied yet. In [24, 25], some particular cases have been investigated, which do not cover our operators. Therefore, in further sections, we study dynamics of $\ell$-Volterra-QSO and permuted $\ell$-Volterra-QSO. In Section 4, we study the behavior of $\ell$-Volterra-QSO $V_{13}$ taken from class $K_1$. In Section 5, we study the behavior of a permuted $\ell$-Volterra-QSO $V_4$ taken from class $K_4$. Note that $V_4$ is a permutation of $V_{13}$. In Section 6, we study the behavior of a permuted Volterra-QSO $V_{28}$ taken from class $K_{19}$. In the last section, we just highlight the dynamics of Volterra-QSO $V_{25}$ taken from class $K_{17}$ which was already studied in [17–19].

## 2. Preliminaries

Recall that a quadratic stochastic operator (QSO) is a mapping of the simplex:

$$S^{m-1} = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = \overline{1,m} \right\} \tag{3}$$

into itself, of the form:

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1,m}, \tag{4}$$

where $V(x) = x' = (x'_1, \ldots, x'_m)$ and $P_{ij,k}$ is a coefficient of heredity, which satisfies the following conditions:

$$P_{ij,k} \geq 0, \qquad P_{ij,k} = P_{ji,k}, \qquad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{5}$$

Thus, each quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ can be uniquely defined by a cubic matrix $\mathcal{P} = (P_{ijk})_{i,j,k=1}^{m}$ with conditions (5).

We denote sets of fixed points and $k$-periodic points of $V : S^{m-1} \to S^{m-1}$ by $\mathrm{Fix}(V)$ and $\mathrm{Per}_k(V)$, respectively. Due to Brouwer's fixed point theorem, one always has that $\mathrm{Fix}(V) \neq \emptyset$ for any QSO $V$. For a given point $x^{(0)} \in S^{m-1}$, a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of $V : S^{m-1} \to S^{m-1}$ starting from $x^{(0)}$ is defined by $x^{(n+1)} = V(x^{(n)})$. By $\omega_V(x^{(0)})$, we denote a set of omega limiting points of the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$. Since $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and $S^{m-1}$ is compact, one has that $\omega_V(x^{(0)}) \neq \emptyset$. Obviously, if $\omega_V(x^{(0)})$ consists of a single point, then the trajectory converges and a limiting point is a fixed point of $V : S^{m-1} \to S^{m-1}$.

Recall that a Volterra-QSO is defined by (4) and (5) and the additional assumption:

$$P_{ij,k} = 0 \quad \text{if } k \notin \{i, j\}. \tag{6}$$

The biological treatment of condition (6) is clear: *the offspring repeats the genotype (trait) of one of its parents*.

One can see that a Volterra-QSO has the following form:

$$x'_k = x_k \left( 1 + \sum_{i=1}^{m} a_{ki} x_i \right), \quad k \in I, \tag{7}$$

where

$$a_{ki} = 2P_{ik,k} - 1 \quad \text{for } i \neq k, \qquad a_{ii} = 0, \quad i \in I. \tag{8}$$

Moreover,

$$a_{ki} = -a_{ik}, \qquad |a_{ki}| \leq 1. \tag{9}$$

This kind of operators is intensively studied in [11, 17–20, 33]. Note that this operator is a discretization of the Lotka-Volterra model [5, 7] which models an interacting competing species in the population system. Such a model has received considerable attention in the fields of biology, ecology, and mathematics (see, e.g., [2, 3, 7, 8]).

In [24], a notion of $\ell$-Volterra-QSO, which generalizes a notion of Volterra-QSO, has been introduced. Let us recall it here.

In order to introduce a new class of QSO, we need some auxiliary notations.

We fix $\ell \in I$ and assume that elements $P_{ij,k}$ of the matrix $(P_{ij,k})_{i,j,k=1}^{m}$ satisfy

$$\begin{aligned} & P_{ij,k} = 0 \quad \text{if } k \notin \{i, j\} \\ & \text{for any } k \in \{1, \ldots, \ell\}, \ i, j \in I, \\ & P_{i_0 j_0,k} > 0 \quad \text{for some } (i_0, j_0), \\ & i_0 \neq k, \ j_0 \neq k, \ k \in \{\ell + 1, \ldots, m\}. \end{aligned} \tag{10}$$

For any fixed $\ell \in I$, the QSO defined by (4), (5), and (10) is called *$\ell$-Volterra-QSO*.

*Remark 1.* Here, we stress the following points.

(1) Note that an $\ell$-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.

(2) It is known [17] that there is not a periodic trajectory for Volterra-QSO. However, there are such trajectories for $\ell$-Volterra-QSO [24].

By following [25], take $k \in \{1, \ldots, \ell\}$; then, $P_{kk,i} = 0$ for $i \neq k$ and

$$1 = \sum_{i=1}^{m} P_{kk,i} = P_{kk,k} + \sum_{i=\ell+1}^{m} P_{kk,i}. \tag{11}$$

By using $P_{ij,k} = P_{ji,k}$ and denoting $a_{ki} = 2P_{ik,k} - 1, k \neq i,\ a_{kk} = P_{kk,k} - 1$, one then gets

$$V: \begin{cases} x_k' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) & \text{if } k = \overline{1, \ell}, \\ x_k' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) + \sum_{\substack{i,j=1 \\ i \neq k, j \neq k}}^{m} P_{ij,k} x_i x_j & \text{if } k = \overline{\ell + 1, m}. \end{cases} \tag{12}$$

$\ell$-Volterra-QSO.

Note that

$$\begin{aligned} & a_{kk} \in [-1, 0], \qquad |a_{ki}| \leq 1, \\ & a_{ki} + a_{ik} = 2\left(P_{ik,i} + P_{ik,k}\right) - 2 \leq 0, \quad i, k \in I. \end{aligned} \tag{13}$$

We call that an operator $V$ is *permuted $\ell$-Volterra-QSO* if there is a permutation $\tau$ of the set $I$ and an $\ell$-Volterra-QSO $V_0$ such that $(V(x))_{\tau(k)} = (V_0(x))_k$ for any $k \in I$. In other words, $V$ can be represented as follows:

$$V_\tau: \begin{cases} x_{\tau(k)}' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) & \text{if } k = \overline{1, \ell}, \\ x_{\tau(k)}' = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right) + \sum_{\substack{i,j=1 \\ i \neq k, j \neq k}}^{m} P_{ij,k} x_i x_j & \text{if } k = \overline{\ell + 1, m}. \end{cases} \tag{14}$$

We remark that if $\ell = m$, then a permuted $\ell$-Volterra-QSO becomes a permuted Volterra-QSO. Some properties of such operators were studied in [19, 34]. The dynamics of certain class of permuted Volterra-QSO have been investigated in [32]. Note that, in [24, 25], a class of $\ell$-Volterra-QSO has been studied. An asymptotic behavior of permuted $\ell$-Volterra-QSO has not been investigated yet. Some particular cases have been considered in [31, 32].

In this paper, we are going to introduce a new class of QSO which contain $\ell$-Volterra-QSO and permuted $\ell$-Volterra-QSO as a particular case.

Note that each element $x \in S^{m-1}$ is a probability distribution of the set $I = \{1, \ldots, m\}$. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be vectors taken from $S^{m-1}$. We say that *$x$ is equivalent to $y$* if $x_k = 0 \Leftrightarrow y_k = 0$. We denote this relation by $x \sim y$.

Let $\operatorname{supp}(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that *$x$ is singular to $y$* and denote by $x \perp y$ if $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$. Note that if $x, y \in S^{m-1}$, then $x \perp y$ if and only if $(x, y) = 0$; here, $(\cdot, \cdot)$ stands for a standard inner product in $\mathbb{R}^m$.

We denote sets of coupled indexes by

$$\begin{aligned} \mathbf{P}_m &= \{(i, j) : i < j\} \subset I \times I, \\ \Delta_m &= \{(i, i) : i \in I\} \subset I \times I. \end{aligned} \tag{15}$$

For a given pair $(i, j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = (P_{ij,1}, \ldots, P_{ij,m})$. It is clear due to the condition (5) that $\mathbb{P}_{ij} \in S^{m-1}$.

Let $\xi_1 = \{A_i\}_{i=1}^N$ and $\xi_2 = \{B_i\}_{i=1}^M$ be some fixed partitions of $\mathbf{P}_m$ and $\Delta_m$, respectively; that is, $A_i \cap A_j = \emptyset$, $B_i \cap B_j = \emptyset$, and $\bigcup_{i=1}^N A_i = \mathbf{P}_m$, $\bigcup_{i=1}^M B_i = \Delta_m$, where $N, M \leq m$.

*Definition 2.* A quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ given by (4) and (5) is called a $\xi^{(\mathrm{as})}$-QSO with respect to the partitions $\xi_1, \xi_2$ (where the letter "as" stands for absolutely continuous-singular) if the following conditions are satisfied:

(i) for each $k \in \{1, \ldots, N\}$ and any $(i, j), (u, v) \in A_k$, one has that $\mathbb{P}_{ij} \sim \mathbb{P}_{uv}$;

(ii) for any $k \neq \ell, k, \ell \in \{1, \ldots, N\}$ and any $(i, j) \in A_k$ and $(u, v) \in A_\ell$, one has that $\mathbb{P}_{ij} \perp \mathbb{P}_{uv}$;

(iii) for each $d \in \{1, \ldots, M\}$ and any $(i, i), (j, j) \in B_d$, one has that $\mathbb{P}_{ii} \sim \mathbb{P}_{jj}$;

(iv) for any $s \neq h$, $s, h \in \{1, \ldots, M\}$ and any $(u, u) \in B_s$, and $(v, v) \in B_h$ one has that $\mathbb{P}_{uu} \perp \mathbb{P}_{vv}$.

*Remark 3.* If $\xi_2$ is the point partition, that is, $\xi_2 = \{\{(1, 1)\}, \ldots \{(m, m)\}\}$, then we call the corresponding QSO by $\xi^{(s)}$-QSO (where the letter "$s$" stands for singularity) since in this case every two different vectors $\mathbb{P}_{ii}$ and $\mathbb{P}_{jj}$ are singular. If $\xi_2$ is the trivial, that is, $\xi_2 = \{\Delta_m\}$, then we call the corresponding QSO by $\xi^{(a)}$-QSO (where the letter "$a$" stands for absolute continuous) since in this case every two vectors $\mathbb{P}_{ii}$ and $\mathbb{P}_{jj}$ are equivalent. We note that some classes of $\xi^{(a)}$-QSO have been studied in [35]. In the present paper, we restrict ourselves to the $\xi^{(s)}$-case. Note that, in general, the class of $\xi^{(\mathrm{as})}$-QSO will be studied elsewhere in the future.

*Remark 4.* For the $\xi^{(s)}$-QSO, that is, in the case $\xi_2 = \{\{(1,1)\},\ldots\{(m,m)\}\}$, the condition (iii) of Definition 2 is trivial and the condition (iv) means that there is a permutation $\pi$ of the set $I = \{1,\ldots,m\}$ such that $\mathbb{P}_{ii} = e_{\pi(i)}$ for any $i = \overline{1,m}$ where $e_k = (0,\ldots,0,\underbrace{1}_{k},0,\ldots,0)$, $k = \overline{1,m}$, are vertices of the simplex $S^{m-1}$.

*A Biological Interpretation of a $\xi^{(s)}$-QSO.* We treat $I = \{1,\ldots,m\}$ as a set of all possible traits of the population system. A coefficient $P_{ij,k}$ is a probability that parents in the $i$th and $j$th traits interbreed to produce a child from the $k$th trait. The condition $P_{ij,k} = P_{ji,k}$ means that the gender of parents do not influence having a child from the $k$th trait. In this sense, $\mathbf{P}_m \cup \Delta_m$ is a set of all possible coupled traits of parents. A vector $\mathbb{P}_{ij} = (P_{ij,1},\ldots,P_{ij,m})$ is a possible distribution of children in a family while parents are carrying traits from the $i$th and $j$th types. A biological meaning of a $\xi^{(s)}$-QSO is as follows: a set $\mathbf{P}_m$ of all differently coupled traits of parents is splitted into $N$ groups $A_1,\ldots,A_N$ (here $N$ is less than the number $m$ of traits) such that the chance (probability) of having a child from any trait in two different families whose parents' coupled traits belong to the same group $A_k$ is simultaneously either positive or zero (the condition (i) of Definition 2); meanwhile, two families whose parents' coupled traits belong to two different groups $A_k$ and $A_l$ cannot have a child from the same trait, simultaneously (condition (ii) of Definition 2). Moreover, the parents who are sharing the same type of traits can have a child from only one type of traits (condition (iv) of Definition 2 and Remark 4).

## 3. Classification of $\xi^{(s)}$-QSO on 2D Simplex

In this section, we are going to study $\xi^{(s)}$-QSO in two-dimensional simplex; that is, $m = 3$. In this case, we have the following possible partitions of $\mathbf{P}_3$:

$$\begin{gathered}\xi_1 = \{\{(1,2)\},\{(1,3)\},\{(2,3)\}\}, \quad |\xi_1| = 3,\\ \xi_2 = \{\{(2,3)\},\{(1,2),(1,3)\}\}, \quad |\xi_2| = 2,\\ \xi_3 = \{\{(1,3)\},\{(1,2),(2,3)\}\}, \quad |\xi_3| = 2,\\ \xi_4 = \{\{(1,2)\},\{(1,3),(2,3)\}\}, \quad |\xi_4| = 2,\\ \xi_5 = \{(1,2),(1,3),(2,3)\}, \quad |\xi_5| = 1.\end{gathered} \tag{16}$$

We note that, in [31, 32], $\xi^{(s)}$-QSO related to the partition $\xi_1$ which is the maximal partition of $\mathbf{P}_3$ has been investigated. In this paper, we are aiming to study $\xi^{(s)}$-QSO related to the partitions $\xi_2$, $\xi_3$, and $\xi_4$. We shall show that these three classes of $\xi^{(s)}$-QSO are conjugate to each other. Therefore, it is enough to study $\xi^{(s)}$-QSO related to the partition $\xi_2$. A class of $\xi^{(s)}$-QSO related to the partition $\xi_5$ will be studied in elsewhere in the future.

Let us recall that two operators $V_1$ and $V_2$ are called *(topologically or linearly) conjugate* if there is a permutation matrix $P$ such that $P^{-1}V_1P = V_2$. Let $\pi$ be a permutation of the set $I = \{1,\ldots,m\}$. For any vector $x$, we define $\pi(x) = (x_{\pi(1)},\ldots,x_{\pi(m)})$. It is easy to check that if $\pi$ is a permutation of the set $I$ corresponding to the given permutation matrix $P$, then one has that $Px = \pi(x)$. Therefore, two operators $V_1$ and $V_2$ are conjugate if and only if $\pi^{-1}V_1\pi = V_2$ for some permutation $\pi$. Throughout this paper, we shall consider "conjugate operators" in this sense. We say that two classes $K_1$ and $K_2$ of operators are conjugate if every operator taken from $K_1$ is conjugate to some operator taken from $K_2$ and vice versa.

**Proposition 5.** *A class of all $\xi^{(s)}$-QSO corresponding to partition $\xi_3$ (or $\xi_4$) is conjugate to a class of all $\xi^{(s)}$-QSO corresponding to partition $\xi_2$.*

*Proof.* We show that two classes of all $\xi^{(s)}$-QSO corresponding to partitions $\xi_2$ and $\xi_3$ are conjugate to each other. Analogously, one can show that two classes of all $\xi^{(s)}$-QSO corresponding to partitions $\xi_2$ and $\xi_4$ are conjugate to each other as well.

Assume that an operator $V : S^2 \to S^2$ given by

$$V : x'_k = \sum_{i,j=1}^{3} P_{ij,k}x_ix_j, \quad k = 1,2,3, \tag{17}$$

is a $\xi^{(s)}$-QSO corresponding to partition $\xi_3 = \{\{(1,3)\},\{(1,2),(2,3)\}\}$. This means that the coefficients $(P_{ij,k})_{i,j,k=1}^{3}$ of $V$ satisfy the following three conditions: (i) $\mathbb{P}_{12} \sim \mathbb{P}_{23}$, (ii) $\mathbb{P}_{13} \perp \mathbb{P}_{12}$, $\mathbb{P}_{13} \perp \mathbb{P}_{23}$, and (iii) $\mathbb{P}_{11} \perp \mathbb{P}_{22} \perp \mathbb{P}_{33}$ where $\mathbb{P}_{ij} = (P_{ij,1}, P_{ij,2}, P_{ij,3})$.

We consider the following operator: $V_\pi = \pi^{-1}V\pi$, where $\pi = \left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$. It is clear that $V_\pi$ is conjugate to $V$, where

$$V_\pi : x'_k = \sum_{i,j=1}^{3} P^{\pi}_{ij,k}x_ix_j, \quad k = 1,2,3 \tag{18}$$

such that $P^{\pi}_{ij,k} = P_{\pi(i)\pi(j),\pi(k)}$ for any $i,j,k = 1,2,3$; equivalently, $\mathbb{P}^{\pi}_{ij} = \pi\mathbb{P}_{\pi(i)\pi(j)}$ (in a vector form) for any $i,j = 1,2,3$. Now, we are going to show that $V_\pi$ is a $\xi^{(s)}$-QSO corresponding to $\xi_2 = \{\{(2,3)\},\{(1,2),(1,3)\}\}$. In order to show it we have to check three conditions.

(i) $\mathbb{P}^{\pi}_{12} \sim \mathbb{P}^{\pi}_{13}$. Indeed, since $\mathbb{P}^{\pi}_{12} = \pi\mathbb{P}_{12}$, $\mathbb{P}^{\pi}_{13} = \pi\mathbb{P}_{23}$, $\mathbb{P}_{12} \sim \mathbb{P}_{23}$, one has $\mathbb{P}^{\pi}_{12} \sim \mathbb{P}^{\pi}_{13}$.

(ii) $\mathbb{P}^{\pi}_{12} \perp \mathbb{P}^{\pi}_{23}$, $\mathbb{P}^{\pi}_{13} \perp \mathbb{P}^{\pi}_{23}$. Indeed, since $\mathbb{P}^{\pi}_{12} = \pi\mathbb{P}_{12}$, $\mathbb{P}^{\pi}_{23} = \pi\mathbb{P}_{13}$, and $\mathbb{P}_{12} \perp \mathbb{P}_{13}$, we obtain that $\mathbb{P}^{\pi}_{12} \perp \mathbb{P}^{\pi}_{23}$. In the same manner, we can get that $\mathbb{P}^{\pi}_{13} \perp \mathbb{P}^{\pi}_{23}$.

(iii) $\mathbb{P}^{\pi}_{11} \perp \mathbb{P}^{\pi}_{22} \perp \mathbb{P}^{\pi}_{33}$. Indeed, since $\mathbb{P}^{\pi}_{11} = \pi\mathbb{P}_{22}$, $\mathbb{P}^{\pi}_{22} = \pi\mathbb{P}_{11}$, $\mathbb{P}^{\pi}_{33} = \pi\mathbb{P}_{33}$, and $\mathbb{P}_{11} \perp \mathbb{P}_{22} \perp \mathbb{P}_{33}$, we have that $\mathbb{P}^{\pi}_{11} \perp \mathbb{P}^{\pi}_{22} \perp \mathbb{P}^{\pi}_{33}$.

This shows that any $\xi^{(s)}$-QSO taken from the class corresponding to partition $\xi_3$ is conjugate to some $\xi^{(s)}$-QSO taken

Table 1

(a)

| Case | $\mathbb{P}_{12}$ | $\mathbb{P}_{13}$ | $\mathbb{P}_{23}$ |
|---|---|---|---|
| $\mathbf{I}_1$ | $(a, 1-a, 0)$ | $(a, 1-a, 0)$ | $(0, 0, 1)$ |
| $\mathbf{I}_2$ | $(0, a, 1-a)$ | $(0, a, 1-a)$ | $(1, 0, 0)$ |
| $\mathbf{I}_3$ | $(a, 0, 1-a)$ | $(a, 0, 1-a)$ | $(0, 1, 0)$ |
| $\mathbf{I}_4$ | $(0, 0, 1)$ | $(0, 0, 1)$ | $(a, 1-a, 0)$ |
| $\mathbf{I}_5$ | $(1, 0, 0)$ | $(1, 0, 0)$ | $(0, a, 1-a)$ |
| $\mathbf{I}_6$ | $(0, 1, 0)$ | $(0, 1, 0)$ | $(a, 0, 1-a)$ |

(b)

| Case | $\mathbb{P}_{11}$ | $\mathbb{P}_{22}$ | $\mathbb{P}_{33}$ |
|---|---|---|---|
| $\mathbf{II}_1$ | $(1, 0, 0)$ | $(0, 1, 0)$ | $(0, 0, 1)$ |
| $\mathbf{II}_2$ | $(0, 1, 0)$ | $(1, 0, 0)$ | $(0, 0, 1)$ |
| $\mathbf{II}_3$ | $(0, 0, 1)$ | $(0, 1, 0)$ | $(1, 0, 0)$ |
| $\mathbf{II}_4$ | $(1, 0, 0)$ | $(0, 0, 1)$ | $(0, 1, 0)$ |
| $\mathbf{II}_5$ | $(0, 0, 1)$ | $(1, 0, 0)$ | $(0, 1, 0)$ |
| $\mathbf{II}_6$ | $(0, 1, 0)$ | $(0, 0, 1)$ | $(1, 0, 0)$ |

from the class corresponding to partition $\xi_2$. Analogously, we can show that any $\xi^{(s)}$-QSO $V$ taken from the class corresponding to partition $\xi_2$ is conjugate to a $\xi^{(s)}$-QSO $V_\pi = \pi^{-1}V\pi$ which belongs to the class corresponding to partition $\xi_3$, where $\pi$ is the same permutation as given above. This completes the proof. □

Therefore, it is enough to study a class of all $\xi^{(s)}$-QSO corresponding to the partition $\xi_2$. Now, we shall consider some subclass of a class of all $\xi^{(s)}$-QSO corresponding to partition $\xi_2$ by choosing coefficients $(P_{ij,k})_{i,j,k=1}^{3}$ in special forms where $a \in [0,1]$ (see Table 1).

The choices of the cases $(\mathbf{I}_i, \mathbf{II}_j)$, where $i, j = \overline{1,6}$, will give 36 operators from the class of $\xi^{(s)}$-QSO corresponding to partition $\xi_2$. Finally, we obtain 36 parametric operators which are defined as follows:

$$V_1 : \begin{cases} x_1' = x_1^2 + 2ax_1(1-x_1), \\ x_2' = x_2^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_3^2 + 2x_2x_3, \end{cases}$$

$$V_2 : \begin{cases} x_1' = x_2^2 + 2ax_1(1-x_1), \\ x_2' = x_1^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_3^2 + 2x_2x_3, \end{cases}$$

$$V_3 : \begin{cases} x_1' = x_3^2 + 2ax_1(1-x_1), \\ x_2' = x_2^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_1^2 + 2x_2x_3, \end{cases}$$

$$V_4 : \begin{cases} x_1' = x_1^2 + 2ax_1(1-x_1), \\ x_2' = x_3^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_2^2 + 2x_2x_3, \end{cases}$$

$$V_5 : \begin{cases} x_1' = x_2^2 + 2ax_1(1-x_1), \\ x_2' = x_3^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_1^2 + 2x_2x_3, \end{cases}$$

$$V_6 : \begin{cases} x_1' = x_3^2 + 2ax_1(1-x_1), \\ x_2' = x_1^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_2^2 + 2x_2x_3, \end{cases}$$

$$V_7 : \begin{cases} x_1' = x_1^2 + 2x_2x_3, \\ x_2' = x_2^2 + 2ax_1(1-x_1), \\ x_3' = x_3^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_8 : \begin{cases} x_1' = x_2^2 + 2x_2x_3, \\ x_2' = x_1^2 + 2ax_1(1-x_1), \\ x_3' = x_3^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_9 : \begin{cases} x_1' = x_3^2 + 2x_2x_3, \\ x_2' = x_2^2 + 2ax_1(1-x_1), \\ x_3' = x_1^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{10} : \begin{cases} x_1' = x_1^2 + 2x_2x_3, \\ x_2' = x_3^2 + 2ax_1(1-x_1), \\ x_3' = x_2^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{11} : \begin{cases} x_1' = x_2^2 + 2x_2x_3, \\ x_2' = x_3^2 + 2ax_1(1-x_1), \\ x_3' = x_1^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{12} : \begin{cases} x_1' = x_3^2 + 2x_2x_3, \\ x_2' = x_1^2 + 2ax_1(1-x_1), \\ x_3' = x_2^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{13} : \begin{cases} x_1' = x_1^2 + 2ax_1(1-x_1), \\ x_2' = x_2^2 + 2x_2x_3, \\ x_3' = x_3^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{14} : \begin{cases} x_1' = x_2^2 + 2ax_1(1-x_1), \\ x_2' = x_1^2 + 2x_2x_3, \\ x_3' = x_3^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{15} : \begin{cases} x_1' = x_3^2 + 2ax_1(1-x_1), \\ x_2' = x_2^2 + 2x_2x_3, \\ x_3' = x_1^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{16} : \begin{cases} x_1' = x_1^2 + 2ax_1(1-x_1), \\ x_2' = x_3^2 + 2x_2x_3, \\ x_3' = x_2^2 + 2(1-a)x_1(1-x_1), \end{cases}$$

$$V_{17} : \begin{cases} x_1' = x_2^2 + 2ax_1(1-x_1), \\ x_2' = x_3^2 + 2x_2x_3, \\ x_3' = x_1^2 + 2(1-a)x(1-x), \end{cases}$$

$$V_{18}:\begin{cases}x_1' = x_3^2 + 2ax_1(1-x_1),\\ x_2' = x_1^2 + 2x_2x_3,\\ x_3' = x_2^2 + 2(1-a)x_1(1-x_1),\end{cases}$$

$$V_{19}:\begin{cases}x_1' = x_1^2 + 2ax_2x_3,\\ x_2' = x_2^2 + 2(1-a)x_2x_3,\\ x_3' = x_3^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{20}:\begin{cases}x_1' = x_2^2 + 2ax_2x_3,\\ x_2' = x_1^2 + 2(1-a)x_2x_3,\\ x_3' = x_3^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{21}:\begin{cases}x_1' = x_3^2 + 2ax_2x_3,\\ x_2' = x_2^2 + 2(1-a)x_2x_3,\\ x_3' = x_1^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{22}:\begin{cases}x_1' = x_1^2 + 2ax_2x_3,\\ x_2' = x_3^2 + 2(1-a)x_2x_3,\\ x_3' = x_2^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{23}:\begin{cases}x_1' = x_2^2 + 2ax_2x_3,\\ x_2' = x_3^2 + 2(1-a)x_2x_3,\\ x_3' = x_1^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{24}:\begin{cases}x_1' = x_3^2 + 2ax_2x_3,\\ x_2' = x_1^2 + 2(1-a)x_2x_3,\\ x_3' = x_2^2 + 2x_1(1-x_1),\end{cases}$$

$$V_{25}:\begin{cases}x_1' = x_1^2 + 2x_1(1-x_1),\\ x_2' = x_2^2 + 2ax_2x_3,\\ x_3' = x_3^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{26}:\begin{cases}x_1' = x_2^2 + 2x_1(1-x_1),\\ x_2' = x_1^2 + 2ax_2x_3,\\ x_3' = x_3^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{27}:\begin{cases}x_1' = x_3^2 + 2x_1(1-x_1),\\ x_2' = x_2^2 + 2ax_2x_3,\\ x_3' = x_1^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{28}:\begin{cases}x_1' = x_1^2 + 2x_1(1-x_1),\\ x_2' = x_3^2 + 2ax_2x_3,\\ x_3' = x_2^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{29}:\begin{cases}x_1' = x_2^2 + 2x_1(1-x_1),\\ x_2' = x_3^2 + 2ax_2x_3,\\ x_3' = x_1^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{30}:\begin{cases}x_1' = x_3^2 + 2x_1(1-x_1),\\ x_2' = x_1^2 + 2ax_2x_3,\\ x_3' = x_2^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{31}:\begin{cases}x_1' = x_1^2 + 2ax_2x_3,\\ x_2' = x_2^2 + 2x_1(1-x_1),\\ x_3' = x_3^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{32}:\begin{cases}x_1' = x_2^2 + 2ax_2x_3,\\ x_2' = x_1^2 + 2x_1(1-x_1),\\ x_3' = x_3^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{33}:\begin{cases}x_1' = x_3^2 + 2ax_2x_3,\\ x_2' = x_2^2 + 2x_1(1-x_1),\\ x_3' = x_1^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{34}:\begin{cases}x_1' = x_1^2 + 2ax_2x_3,\\ x_2' = x_3^2 + 2x_1(1-x_1),\\ x_3' = x_2^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{35}:\begin{cases}x_1' = x_2^2 + 2ax_2x_3,\\ x_2' = x_3^2 + 2x_1(1-x_1),\\ x_3' = x_1^2 + 2(1-a)x_2x_3,\end{cases}$$

$$V_{36}:\begin{cases}x_1' = x_3^2 + 2ax_2x_3,\\ x_2' = x_1^2 + 2x_1(1-x_1),\\ x_3' = x_2^2 + 2(1-a)x_2x_3.\end{cases} \tag{19}$$

**Theorem 6.** *All 36 operators from the class of $\xi^{(s)}$-QSO corresponding to partition $\xi_2$ defined as above are classified into 20 nonconjugate classes:*

$$\begin{gathered}K_1 = \{V_1, V_{13}\}, \qquad K_2 = \{V_2, V_{15}\},\\ K_3 = \{V_3, V_{14}\}, \qquad K_4 = \{V_4, V_{16}\},\\ K_5 = \{V_5, V_{18}\}, \qquad K_6 = \{V_6, V_{17}\},\\ K_7 = \{V_7\}, \qquad K_8 = \{V_8, V_9\},\\ K_9 = \{V_{10}\}, \qquad K_{10} = \{V_{11}, V_{12}\},\\ K_{11} = \{V_{19}, V_{31}\}, \qquad K_{12} = \{V_{20}, V_{33}\},\\ K_{13} = \{V_{21}, V_{32}\}, \qquad K_{14} = \{V_{22}, V_{34}\},\\ K_{15} = \{V_{23}, V_{36}\}, \qquad K_{16} = \{V_{24}, V_{35}\},\\ K_{17} = \{V_{25}\}, \qquad K_{18} = \{V_{26}, V_{27}\},\\ K_{19} = \{V_{28}\}, \qquad K_{20} = \{V_{29}, V_{30}\}.\end{gathered} \tag{20}$$

*Proof.* It is easy to check that partition $\xi_2 = \{\{(2,3)\}, \{(1,2),(1,3)\}\}$ is invariant under only one permutation $\pi = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$. The proof of the theorem can be easily insured with respect to this permutation. It is straightforward. □

The main problem is to investigate the dynamics of these classes of operators. In what follows, we are going to study three classes $K_1$, $K_4$, and $K_{19}$. From the list, one can conclude that these three classes of operators are either $\ell$-Volterra-QSO or permuted $\ell$-Volterra-QSO. The class $K_{17}$ was already studied in [17, 18]. The rest of the classes of operators would be studied in elsewhere in the future.

## 4. Dynamics of $\xi^{(s)}$-QSO from Class $K_1$

In this section we are going to study dynamics of $\xi^{(s)}$-QSO from class $K_1$.

We need some auxiliary facts about properties of the function $f_a : [0,1] \to [0,1]$ given by

$$f_a(x) = x^2 + 2ax(1-x), \tag{21}$$

where $a \in [0,1]$. If $a = 1/2$, then the function becomes the identity mapping. Therefore, we shall consider only the case of $a \neq 1/2$.

**Proposition 7.** *Let $f_a : [0,1] \to [0,1]$ be a function given by (21) where $a \neq 1/2$. Then, the following statements hold true.*

(i) *One has that* $\mathrm{Fix}(f_a) = \{0,1\}$.

(ii) *The function $f_a$ is increasing.*

(iii) *One has that $(a-(1/2))(f_a(x)-x) > 0$ for any $x \in (0,1)$.*

(iv) *One has that $\omega_{f_a}(x_0) = \begin{cases}\{0\} & \text{if } 0\le a<1/2\\ \{1\} & \text{if } 1/2<a\le 1\end{cases}$ for any $x_0 \in (0,1)$.*

*Proof.* Let $f_a : [0,1] \to [0,1]$ be a function given by (21) where $a \neq 1/2$.

(i) In order to find fixed points of the function $f_a$, we should solve the following equation: $x^2 + 2ax(1-x) = x$. It follows from the last equation that $(1-2a)(x^2-x) = 0$. Since $a \neq 1/2$, we get that $x = 0$ or $x = 1$. Therefore, $\mathrm{Fix}(f_a) = \{0,1\}$.

(ii) Since $f_a'(x) = 2x(1-a) + 2a(1-x) \geq 0$, the function $f_a$ is increasing.

(iii) Since $f_a(x) - x = (1-2a)(x^2-x)$, we may get that $(a-(1/2))(f_a(x)-x) > 0$.

(iv) Let $0 \leq a < 1/2$ and $x_0 \in (0,1)$. Due to (iii), we have that $f_a(x_0) < x_0$. Since $f_a$ is increasing, we obtain that $f_a^{(n+1)}(x_0) < f_a^{(n)}(x_0)$ for any $n \in \mathbb{N}$. This means that $\{f_a^{(n)}(x_0)\}_{n=1}^{\infty}$ is a bounded decreasing sequence. Consequently, it converges to some point $x^*$, and $x^*$ should be a fixed point; that is, $x^* = 0$. This means that $w_{f_a}(x_0) = \{0\}$. Similarly, one can show that if $1/2 < a \leq 1$, then $\omega_{f_a}(x_0) = \{1\}$ for any $x_0 \in (0,1)$. This completes the proof. □

Now, we are going to study dynamics of a $\xi^{(s)}$-QSO $V_{13} : S^2 \to S^2$ taken from $K_1$:

$$V_{13} : \begin{cases} x_1' = x_1^2 + 2x_1 a(1-x_1) \\ x_2' = x_2^2 + 2x_2 x_3 \\ x_3' = x_3^2 + 2(1-a)x_1(1-x_1), \end{cases} \tag{22}$$

where $0 \leq a \leq 1$. This operator is an $\ell$-Volterra-QSO, and its behavior was not studied in [24, 25].

Let $e_1$, $e_2$, and $e_3$ be the vertices of the simplex $S^2$ and let $\Gamma_i = \{x \in S^2 : x_i = 0\}$ be an edge of the simplex $S^2$, where $i = 1,2,3$. Let $S_{1\leq 3} = \{x \in S^2 : x_1 \leq x_3\}$, $S_{1\geq 3} = \{x \in S^2 : x_1 \geq x_3\}$, and $l_{13} = \{x \in S^2 : x_1 = x_3\}$.

**Theorem 8.** *Let $V_{13} : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (22) and let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_{13})$ be an initial point. Then the following statements hold true.*

(i) *One has that*

$$\mathrm{Fix}(V_{13}) = \begin{cases} \{e_1, e_2, e_3\} & \text{if } a \neq \dfrac{1}{2}, \\ \Gamma_2 \cup l_{13} & \text{if } a = \dfrac{1}{2}. \end{cases} \tag{23}$$

(ii) *If $0 \leq a < 1/2$, then*

$$\omega_{V_{13}}(x^{(0)}) = \begin{cases} \{e_2\} & \text{if } x_2^{(0)} \neq 0, \\ \{e_3\} & \text{if } x_2^{(0)} = 0. \end{cases} \tag{24}$$

(iii) *If $1/2 < a \leq 1$, then*

$$\omega_{V_{13}}(x^{(0)}) = \begin{cases} \{e_1\} & \text{if } x_1^{(0)} \neq 0, \\ \{e_2\} & \text{if } x_1^{(0)} = 0. \end{cases} \tag{25}$$

(iv) *If $a = 1/2$, then*

$$\omega_{V_{13}}(x^{(0)}) = \begin{cases} \left\{\left(x_1^{(0)}, 0, 1-x_1^{(0)}\right)\right\} & \text{if } x_1^{(0)} > \dfrac{1}{2}, \\ \left\{\left(x_1^{(0)}, 1-2x_1^{(0)}, x_1^{(0)}\right)\right\} & \text{if } x_1^{(0)} \leq \dfrac{1}{2}. \end{cases} \tag{26}$$

*Proof.* Let $V_{13} : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (22), let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_{13})$ be an initial point, and let $\{x^{(n)}\}_{n=0}^{\infty}$ be a trajectory of $V_{13}$ starting from the point $x^{(0)}$.

(i) In order to find fixed points of (22), we should solve the following system of equations:

$$\begin{aligned} x_1 &= x_1^2 + 2x_1 a(1-x_1), \\ x_2 &= x_2^2 + 2x_2 x_3, \\ x_3 &= x_3^2 + 2(1-a)x_1(1-x_1). \end{aligned} \tag{27}$$

We shall separately consider two cases: $a = 1/2$ and $a \neq 1/2$.

Let $a \neq 1/2$. From the first equation of (27), we get that $x_1 = 0$ and $x_1 = 1$ (see Proposition 7(i)). It follows from the second equation of (27) that if $x_1 = 0$, then $x_2 = 0$, $x_3 = 1$ or $x_2 = 1$, $x_3 = 0$ and if $x_1 = 1$, then $x_2 = x_3 = 0$. This means that $\mathrm{Fix}(V_{13}) = \{e_1, e_2, e_3\}$.

Let $a = 1/2$. The first equation of (27) takes the form $x_1 = x_1$. From the second equation of (27), we get that $x_2(x_1 - x_3) = 0$. This yields that $x_2 = 0$ or $x_1 = x_3$. In both cases, the third equation of (27) holds true. Therefore, we have that $\mathrm{Fix}(V_{13}) = \Gamma_2 \cup l_{13}$.

(ii) Let $0 \le a < 1/2$. It is clear that $x_1^{(n)} = f_a^{(n)}(x_1^{(0)})$. Therefore, due to Proposition 7(iv), we have that $\lim_{n\to\infty} x_1^{(n)} = 0$. Hence, $\omega_{V_{13}}(x^{(0)}) \subset \Gamma_1 = \{x \in S^2 : x_1 = 0\}$. Now, we shall separately consider two cases: $x_2^{(0)} = 0$ and $x_2^{(0)} \neq 0$.

Let $x_2^{(0)} = 0$. In this case, $x_2^{(n)} = 0$ and $\lim_{n\to\infty} x_3^{(n)} = 1$. Hence, $\omega_{V_{13}}(x^{(0)}) = \{e_3\}$.

Let $x_2^{(0)} \neq 0$. We need the following result.

*Claim*. One has that $V_{13}(S_{1\le 3}) \subset S_{1\le 3}$. Moreover, for any $x^{(0)} \notin \mathrm{Fix}(V_{13})$, there exists $n_0$ (depending on $x^{(0)}$) such that $\{x^{(n)}\}_{n=n_0}^{\infty} \subset S_{1\le 3}$.

*Proof of Claim*. If $x_1 \le x_3$, then $x_1' = x_1^2 + 2ax_1(1-x_1) \le x_3^2 + 2(1-a)x_1(1-x_1) = x_3'$. This means that $V_{13}(S_{1\le 3}) \subset S_{1\le 3}$.

Let $x^{(0)} \notin S_{1\le 3}$, and suppose that all elements of the trajectory belong to set $S^2 \setminus S_{1\le 3}$; that is, $\{x^{(n)}\}_{n=0}^{\infty} \subset S^2 \setminus S_{1\le 3}$. It follows from $x_1^{(n)} \to 0$ and $x_3^{(n)} < x_1^{(n)}$ that $x_3^{(n)} \to 0$. This with $\sum_{i=1}^{3} x_i^{(n)} = 1$ implies that $x_2^{(n)} \to 1$. On the other hand, we have that $x_2^{(n+1)} = x_2^{(n)}(1 - (x_1^{(n)} - x_3^{(n)})) \le x_2^{(n)}$. It yields that $\{x_2^{(n)}\}_{n=0}^{\infty}$ is decreasing; hence, it converges to some point $x^* < 1$. This is a contradiction. This completes the proof of Claim.

Due to Claim, there exists $n_0$ such that $x_1^{(n)} \le x_3^{(n)}$ for all $n \ge n_0$. Therefore, $x_2^{(n+1)} = x_2^{(n)}(1 - (x_1^{(n)} - x_3^{(n)})) \ge x_2^{(n)}$, and $\{x_2^{(n)}\}_{n=n_0}^{\infty}$ is an increasing sequence which converges to $x_2^*$. This yields that $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)})$ converges to $(0, x_2^*, 1 - x_2^*)$, where $x_2^* > 0$. We know that $(0, x_2^*, 1 - x_2^*)$ should be a fixed point. Consequently, $(0, x_2^*, 1 - x_2^*) = (0,1,0)$ and $\omega_{V_{13}}(x^{(0)}) = \{e_2\}$.

(iii) Let $1/2 < a \le 1$. Due to Proposition 7(iv), we have that $\lim_{n\to\infty} x_1^{(n)} = 1$, whenever $0 < x_1^{(0)} < 1$. Therefore, if $x_1^{(0)} \neq 0$, then $\omega_{V_{13}}(x^{(0)}) \subset \{e_1\}$. Since $\omega_{V_{13}}(x^{(0)})$ is not empty, we obtain that $\omega_{V_{13}}(x^{(0)}) = \{e_1\}$. Let $x_1^{(0)} = 0$, then $x_1^{(n)} = 0$ for all $n \in \mathbb{N}$. Moreover, we have that $x_3^{(n+1)} = (x_3^{(n)})^2$ and $\lim_{n\to\infty} x_3^{(n)} = 0$. This means that $\lim_{n\to\infty} x_2^{(n)} = 1$. Therefore, if $x_1^{(0)} = 0$, then $\omega_{V_{13}}(x^{(0)}) = \{e_2\}$.

(iv) Let $a = 1/2$ and $x_1^{(0)} > 1/2$. Then $x_1^{(n)} = x_1^{(0)} > 1/2$ for any $n \in \mathbb{N}$. Since $x_3^{(n)} = 1 - x_1^{(n)} - x_2^{(n)}$, one gets that $x_3^{(n)} < 1/2 < x_1^{(n)}$. This implies that $\{x_2^{(n)}\}_{n=0}^{\infty}$ is decreasing, and hence it converges to $x_2^*$. Consequently, $(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) \to (x_1^{(0)}, x_2^*, x_3^*)$. We know that $(x_1^{(0)}, x_2^*, x_3^*)$ should be a fixed point. Since $x_1^{(0)} > 1/2 \ge x_3^{(*)}$, we find that $x_2^* = 0$ and $x_3^* = 1 - x_1^{(0)}$. This means that $\omega_{V_{13}}(x^{(0)}) = \{(x_1^{(0)}, 0, 1 - x_1^{(0)})\}$.

Let $x_1^{(0)} \le 1/2$. Then $x_1^{(n)} = x_1^{(0)} \le 1/2$ for any $n \in \mathbb{N}$. Since $x_3' - x_1' = x_3^2 - x_1^2$, we have that $V(S_{1\le 3}) \subset S_{1\le 3}$ and $V(S_{1\ge 3}) \subset S_{1\ge 3}$. If $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \in S_{1\le 3}$, then $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) \in S_{1\le 3}$. This yields that $\{x_2^{(n)}\}_{n=0}^{\infty}$ is decreasing, and hence it converges to $x_2^*$. Therefore, $(x_1^{(n)}, x_2^{(n)}, x_3^{(n)})$ converges to $(x_1^{(0)}, x_2^{(*)}, x_3^{(*)})$. Since $(x_1^{(0)}, x_2^*, x_3^*) \in S_{1\le 3}$ is a fixed point, we have that $x_2^{(*)} = 1 - 2x_1^{(0)}$ and $x_3^{(*)} = x_1^{(0)}$. In the similar manner, one may have that if $x^{(0)} \in S_{1\ge 3}$, then $\omega_{V_{13}}(x^{(0)}) = \{(x_1^{(0)}, 1 - 2x_1^{(0)}, x_1^{(0)})\}$.

This completes the proof. □

## 5. Dynamics of $\xi^{(s)}$-QSO from Class $K_4$

We are going to study dynamics of a $\xi^{(s)}$-QSO $V_4 : S^2 \to S^2$ taken from $K_4$:

$$V_4 : \begin{cases} x_1' = x_1^2 + 2ax_1(1-x_1), \\ x_2' = x_3^2 + 2(1-a)x_1(1-x_1), \\ x_3' = x_2^2 + 2x_2x_3, \end{cases} \tag{28}$$

where $0 \le a \le 1$. One can immediately see that this operator is a permuted $\ell$-Volterra-QSO. As we mentioned, the behavior of such kinds of operators is not studied yet. It is worth mentioning that $V_4$ is a permutation of $V_{13}$.

Let

$$\begin{aligned} B(b) &= \frac{3 - 2b - \sqrt{4b^2 - 8b + 5}}{2}, \quad b \in [0,1], \\ C_{\pm}(c) &= \frac{1 - 2c \pm \sqrt{4c^2 - 8c + 1}}{2}, \quad c \in \left[0, \frac{2-\sqrt{3}}{2}\right]. \end{aligned} \tag{29}$$

It is clear that

$$\begin{aligned} 0 \le B(b) &= \frac{2(1-b)}{1 + 2(1-b) + \sqrt{1 + 4(1-b)^2}} \le 1, \\ 0 &\le \frac{1 - 2c \pm \sqrt{(1-2c)^2 - 4c}}{2} = C_{\pm}(c) \\ &\le \frac{1 - 2c + \sqrt{4c^2 + 4c + 1}}{2} = 1 \end{aligned} \tag{30}$$

for any $b \in [0,1]$ and $c \in [0, (2-\sqrt{3})/2] \subset [0, 1/2]$.

**Theorem 9.** *Let $V_4 : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (28) and let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_4) \cup \mathrm{Per}_2(V_4)$ be an initial point. Then, the following statements hold true.*

(i) *One has that*

$$\mathrm{Fix}(V_4) = \begin{cases} \left\{e_1, \left(0, \dfrac{3-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right)\right\} & \text{if } a \neq \dfrac{1}{2}, \\ \{b, B(b), 1 - B(b)\}_{b\in[0,1]} & \text{if } a = \dfrac{1}{2}. \end{cases} \tag{31}$$

(ii) *One has that*

$$\mathrm{Per}_2(V_4) = \begin{cases} \{e_2, e_3\} & \text{if } a \neq \dfrac{1}{2}, \\ \{c, C_{\pm}(c), 1 - C_{\pm}(c)\}_{c\in[0,(2-\sqrt{3})/2)} & \text{if } a = \dfrac{1}{2}. \end{cases} \tag{32}$$

(iii) *If* $1/2 < a \le 1$*, then*

$$\omega_{V_4}\left(x^{(0)}\right) = \begin{cases} \{e_1\} & if\ x_1^{(0)} \neq 0, \\ \{e_2, e_3\} & if\ x_1^{(0)} = 0. \end{cases} \tag{33}$$

(iv) *If* $a = 1/2$*, then*

$$\omega_{V_4}\left(x^{(0)}\right) = \begin{cases} \left\{\left(x_1^{(0)}, C_\pm\left(x_1^{(0)}\right), 1 - C_\pm\left(x_1^{(0)}\right)\right)\right\} & if\ x_1^{(0)} \in \left[0, \dfrac{2-\sqrt{3}}{2}\right), \\ \left\{\left(x_1^{(0)}, B\left(x_1^{(0)}\right), 1 - B\left(x_1^{(0)}\right)\right)\right\} & if\ x_1^{(0)} \in \left[\dfrac{2-\sqrt{3}}{2}, 1\right]. \end{cases} \tag{34}$$

*Proof.* Let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_4) \cup \mathrm{Per}_2(V_4)$ be an initial point and let $\{x^{(n)}\}_{n=0}^{\infty}$ be a trajectory of $V_4$ starting from the point $x^{(0)}$.

(i) In order to find fixed points of $V_4$, we have to solve the following system:

$$\begin{aligned} x_1 &= x_1^2 + 2ax_1(1-x_1), \\ x_2 &= x_3^2 + 2(1-a)x_1(1-x_1), \\ x_3 &= x_2^2 + 2x_2x_3. \end{aligned} \tag{35}$$

Let $a \neq 1/2$. From the first equation of the system (35), one can find that $x_1 = 0$ or $x_1 = 1$ (see Proposition 7(i)). If $x_1 = 1$, then $x_2 = x_3 = 0$. If $x_1 = 0$, then the second equation of the system (35) becomes as follows:

$$x_2^2 - 3x_2 + 1 = 0. \tag{36}$$

So, the solutions of this quadratic equation are $x_2^\pm = (3 \pm \sqrt{5})/2$. We can verify that the only solution $x_2 = (3-\sqrt{5})/2$ belongs to $[0,1]$. Therefore, one has $x_3 = (-1+\sqrt{5})/2$. Hence, $\mathrm{Fix}(V_4) = \{e_1, (0, (3-\sqrt{5})/2, (-1+\sqrt{5})/2)\}$ whenever $a \neq 1/2$.

Let $a = 1/2$. The system (35) then takes the following form

$$\begin{aligned} x_1 &= x_1, \\ x_2 &= x_3^2 + x_1(1-x_1), \\ x_3 &= x_2^2 + 2x_2x_3. \end{aligned} \tag{37}$$

So, by letting $x_1 = b$ (any $b \in [0,1]$), the second equation of the system (37) can be written as follows

$$x_2^2 - (3-2b)x_2 + (1-b) = 0. \tag{38}$$

The solutions of the last equation are $x_2^\pm = (3 - 2b \pm \sqrt{4b^2 - 8b + 5})/2$. One can check that the only solution $x_2 = (3 - 2b - \sqrt{4b^2 - 8b + 5})/2 = B(b)$ belongs to $[0,1]$. Therefore, one has that $\mathrm{Fix}(V_4) = \{(b, B(b), 1 - B(b))\}_{b\in[0,1]}$ whenever $a = 1/2$.

(ii) Let $a \neq 1/2$. Now, we are going to show that the operator $V_4$ given by (28) does not have any order periodic points in the set $S^2 \setminus \Gamma_1$, where $\Gamma_1 = \{x \in S^2 : x_1 = 0\}$. In fact, since the function $f_a(x) = x^2 + 2ax(1-x)$ is increasing (due to Proposition 7(ii)), the first coordinate of $V_4$ increases along the iteration of $V_4$ in the set $S^2 \setminus \Gamma_1$. This means that $V_4$ does not have any order periodic points in set $S^2 \setminus \Gamma_1$. Therefore, it is enough to find periodic points of $V_4$ in $\Gamma_1$. In this case, in order to find 2-periodic points, we have to solve the following system of equations:

$$\begin{aligned} x_1 &= 0, \\ x_2 &= \left(1 - (1-x_2)^2\right)^2, \\ x_3 &= 1 - \left(1 - x_3^2\right)^2. \end{aligned} \tag{39}$$

The solutions of the second equation of the last system are $0, 1, (3 \pm \sqrt{5})/2$. Hence, we have that $\mathrm{Per}(V_4) = \{e_2, e_3\}$ whenever $a \neq 1/2$.

Let $a = 1/2$. In order to find 2-periodic points of $V_4$, we should solve the following system of equations:

$$\begin{aligned} x_1 &= x_1, \\ x_2 &= x_2^2(1 - x_1 + x_3)^2 + x_1(1-x_1), \\ x_3 &= \left(x_3^2 + x_1(1-x_1)\right)(1 - x_1 + x_2(1 - x_1 + x_3)). \end{aligned} \tag{40}$$

By letting $x_1 = c$, where $c \in [0,1]$, the second equation of system (40) reduces to the following equation:

$$x_2^2(2 - 2c - x_2)^2 + c(1-c) = x_2. \tag{41}$$

One can easily check that the solutions of the last equation which belong to $[0,1]$ are only $B(c)$ and $C_\pm(c)$ whenever $c \in [0, (2-\sqrt{3})/2)$ and $B(c)$ whenever $c \in [(2-\sqrt{3})/2, 1]$. Consequently, we get that $\mathrm{Per}_2(V_4) = \{(c, C_\pm(c), 1 - C_\pm(c))\}_{c\in[0,(2-\sqrt{3})/2)}$.

(iii) Let $1/2 < a \le 1$ and $x_1^{(0)} = 0$. In this case, the second coordinate of $V_4$ has the form $x_2' = h(x_2)$, where $h(x_2) = (1-x_2)^2$. It is clear that function $h$ is decreasing on $[0,1]$. This yields that function $h^{(2)}$ is increasing on $[0,1]$, As we already discussed in (i) and (ii) that $\mathrm{Fix}(h) \cap [0,1] = \{(3-\sqrt{5})/2\}$ and $\mathrm{Fix}(h^{(2)}) \cap [0,1] = \{0, (3-\sqrt{5})/2, 1\}$. This means that the sets $[0, (3-\sqrt{5})/2]$ and $[(3-\sqrt{5})/2, 1]$ are invariant function $h^{(2)}$. We immediately find (see the above discussion (ii)) that $h^{(2)}(x_2) > x_2$ whenever $x_2 > (3-\sqrt{5})/2$ and $h^{(2)}(x_2) < x_2$ whenever $x_2 < (3-\sqrt{5})/2$. Consequently, one has that if $x_2^{(0)} \in [0, (3-\sqrt{5})/2)$, then $\omega_{h^{(2)}}(x_2^{(0)}) = \{0\}$ and if $x_2^{(0)} \in ((3-\sqrt{5})/2, 1]$, then $\omega_{h^{(2)}}(x_2^{(0)}) = \{1\}$. On the other hand, we have that

$$V_4^{(n)}\left(x^{(0)}\right) = \begin{cases} \left(0, h^{(2k)}\left(x_2^{(0)}\right), 1 - h^{(2k)}\left(x_2^{(0)}\right)\right) & \text{if } n = 2k, \\ \left(0, h^{(2k)}\left(h\left(x_2^{(0)}\right)\right), 1 - h^{(2k)}\left(h\left(x_2^{(0)}\right)\right)\right) & \text{if } n = 2k+1. \end{cases} \tag{42}$$

Therefore, we obtain that $\omega_{V_4}(x^{(0)}) = \{e_2, e_3\}$ if $x_1^{(0)} = 0$.

Let $1/2 < a \leq 1$ and $0 < x_1^{(0)} < 1$. In this case, it is clear that $x_1^{(n)} = f_a^{(n)}(x_1^{(0)})$. Therefore, due to Proposition 7(iv), we have that $\lim_{n\to\infty} x_1^{(n)} = 1$. This means that $\omega_{V_4}(x^{(0)}) \subset \{e_1\}$. Since $\omega_{V_4}(x^{(0)}) \neq \emptyset$, one has that $\omega_{V_4}(x^{(0)}) = \{e_1\}$.

(iv) Let $a = 1/2$. Then the operator $V_4$ takes the following form:

$$\begin{aligned} x_1' &= x_1, \\ x_2' &= x_3^2 + x_1(1-x_1), \\ x_3' &= x_2(1 - x_1 + x_3). \end{aligned} \tag{43}$$

It is clear that $L_c = \{x \in S^2 : x_1 = c\}$ is invariant under $V_4$ where $c \in [0,1]$. Therefore, we shall study the dynamics of $V_4$ over $L_c$.

Let $x_1^{(0)} = c \in [0, (2-\sqrt{3})/2)$ be a fixed number. Let us consider function $h_c : [0, 1-c] \to [0, 1-c]$,

$$h_c(x_2) = (1 - c - x_2)^2 + c(1-c). \tag{44}$$

One can show that the function $h_c$ is decreasing on $[0, 1-c]$. This yields that the function $h_c^{(2)}$ is increasing. It follows from the discussion presented above (see (ii)) that $\mathrm{Fix}(h_c) \cap [0, 1-c] = \{B(c)\}$ and $\mathrm{Fix}(h_c^{(2)}) \cap [0, 1-c] = \{B(c), C_\pm(c)\}$ where $C_-(c) < B(c) < C_+(c)$. Moreover, one has that

$$\begin{aligned} h_c[0, C_-(c)] &\subset [C_+(c), 1-c], \\ h_c[C_-(c), B(c)] &\subset [B(c), C_+(c)], \\ h_c[B(c), C_+(c)] &\subset [C_-(c), B(c)], \\ h_c[C_+(c), 1-c] &\subset [0, C_-(c)], \end{aligned} \tag{45}$$

Therefore, the sets $[0, C_-(c)]$, $[C_-(c), B(c)]$, $[B(c), C_+(c)]$, and $[C_+(c), 1-c]$ are invariant under function $h_c^{(2)}$. Simple calculations show that

$$\begin{aligned} h_c^{(2)}(x_2) &> x_2, \quad \forall x_2 \in [0, C_-(c)) \cup (B(c), C_+(c)), \\ h_c^{(2)}(x_2) &< x_2, \quad \forall x_2 \in (C_-(c), B(c)) \cup (C_+(c), 1-c]. \end{aligned} \tag{46}$$

Consequently, we get that

$$\begin{aligned} \omega_{h_c^{(2)}}(x_2^{(0)}) &= \{C_-(c)\}, \quad \forall x_2^{(0)} \in [0, B(c)), \\ \omega_{h_c^{(2)}}(x_2^{(0)}) &= \{C_+(c)\}, \quad \forall x_2^{(0)} \in (B(c), 1-c]. \end{aligned} \tag{47}$$

On the other hand, we have that

$$V_4^{(n)}(x^{(0)}) = \begin{cases} \left(c, h_c^{(2k)}(x_2^{(0)}), 1 - h_c^{(2k)}(x_2^{(0)})\right) & \text{if } n = 2k, \\ \left(c, h_c^{(2k)}(h_c(x_2^{(0)})), 1 - h_c^{(2k)}(h_c(x_2^{(0)}))\right) & \text{if } n = 2k+1. \end{cases} \tag{48}$$

Therefore, we obtain that

$$\omega_{V_4}(x^{(0)}) = \{(c, C_\pm(c), 1 - C_\pm(c))\} \quad \text{if } x_1^{(0)} = c \in \left[0, \frac{2-\sqrt{3}}{2}\right). \tag{49}$$

In the same manner, one can show that $\omega_{V_4}(x^{(0)}) = \{(c, B(c), 1 - B(c))\}$ whenever $x_1^{(0)} = c \in [(2-\sqrt{3})/2, 1]$.

This completes the proof. □

## 6. Dynamics of $\xi^{(s)}$-QSO from Class $K_{19}$

We are going to study dynamics of a $\xi^{(s)}$-QSO $V_{28} : S^2 \to S^2$ taken from $K_{19}$:

$$V_{28} : \begin{cases} x_1' = x_1^2 + 2x_1(1-x_1), \\ x_2' = x_3^2 + 2a x_2 x_3, \\ x_3' = x_2^2 + 2(1-a) x_2 x_3, \end{cases} \tag{50}$$

where $0 \leq a \leq 1$. One can see that this operator is a permuted Volterra-QSO. The behavior of this operator was not studied in [21, 22, 34]. It is worth mentioning that $V_{28}$ is a permutation of $V_{25}$.

Let $A = (3 - 2a - \sqrt{4 + (2a-1)^2})/2(1-2a)$ for any $a \neq 1/2$. Then $0 \leq A \leq 1$. In fact, one has that

$$0 \leq A = \frac{2}{2 + (1-2a) + \sqrt{4 + (2a-1)^2}} \leq 1. \tag{51}$$

**Theorem 10.** *Let $V_{28} : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by* (50) *and let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_{28}) \cup \mathrm{Per}_2(V_{28})$ be an initial point. Then the following statements hold true.*

(i) *One has that*

$$\mathrm{Fix}(V_{28}) = \begin{cases} \{e_1, (0, A, 1-A)\} & \text{if } a \neq \dfrac{1}{2}, \\ \left\{e_1, \left(0, \dfrac{1}{2}, \dfrac{1}{2}\right)\right\} & \text{if } a = \dfrac{1}{2}. \end{cases} \tag{52}$$

(ii) *One has that*

$$\mathrm{Per}_2(V_{28}) = \begin{cases} \{e_2, e_3\} & \text{if } a \neq \dfrac{1}{2}, \\ \Gamma_1 \setminus \left\{\left(0, \dfrac{1}{2}, \dfrac{1}{2}\right)\right\} & \text{if } a = \dfrac{1}{2}. \end{cases} \tag{53}$$

(iii) *If $a \neq 1/2$, then*

$$\omega_{V_{28}}(x^{(0)}) = \begin{cases} \{e_2, e_3\} & \text{if } x_1^{(0)} = 0, \\ \{e_1\} & \text{if } x_1^{(0)} \neq 0. \end{cases} \tag{54}$$

(iv) *If $a = 1/2$, then $\omega_{V_{28}}(x^{(0)}) = \{e_1\}$.*

*Proof.* Let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_{28}) \cup \mathrm{Per}_2(V_{28})$ be an initial point and let $\{x^{(n)}\}_{n=0}^{\infty}$ be a trajectory of $V_{28}$ starting from point $x^{(0)}$.

(i) In order to find fixed points of $V_{28}$, we need to solve the following system of equations:

$$\begin{aligned} x_1 &= x_1^2 + 2x_1(1-x_1), \\ x_2 &= x_3^2 + 2a x_2 x_3, \\ x_3 &= x_2^2 + 2(1-a) x_2 x_3. \end{aligned} \tag{55}$$

From the first equation of (55), one can find that $x_1 = 0$ or $x_1 = 1$. If $x_1 = 1$, then $x_2 = x_3 = 0$. If $x_1 = 0$, then $x_2 + x_3 = 1$. So, the second equation of (55) becomes as follows:

$$(1 - x_2)^2 + 2ax_2(1 - x_2) = x_2. \tag{56}$$

Let $a \neq 1/2$. Then, one can find that solutions of (56) are $x_2^{\pm} = (3 - 2a \pm \sqrt{4 + (2a-1)^2})/2(1-2a)$. We can verify that the only solution which lies in the interval $[0,1]$ is $x_2 = (3 - 2a - \sqrt{4 + (2a-1)^2})/2(1-2a) = A$. Therefore, we have $x_3 = 1 - A = (-1 - 2a + \sqrt{4 + (2a-1)^2})/2(1-2a)$. Hence, $\mathrm{Fix}(V_{28}) = \{e_1, (0, A, 1 - A)\}$ whenever $a \neq 1/2$.

Let $a = 1/2$. Then, (56) has a solution $x_2 = 1/2$. This yields that $x_3 = 1/2$. Therefore, one has that $\mathrm{Fix}(V_{28}) = \{e_1, (0, 1/2, 1/2)\}$ whenever $a = 1/2$.

(ii) It is clear that $V_{28}$ does not have any order periodic points in $S^2 \setminus \Gamma_1$ (see Proposition 7(ii)), where $\Gamma_1 = \{x \in S^2 : x_1 = 0\}$. So, any order periodic points of $V_{28}$ lie on $\Gamma_1$ (if any). In order to find 2-periodic points of $V_{28}$, we have to solve the equation $V_{28}^2(x) = x$ with the condition $x_1 = 0$. Then, by taking into account $x_2 + x_3 = 1$, we may get the following equation:

$$\begin{aligned}&\left[x_2^2 + 2(1-a)x_2(1-x_2)\right]^2 \\ &\quad + 2a\left[(1-x_2)^2 + 2ax_2(1-x_2)\right] \\ &\quad \times \left[x_2^2 + 2(1-a)x_2(1-x_2)\right] = x_2.\end{aligned} \tag{57}$$

Let $a \neq 1/2$. Then, the last equation has the solutions $\{0, 1, \pm A\}$. So, 2-periodic points of $V_{28}$ are only $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

Let $a = 1/2$. Then, the equation given above becomes an identity $x_2 = x_2$. This means that all points of the edge $\Gamma_1$ except $(0, 1/2, 1/2)$ are 2-periodic points.

(iii) Let $a \neq 1/2$. It is clear that the edge $\Gamma_1$ is invariant under $V_{28}$. We want to study the behavior of $V_{28}$ over this line. In this case, $V_{28}|_{\Gamma_1}$ takes the following form:

$$V_{28}|_{\Gamma_1} : \begin{cases} x_1' = 0, \\ x_2' = x_3^2 + 2ax_2x_3, \\ x_3' = x_2^2 + 2(1-a)x_2x_3. \end{cases} \tag{58}$$

Let us consider the function $g_a(x_2) = (1-x_2)^2 + 2ax_2(1-x_2)$, where $a \neq 1/2$. One can easily check that $g_a$ is decreasing on $[0,1]$. This yields that $g_a^{(2)}$ is increasing on $[0,1]$, as we already discussed that $\mathrm{Fix}(g_a) \cap [0,1] = \{A\}$ and $\mathrm{Fix}(g_a^{(2)}) \cap [0,1] = \{0, A, 1\}$. This means that sets $[0, A]$ and $[A, 1]$ are invariant under function $g_a^{(2)}$. We immediately find (see the above discussion (ii)) that $g_a^{(2)}(x_2) > x_2$ whenever $x_2 > A$ and $g_a^{(2)}(x_2) < x_2$ whenever $x_2 < A$. Consequently, one has that if $x_2^{(0)} \in [0, A)$, then $\omega_{g_a^{(2)}}(x_2^{(0)}) = \{0\}$ and if $x_2^{(0)} \in (A, 1]$, then $\omega_{g_a^{(2)}}(x_2^{(0)}) = \{1\}$. On the other hand, we have that

$$\begin{aligned}&V_{28}^{(n)}(x^{(0)}) \\ &= \begin{cases} \left(0, g_a^{(2k)}(x_2^{(0)}), 1 - g_a^{(2k)}(x_2^{(0)})\right) & \text{if } n = 2k, \\ \left(0, g_a^{(2k)}(g_a(x_2^{(0)})), 1 - g_a^{(2k)}(g_a(x_2^{(0)}))\right) & \text{if } n = 2k+1. \end{cases}\end{aligned} \tag{59}$$

Therefore, we obtain that $\omega_{V_{28}}(x^{(0)}) = \{e_2, e_3\}$ if $x_1^{(0)} = 0$.

Let $x_1^{(0)} \neq 0$. It is clear that $x_1^{(n)} = f_1^{(n)}(x_1^{(0)})$. Therefore, due to Proposition 7(iv), we have that $\lim_{n\to\infty} x_1^{(n)} = 1$. This means that $\omega_{V_{28}}(x^{(0)}) \subset \{e_1\}$. Since $\omega_{V_{28}}(x^{(0)})$ is not empty, we obtain that $\omega_{V_{28}}(x^{(0)}) = \{e_1\}$.

(iv) Let $a = 1/2$. Since $x^{(0)} \notin \mathrm{Fix}(V_{28}) \bigcup \mathrm{Per}_2(V_{28})$, we have that $x_1^{(0)} > 0$. Then, due to Proposition 7(iv), we again have that $x_1^{(n)} = f_1^{(n)}(x_1^{(0)})$ and $\lim_{n\to\infty} x_1^{(n)} = 1$. Since $\omega_{V_{28}}(x^{(0)})$ is not empty, we obtain that $\omega_{V_{28}}(x^{(0)}) = \{e_1\}$.

This completes the proof. □

## 7. Dynamics of $\xi^{(s)}$-QSO from the Class $K_{17}$

We are going to highlight the dynamics of a $\xi^{(s)}$-QSO $V_{25} : S^2 \to S^2$ taken from $K_{17}$:

$$V_{25} : \begin{cases} x_1' = x_1^2 + 2x_1(1-x_1), \\ x_2' = x_2^2 + 2ax_2x_3, \\ x_3' = x_3^2 + 2(1-a)x_2x_3, \end{cases} \tag{60}$$

where $0 \le a \le 1$. One can immediately see that operator (60) is a Volterra-QSO. The dynamics of such kinds of operators have been studied in [17–19]. By means of the results of the mentioned papers, one can formulate the following.

**Theorem 11.** *Let $V_{25} : S^2 \to S^2$ be a $\xi^{(s)}$-QSO given by (60) and let $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) \notin \mathrm{Fix}(V_{25})$ be an initial point. Then, the following statements hold true.*

(i) *One has that*

$$\mathrm{Fix}(V_{25}) = \begin{cases} \{e_1, e_2, e_3\} & \text{if } a \neq \frac{1}{2}, \\ \Gamma_1 & \text{if } a = \frac{1}{2}. \end{cases} \tag{61}$$

(ii) *If $0 \le a < 1/2$, then*

$$\omega_{V_{25}}(x^{(0)}) = \begin{cases} \{e_3\} & \text{if } x_1^{(0)} = 0, \\ \{e_1\} & \text{if } x_1^{(0)} \neq 0. \end{cases} \tag{62}$$

(iii) *If $1/2 < a \le 1$, then*

$$\omega_{V_{25}}(x^{(0)}) = \begin{cases} \{e_2\} & \text{if } x_1^{(0)} = 0, \\ \{e_1\} & \text{if } x_1^{(0)} \neq 0. \end{cases} \tag{63}$$

(iv) *If $a = 1/2$, then $\omega_{V_{25}}(x^{(0)}) = \{e_1\}$.*

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## References

[1] S. Bernstein, "Solution of a mathematical problem connected with the theory of heredity," *Annals of Mathematical Statistics*, vol. 13, pp. 53–61, 1942.

[2] J. Hofbauer, V. Hutson, and W. Jansen, "Coexistence for systems governed by difference equations of Lotka-Volterra type," *Journal of Mathematical Biology*, vol. 25, no. 5, pp. 553–570, 1987.

[3] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*, Mathematical Aspects of Selection, Cambridge University Press, Cambridge, UK, 1988.

[4] S.-T. Li, D.-M. Li, and G.-K. Qu, "On stability and chaos of discrete population model for a singlespecies with harvesting," *Harbin University of Science and Technology*, vol. 6, article 021, 2006.

[5] A. J. Lotka, "Undamped oscillations derived from the law of mass action," *Journal of the American Chemical Society*, vol. 42, pp. 1595–1599, 1920.

[6] Y. I. Lyubich, *Mathematical Structures in Population Genetics*, Springer, Berlin, Germany, 1992.

[7] V. Volterra, *Lois de fluctuation de la population de plusieurs espèces coexistant dans le même milieu*, Association Française pour l'Avancement des Sciences, Lyon, France, 1926.

[8] M. Plank and V. Losert, "Hamiltonian structures for the n-dimensional Lotka-Volterra equations," *Journal of Mathematical Physics*, vol. 36, no. 7, pp. 3520–3534, 1995.

[9] F. E. Udwadia and N. Raju, "Some global properties of a pair of coupled maps: quasi-symmetry, periodicity, and synchronicity," *Physica D*, vol. 111, no. 1–4, pp. 16–26, 1998.

[10] H. Kesten, "Quadratic transformations: a model for population growth. I, II," *Advances in Applied Probability*, vol. 2, pp. 1–82, 179–228, 1970.

[11] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.

[12] R. Ganikhodzhaev, F. Mukhamedov, and U. Rozikov, "Quadratic stochastic operators and processes: results and open problems," *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, vol. 14, no. 2, pp. 279–335, 2011.

[13] R. N. Ganikhodzhaev, "A family of quadratic stochastic operators that act in $S^2$," *Doklady Akademii Nauk UzSSR*, no. 1, pp. 3–5, 1989 (Russian).

[14] M. Saburov, "Some strange properties of quadratic stochastic Volterra operators," *World Applied Sciences Journal*, vol. 21, pp. 94–97, 2013.

[15] P. R. Stein and S. M. Ulam, *Non-Linear Transformation Studies on Electronic Computers*, Los Alamos National Laboratory, Los Alamos, NM, USA, 1962.

[16] M. I. Zakharevich, "The behavior of trajectories and the ergodic hypothesis for quadratic mappings of a simplex," *Russian Mathematical Surveys*, vol. 33, pp. 207–208, 1978.

[17] R. N. Ganikhodzhaev, "Quadratic stochastic operators, Lyapunov functions and tournaments," *Russian Academy of Sciences. Sbornik Mathematics*, vol. 76, no. 2, pp. 489–506, 1993.

[18] R. N. Ganikhodzhaev, "A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems," *Mathematical Notes*, vol. 56, no. 5, pp. 1125–1131, 1994.

[19] R. N. Ganikhodzhaev and D. B. Èshmamatova, "Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories," *Vladikavkaz Mathematical Journal*, vol. 8, no. 2, pp. 12–28, 2006 (Russian).

[20] R. D. Jenks, "Quadratic differential systems for interactive population models," *Journal of Differential Equations*, vol. 5, pp. 497–514, 1969.

[21] R. N. Ganikhodzhaev and A. M. Dzhurabaev, "The set of equilibrium states of quadratic stochastic operators of type $V_\pi$," *Uzbek Mathematical Journal*, no. 3, pp. 23–27, 1998 (Russian).

[22] R. N. Ganikhodzhaev and R. È. Abdirakhmanova, "Description of quadratic automorphisms of a finite-dimensional simplex," *Uzbek Mathematical Journal*, no. 1, pp. 7–16, 2002 (Russian).

[23] N. N. Ganikhodzhaev and R. T. Mukhitdinov, "On a class of measures corresponding to quadratic operators," *Doklady Akademii Nauk Respubliki Uzbekistan*, no. 3, pp. 3–6, 1995 (Russian).

[24] U. A. Rozikov and A. Zada, "On $\ell$-Volterra quadratic stochastic operators," *International Journal of Biomathematics*, vol. 3, no. 2, pp. 143–159, 2010.

[25] U. A. Rozikov and A. Zada, "$\ell$-Volterra quadratic stochastic operators: Lyapunov functions, trajectories," *Applied Mathematics & Information Sciences*, vol. 6, no. 2, pp. 329–335, 2012.

[26] U. A. Rozikov and U. U. Zhamilov, "On the dynamics of strictly non-Volterra quadratic stochastic operators on a two-dimensional simplex," *Sbornik*, vol. 200, no. 9, pp. 81–94, 2009.

[27] U. A. Rozikov and U. U. Zhamilov, "$F$-quadratic stochastic operators," *Mathematical Notes*, vol. 83, no. 3-4, pp. 554–559, 2008.

[28] N. N. Ganikhodzhaev and U. A. Rozikov, "On quadratic stochastic operators generated by Gibbs distributions," *Regular & Chaotic Dynamics*, vol. 11, no. 4, pp. 467–473, 2006.

[29] N. N. Ganikhodjaev, "An application of the theory of Gibbs distributions to mathematical genetics," *Doklady Mathematics*, vol. 61, pp. 321–323, 2000.

[30] U. A. Rozikov and N. B. Shamsiddinov, "On non-Volterra quadratic stochastic operators generated by a product measure," *Stochastic Analysis and Applications*, vol. 27, no. 2, pp. 353–362, 2009.

[31] F. Mukhamedov and A. H. M. Jamal, "On $\xi^s$-quadratic stochastic operators in 2-dimensional simplex," in *Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and Its Application (ICMSA '10)*, pp. 159–172, Universiti Tunku Abdul Rahman, Kuala Lumpur, Malaysia, November 2010.

[32] F. Mukhamedov, M. Saburov, and A. H. M. Jamal, "On dynamics of $\xi^s$-quadratic stochastic operators," *International Journal of Modern Physics: Conference Series*, vol. 9, pp. 299–307, 2012.

[33] A. Dohtani, "Occurrence of chaos in higher-dimensional discrete-time systems," *SIAM Journal on Applied Mathematics*, vol. 52, no. 6, pp. 1707–1721, 1992.

[34] R. N. Ganikhodzhaev and A. Z. Karimov, "Mappings generated by a cyclic permutation of the components of Volterra quadratic stochastic operators whose coefficients are equal in absolute magnitude," *Uzbek Mathematical Journal*, no. 4, pp. 16–21, 2000 (Russian).

[35] F. Mukhamedov, I. Qaralleh, and W. N. F. A. W. Rozali, "On $\xi^a$-quadratic stochastic operators on 2-D simplex," accepted in *Sains Malaysiana*.