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Research Article

The Shared Set and Uniqueness of Meromorphic Functions on Annuli

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The purpose of this paper is to deal with the shared set and uniqueness of meromorphic functions on annulus. The set of this paper is different from the set of the paper by Cao and Deng, and our theorems are improvement of the results given by Cao and Deng.

1. Introduction

In 1929, Nevanlinna [1] first investigated the uniqueness of meromorphic functions in the whole complex plane and obtained the well-known result—5 *IM* theorem of two meromorphic functions sharing five distinct values.

After his theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane, the unit disc; and angular domain (see [2–8]).

The notations of the Nevanlinna theory such as $T(r, f)$, $m(r, f)$, and $N(r, f)$ were usually used in those papers (see [5, 9, 10]). We use $\mathbb{C}$ to denote the open complex plane, $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ to denote the extended complex plane, and $\mathbb{X}$ to denote the subset of $\mathbb{C}$. Let S be a set of distinct elements in $\widetilde{\mathbb{C}}$ and $\mathbb{X} \subseteq \mathbb{C}$. Define

$$E^{\mathbb{X}}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{X} \mid f_a(z) = 0, \text{counting multiplicities}\},$$
$$\overline{E}^{\mathbb{X}}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{X} \mid f_a(z) = 0, \text{ignoring multiplicities}\}, \tag{1}$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = 1/f(z)$. We also define

$$\overline{E}_1^{\mathbb{X}}(S, f) = \bigcup_{a \in S} \{z \in \mathbb{X} : \text{all the simple zeros of } f_a(z)\}. \tag{2}$$

For $a \in \widetilde{\mathbb{C}}$, we say that two meromorphic functions f and g share the value $aCM(IM)$ in $\mathbb{X}$ (or $\mathbb{C}$), if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (ignoring multiplicities) in $\mathbb{X}$ (or $\mathbb{C}$).

The whole complex plane $\mathbb{C}$, the unit disc, and angular domain all can be regarded as simply connected regions those results of the uniqueness of shared values and sets in the above cases can also be regarded as the uniqueness of meromorphic functions in simply connected regions.

Thus, it raises naturally an interesting subject on the uniqueness of the meromorphic functions in the multiply connected region.

The main purpose of this paper is to study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. From the doubly connected mapping theorem [11], we can get that each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. There are two cases: (1) $r = 0$ and $R = +\infty$ and (2) $0 < r < R < +\infty$; for case (2) the homothety $z \mapsto z/\sqrt{rR}$ reduces the given domain to the annulus $\{z : 1/R_0 < |z| < R_0\}$, where $R_0 = \sqrt{R/r}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto 1/z$ in two cases. The basic notions of the Nevanlinna theory on annuli will be showed in the next section.

Recently, there are some results on the Nevanlinna Theory of meromorphic functions on the annulus (see [12–19]). In 2005, Khrystiyanyn and Kondratyuk [13, 14] proposed the

Nevanlinna theory for meromorphic functions on annuli (see also [20]). Lund and Ye [16] in 2009 studied meromorphic functions on the annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geq 0$ and $R_2 \leq \infty$. However, there are few results about the uniqueness of meromorphic functions on the annulus. In 2009 and 2011, Cao et al. [21, 22] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna's famous five-value theorem as follows.

Theorem 1 ([22, Theorem 3.2] or [21, Corollary 3.3]). *Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a_j ($j = 1, 2, 3, 4, 5$) be five distinct complex numbers in $\widetilde{\mathbb{C}}$. If f_1, f_2 share $a_j IM$ for $j = 1, 2, 3, 4, 5$, then $f_1(z) \equiv f_2(z)$.*

Remark 2. For the case $R_0 = +\infty$, the assertion was proved by Kondratyuk and Laine [20].

In 2012, Cao and Deng [23] investigated the uniqueness of two meromorphic functions in $\mathbb{A}$ sharing three or two finite sets; we obtain the following theorems which are an analog of results on $\mathbb{C}$ according to Lin and Yi [24].

Theorem 3. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. Put $S_1 = \{0\}$, $S_2 = \{\infty\}$, and $S_3 = \{w : P(w) = 0\}$, where*

$$\begin{aligned} P(w) = {} & aw^n + n(n-1)w^2 + 2n(n-2)bw \\ & - (n-1)(n-2)b^2, \end{aligned} \tag{3}$$

where $n \geq 5$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 1, 2$. If $\overline{E}(S_2, f) = \overline{E}(S_2, g)$ and $E(S_j, f) = E(S_j, g)$ ($j = 1, 3$), then $f \equiv g$.

Theorem 4. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. Put $S_1 = \{\infty\}$ and $S_2 = \{w : P(w) = 0\}$, where $P(w)$ is stated as in Theorem 3, and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$, $n \geq 8$ is an integer. If $\overline{E}(S_1, f) = \overline{E}(S_1, g)$ and $E(S_2, f) = E(S_2, g)$, then $f \equiv g$.*

In this paper, we will focus on the uniqueness problem of shared set of meromorphic functions on the annuli. In fact, we will study the uniqueness of meromorphic functions on the annuli sharing one set $S = \{w \in \mathbb{A} : P_1(w) = 0\}$, where

$$\begin{aligned} P_1(w) = {} & \frac{(n-1)(n-2)}{2} w^n - n(n-2) w^{n-1} \\ & + \frac{n(n-1)}{2} w^{n-2} - c, \end{aligned} \tag{4}$$

and c is a complex number satisfying $c \neq 0, 1$ and we obtain the following results.

Theorem 5. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. If $E^{\mathbb{A}}(S, f) = E^{\mathbb{A}}(S, g)$ and n is an integer ≥ 11, then $f \equiv g$.*

A set S is called a unique range set for meromorphic functions on annulus $\mathbb{A}$, if, for any two nonconstant meromorphic functions f and g, the condition $E^{\mathbb{A}}(S, f) = E^{\mathbb{A}}(S, g)$ implies $f \equiv g$. We denote by $\sharp S$ the cardinality of a set S. Thus, from Theorem 5, we can get the following corollary.

Corollary 6. *There exists one finite set S with $\sharp S = 7$, such that any two admissible meromorphic functions f and g on $\mathbb{A}$ must be identical if $E^{\mathbb{A}}(S, f) = E^{\mathbb{A}}(S, g)$.*

Theorem 7. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. If n is an integer ≥ 7, $E^{\mathbb{A}}(S, f) = E^{\mathbb{A}}(S, g)$, and $\Theta_0(\infty, f) > 3/4$, $\Theta_0(\infty, g) > 3/4$, then $f \equiv g$.*

Corollary 8. *There exists one finite set S with $\sharp S = 7$, such that any two admissible analytic functions f and g on $\mathbb{A}$ must be identical if $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$.*

Theorem 9. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. If $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$ and n is an integer ≥ 15, then $f \equiv g$.*

A set S is called a unique range set, with weight 1 for meromorphic functions on annulus $\mathbb{A}$, if for any two nonconstant meromorphic functions f and g, the condition $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$ implies $f \equiv g$. Thus, from Theorem 9, we can get the following corollary.

Corollary 10. *There exists one finite set S with $\sharp S = 15$, such that any two admissible meromorphic functions f and g on $\mathbb{A}$ must be identical if $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$.*

Theorem 11. *Let f and g be two admissible meromorphic functions in the annulus $\mathbb{A}$. Let n be an integer ≥ 9 and $S = \{w \in \mathbb{A} : P_1(w) = 0\}$, where $P_1(w)$ and c are stated as in Theorem 5. If $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$ and $\Theta_0(\infty, f) > 5/6$, $\Theta_0(\infty, g) > 5/6$, then $f \equiv g$.*

From Theorem 11, we can get the corollary as follows.

Corollary 12. *There exists one finite set S with $\sharp S = 9$, such that any two admissible analytic functions f and g on $\mathbb{A}$ must be identical if $E_1^{\mathbb{A}}(S, f) = E_1^{\mathbb{A}}(S, g)$.*

2. Preliminaries and Some Lemmas

Letting f be a meromorphic function on whole plane $\mathbb{C}$, the classical notations of the Nevanlinna theory are denoted as follows:

$$\begin{aligned} N(R, f) &= \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R, \\ m(R, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(Re^{i\theta}\right) \right| d\theta, \\ T(R, f) &= N(R, f) + m(R, f), \end{aligned} \tag{5}$$

where $\log^+ x = \max\{\log x, 0\}$ and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \leq t\}$.

Letting f be a meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$, the notations of the Nevanlinna theory on annuli had been introduced in [13, 20], such as $N_0(R, f), m_0(R, f), T_0(R, f), \ldots$. In addition, we define

$$\Theta_0(\infty, f) = 1 - \limsup_{R\to\infty} \frac{\overline{N}_0(R, f)}{T_0(R, f)}. \quad (6)$$

We also use $\overline{n}_1^{k)}(t, 1/(f-a))$ (or $\overline{n}_1^{(k}(t, 1/(f-a))$) to denote the counting function of poles of the function $1/(f-a)$ with multiplicities $\leq k$ (or $> k$) in $\{z : t < |z| \leq 1\}$, with each point being counted only once. Similarly, we have the notations $\overline{N}_1^{k)}(t, f), \overline{N}_1^{(k}(t, f), \overline{N}_2^{k)}(t, f), \overline{N}_2^{(k}(t, f), \overline{N}_0^{k)}(t, f), \overline{N}_0^{(k}(t, f)$.

For a nonconstant meromorphic function f on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$, the following properties will be used in this paper (see [13]):

(i)

$$T_0(R, f) = T_0\left(R, \frac{1}{f}\right), \quad (7)$$

(ii)

$$\begin{aligned} &\max\left\{T_0(R, f_1 \cdot f_2), T_0\left(R, \frac{f_1}{f_2}\right), T_0(R, f_1 + f_2)\right\} \\ &\quad \leq T_0(R, f_1) + T_0(R, f_2) + O(1), \end{aligned} \quad (8)$$

(iii)

$$T_0\left(R, \frac{1}{f-a}\right) = T_0(R, f) + O(1), \quad \text{for every fixed } a \in \mathbb{C}. \quad (9)$$

Khrystiyanyn and Kondratyuk [14] also obtained the lemma on the logarithmic derivative on the annulus $\mathbb{A}$.

Lemma 13 (see [14], lemma on the logarithmic derivative). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then,*

(i) *in the case $R_0 = +\infty$,*

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(RT_0(R, f)\right)\right), \quad (10)$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\lambda-1} dR < +\infty$;

(ii) *if $R_0 < +\infty$, then*

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right), \quad (11)$$

for $R \in (1, R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R} (dR/(R_0 - R)^{\lambda-1}) < +\infty$.

Remark 14. From [14, 20], the conclusions still hold if $m_0(R, f'/f)$ is replaced by $m_0(R, f^{(k)}/f)$, $k \in N_+$.

In 2005, the second fundamental theorem on the annulus $\mathbb{A}$ was first obtained by Khrystiyanyn and Kondratyuk [14]. Later, Cao et al. [22] introduced other forms of the second fundamental theorem on annuli as follows.

Lemma 15 ([22, Theorem 2.3], the second fundamental theorem). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then,*

$$(q-2)T_0(R, f) < \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{f - a_j}\right) + S(R, f), \quad (12)$$

where (i) in the case $R_0 = +\infty$,

$$S(R, f) = O\left(\log\left(RT_0(R, f)\right)\right), \quad (13)$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\lambda-1} dR < +\infty$.

(ii) If $R_0 < +\infty$, then

$$S(R, f) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right), \quad (14)$$

for $R \in (1, R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R} (dR/(R_0 - R)^{\lambda-1}) < +\infty$.

Remark 16. In fact, from the proof of Theorem 2.3 in [22], under the assumptions of Lemma 15, we can get the following conclusion:

$$\begin{aligned} &(q-2)T_0(R, f) \\ &\quad < \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{f - a_j}\right) - N_0^0\left(R, \frac{1}{f'}\right) + S(R, f), \end{aligned} \quad (15)$$

where $S(R, f)$ is stated as in Lemma 15 and $N_0^0(R, 1/f')$ is the counting function for the zeros of f' in $\mathbb{A}$, where f does not take one of the values a_j $(j = 1, 2, \ldots, q)$.

Definition 17. Let $f(z)$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. The function f is called an admissible meromorphic function on the annulus $\mathbb{A}$ provided that

$$\limsup_{R\to\infty} \frac{T_0(R, f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty, \quad (16)$$

or

$$\limsup_{R\to R_0} \frac{T_0(R, f)}{-\log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty, \quad (17)$$

respectively.

Thus, for an admissible meromorphic function on the annulus $\mathbb{A}$, $S(R,f) = o(T_0(R,f))$ holds for all $1 < R < R_0$ except for the set Δ_R or the set Δ'_R mentioned in Lemma 13, respectively.

The following result can be derived from the proof of Frank-Reinders' theorem in [25].

Lemma 18. *Let $n \geq 6$ and*

$$H(w) = \frac{(n-1)(n-2)}{2}w^n - n(n-2)w^{n-1} + \frac{n(n-1)}{2}w^{n-2}. \tag{18}$$

Then, $H(w)$ is a unique polynomial for admissible meromorphic functions; that is, for any two admissible meromorphic functions f and g in $\mathbb{A}$, $H(f) \equiv H(g)$ implies $f \equiv g$.

By a similar discussion to the one in [26], one can obtain a stand and Valiron-Mohoko type result in $\mathbb{A}$ as follows.

Lemma 19 (see [23]). *Let f be a nonconstant meromorphic function in $\mathbb{A}$, $Q_1(f)$ and let $Q_2(f)$ be two mutually prime polynomials in f with degree m and n, respectively. Then,*

$$T_0\left(R, \frac{Q_1(f)}{Q_2(f)}\right) = \max\{m,n\}\, T_0(R,f) + S(R,f). \tag{19}$$

Lemma 20. *Suppose f is a nonconstant meromorphic function in $\mathbb{A}$. Then,*

$$N_0\left(R, \frac{1}{f'}\right) \leq N_0\left(R, \frac{1}{f}\right) + \overline{N}_0(R,f) + S(R,f) + O(1), \tag{20}$$

where $S(R,f)$ is stated as in Lemma 15.

Proof. Since

$$m_0\left(R, \frac{1}{f}\right) \leq m_0\left(R, \frac{1}{f'}\right) + m_0\left(R, \frac{f'}{f}\right) = m_0\left(R, \frac{1}{f'}\right) + S(R,f). \tag{21}$$

Then, from properties of $T_0(R,f)$, we have

$$T_0(R,f) - N_0\left(R, \frac{1}{f}\right) \leq T_0(R,f') - N_0\left(R, \frac{1}{f'}\right) + S(r,f) + O(1); \tag{22}$$

that is,

$$N_0\left(R, \frac{1}{f'}\right) \leq T_0(R,f') - T_0(R,f) + N_0\left(R, \frac{1}{f}\right) + S(r,f) + O(1). \tag{23}$$

Since

$$\begin{aligned} T_0(R,f') &= m_0(R,f') + N_0(R,f') - 2m(1,f') \\ &\leq m_0(R,f) + m_0\left(R, \frac{f'}{f}\right) + N_0(R,f) \\ &\quad + \overline{N}_0(R,f) - 2m(1,f') \\ &\leq T_0(R,f) + \overline{N}_0(R,f) + S(R,f) + O(1). \end{aligned} \tag{24}$$

Then, from (23) and (24), we can get the conclusion of this lemma. □

Next, we will give the two main lemmas of this paper as follows.

Lemma 21. *Let F and G be admissible meromorphic functions in A satisfying $E^{\mathbb{A}}(F,0) = E^{\mathbb{A}}(G,0)$ and let $c_1, c_2, \ldots, c_q$ be q (≥ 2) distinct nonzero complex numbers. If*

$$\begin{aligned} &\limsup_{R\to\infty, R\in I}\left(\left(3\overline{N}_0(R,F) + \sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R, \frac{1}{F-c_j}\right) + \overline{N}_0\left(R, \frac{1}{F'}\right)\right) \times (T_0(R,F))^{-1}\right) < q, \\ &\limsup_{R\to\infty, R\in I}\left(\left(3\overline{N}_0(R,G) + \sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0\left(R, \frac{1}{G'}\right)\right) \times (T_0(R,G))^{-1}\right) < q, \end{aligned} \tag{25}$$

where $\overline{N}_0^{(2)}(R,\cdot) = \overline{N}_0(R,\cdot) + \overline{N}_0^{(2}(R,\cdot)$, $\overline{N}_0^{(2}(r,\cdot) = \overline{N}_0(r,\cdot) - \overline{N}_0^{1)}(r,\cdot)$, and I is some set of R of infinite linear measure, then

$$F = \frac{aG+b}{cG+d}, \tag{26}$$

where $a,b,c,d \in \mathbb{C}$ are constants with $ad - bc \neq 0$.

Proof. Set

$$H \equiv \frac{F''}{F'} - 2\frac{F'}{F} - \left(\frac{G''}{G'} - 2\frac{G'}{G}\right). \tag{27}$$

Supposing that $H \not\equiv 0$, from Lemma 13 and Remark 14, we have

$$m_0(R,H) = S(R), \tag{28}$$

where $S(R) := o\{T_0(R)\}$, $T_0(R) = \max\{T_0(R,F), T_0(R,G)\}$. Since $E^{\mathbb{A}}(F,0) = E^{\mathbb{A}}(G,0)$, and by an elementary calculation, we can conclude that if z_0 is a common simple zero of F and G in $\mathbb{A}$, then $H(z_0) = 0$. Thus, we have

$$\begin{aligned} N_0^{1)}(R) &\le N_0\left(R, \frac{1}{H}\right) \le T_0(R,H) + O(1) \\ &\le N_0(R,H) + S(R), \end{aligned} \tag{29}$$

where $N_0^{1)}(R) = N_0^{1)}(R, 1/F) = N_0^{1)}(R, 1/G)$. The poles of H in $\mathbb{A}$ can only occur at zeros of F' and G' in $\mathbb{A}$ or poles of F and G in $\mathbb{A}$. Moreover, H only has simple zeros in $\mathbb{A}$. Hence, from (29), we have

$$\begin{aligned} &N_0^{1)}(R) \\ &\quad\le \overline{N}_0(R,F) + \overline{N}_0(R,G) + \overline{N}_0^0\left(R, \frac{1}{F'}\right) + \overline{N}_0^0\left(R, \frac{1}{G'}\right) \\ &\qquad + \sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{F-c_j}\right) + \sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{G-c_j}\right) + S(R), \end{aligned} \tag{30}$$

where $\overline{N}_0^0(R, 1/F')$ is the reduced counting function for the zeros of F' in A, where F does not take one of the values $0, c_1, c_2, \ldots, c_q$.

Since

$$\begin{aligned} &\overline{N}_0\left(R, \frac{1}{F}\right) + \overline{N}_0\left(R, \frac{1}{G}\right) \\ &\quad = 2N_0^{1)}(R) + \overline{N}_0^{(2}\left(R, \frac{1}{F}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{G}\right). \end{aligned} \tag{31}$$

Then, from (30) and (31), we have

$$\begin{aligned} &\overline{N}_0\left(R, \frac{1}{F}\right) + \overline{N}_0\left(R, \frac{1}{G}\right) \\ &\quad \le 2\overline{N}_0(R,F) + 2\overline{N}_0(R,G) + 2\overline{N}_0^0\left(R, \frac{1}{F'}\right) \\ &\qquad + 2\overline{N}_0^0\left(R, \frac{1}{G'}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{F}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{G}\right) \\ &\qquad + 2\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{F-c_j}\right) + 2\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{G-c_j}\right) \\ &\qquad + S(R). \end{aligned} \tag{32}$$

From Remark 16, we have

$$\begin{aligned} qT_0(R,F) &\le \overline{N}_0(R,F) + \overline{N}_0\left(R, \frac{1}{F}\right) + \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{F-c_j}\right) \\ &\quad - N_0^0\left(R, \frac{1}{F'}\right) + S(R), \quad r \notin E, \\ qT_0(R,G) &\le \overline{N}_0(R,G) + \overline{N}_0\left(R, \frac{1}{G}\right) + \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{G-c_j}\right) \\ &\quad - N_0^0\left(R, \frac{1}{G'}\right) + S(R), \quad r \notin E, \end{aligned} \tag{33}$$

where E is a set of r of finite linear measure and it needs not to be the same at each occurrence. From (32)-(33), it follows that, for $r \notin E$,

$$\begin{aligned} &q\{T_0(R,F) + T_0(R,G)\} \\ &\quad \le 3\overline{N}_0(R,F) + 3\overline{N}_0(R,G) + \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{F-c_j}\right) \\ &\qquad + \sum_{j=1}^{q} \overline{N}_0\left(R, \frac{1}{G-c_j}\right) + 2\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{F-c_j}\right) \\ &\qquad + 2\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0^0\left(R, \frac{1}{F'}\right) \\ &\qquad + \overline{N}_0^0\left(R, \frac{1}{G'}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{F}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{G}\right) + S(R). \end{aligned} \tag{34}$$

Since

$$\begin{aligned} &\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{F-c_j}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{F}\right) + \overline{N}_0^0\left(R, \frac{1}{F'}\right) \\ &\quad = \overline{N}_0\left(R, \frac{1}{F'}\right), \\ &\sum_{j=1}^{q} \overline{N}_0^{(2}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0^{(2}\left(R, \frac{1}{G}\right) + \overline{N}_0^0\left(R, \frac{1}{G'}\right) \\ &\quad = \overline{N}_0\left(R, \frac{1}{G'}\right). \end{aligned} \tag{35}$$

From (34)-(35), we can get that, for $R \notin E$,

$$\begin{aligned} &q\{T_0(R,F) + T_0(R,G)\} \\ &\quad \le 3\overline{N}_0(R,F) + 3\overline{N}_0(R,G) + \sum_{j=1}^{q} \overline{N}_0^{(2)}\left(R, \frac{1}{F-c_j}\right) \\ &\qquad + \sum_{j=1}^{q} \overline{N}_0^{(2)}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0\left(R, \frac{1}{F'}\right) + \overline{N}_0\left(R, \frac{1}{G'}\right) \\ &\qquad + S(R). \end{aligned} \tag{36}$$

From (25) and (36), since f, g are admissible functions in $\mathbb{A}$, we can get that

$$\begin{aligned} T_0(R,F) + T_0(R,G) &\le o\{T_0(R,F) + T_0(R,G)\}, \\ &\qquad R \notin E, \; R \in I. \end{aligned} \tag{37}$$

Thus, we can get a contradiction. Therefore, $H(z) \equiv 0$; that is,

$$\frac{F''}{F'} - 2\frac{F'}{F} \equiv \frac{G''}{G'} - 2\frac{G'}{G}. \tag{38}$$

For the above equality, by integration, we can get

$$F \equiv \frac{aG+b}{cG+d}, \tag{39}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. □

Lemma 22. *Let F and G be admissble meromorphic functions in $\mathbb{A}$ satisfying $E_1^{\mathbb{A}}(F,0)=E_1^{\mathbb{A}}(G,0)$ and let $c_1,c_2,\ldots,c_q$ be q (≥ 2) distinct nonzero complex numbers. If*

$$\limsup_{R\to\infty,R\in I}\left(\left(3\overline{N}_0(R,F)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_j}\right)+\overline{N}_0\left(R,\frac{1}{F'}\right)+2\overline{N}_0^{(2}\left(R,\frac{1}{F}\right)\right)\times\left(T_0(R,F)\right)^{-1}\right)<q,$$
$$\limsup_{R\to\infty,R\in I}\left(\left(3\overline{N}_0(R,G)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R,\frac{1}{G-c_j}\right)+\overline{N}_0\left(R,\frac{1}{G'}\right)+2\overline{N}_0^{(2}\left(R,\frac{1}{G}\right)\right)\times\left(T_0(R,G)\right)^{-1}\right)<q, \tag{40}$$

where $\overline{N}_0^{(2)}(R,\cdot)$, $\overline{N}_0^{(2}(r,\cdot)$, and I are stated as in Lemma 21; then

$$F=\frac{aG+b}{cG+d}, \tag{41}$$

where $a,b,c,d\in\mathbb{C}$ are constants with $ad-bc\neq 0$.

Proof. Let H be stated as in the proof of Lemma 21, since $E_1^{\mathbb{A}}(F,0)=E_1^{\mathbb{A}}(G,0)$, we can get that

$$N_0^{1)}(R)\leq\overline{N}_0(R,F)+\overline{N}_0(R,G)+\overline{N}_0^{0}\left(R,\frac{1}{F'}\right)+\overline{N}_0^{0}\left(R,\frac{1}{G'}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{F}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{G}\right)+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(R,\frac{1}{F-c_j}\right)+\sum_{j=1}^{q}\overline{N}_0^{(2}\left(R,\frac{1}{G-c_j}\right). \tag{42}$$

Similar to the argument in Lemma 21, we can get that, for $R\notin E$

$$\begin{aligned}q\{T_0(R,F)+T_0(R,G)\}&\leq 3\overline{N}_0(R,F)+3\overline{N}_0(R,G)+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_j}\right)\\&\quad+\sum_{j=1}^{q}\overline{N}_0^{(2)}\left(R,\frac{1}{G-c_j}\right)+2\overline{N}_0^{(2}\left(R,\frac{1}{F}\right)\\&\quad+2\overline{N}_0^{(2}\left(R,\frac{1}{G}\right)+\overline{N}_0\left(R,\frac{1}{F'}\right)+\overline{N}_0\left(R,\frac{1}{G'}\right)+S(R).\end{aligned} \tag{43}$$

From (40) and (43), since f, g are admissible functions in $\mathbb{A}$, we can get that

$$T_0(R,F)+T_0(R,G)\leq o\{T_0(R,F)+T_0(R,G)\},\quad R\notin E,\ R\in I. \tag{44}$$

Thus, we can get a contradiction. Therefore, $H(z)\equiv 0$; that is,

$$\frac{F''}{F'}-2\frac{F'}{F}\equiv\frac{G''}{G'}-2\frac{G'}{G}. \tag{45}$$

From the above equality, by integration, we can get

$$F\equiv\frac{aG+b}{cG+d}, \tag{46}$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. □

3. Proofs of Theorems 5 and 7

3.1. The Proof of Theorem 5. From the definition of $P_1(w)$, we can get that $P_1(1)=1-c:=c_1\neq 0$, $P_1(0)=-c:=c_2\neq 0$, and

$$P_1'(w)=\frac{n(n-1)(n-2)}{2}(w-1)^2w^{n-3}, \tag{47}$$

$$P_1(w)-c_1=(w-1)^3Q_1(w),\quad Q_1(1)\neq 0, \tag{48}$$

$$P_1(w)-c_2=w^{n-2}Q_2(w),\quad Q_2(0)\neq 0, \tag{49}$$

where Q_1,Q_2 are polynomials of degrees $n-3$ and 2, respectively. We also see that Q_i ($i=1,2$) and P_1 have only simple zeros.

Let F and G be defined as $F=P_1(f)$ and $G=P_1(g)$. Since $E^{\mathbb{A}}(f,S)=E^{\mathbb{A}}(g,S)$, we have $E^{\mathbb{A}}(F,0)=E^{\mathbb{A}}(G,0)$. From (48) and (49), we have

$$\begin{aligned}\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_1}\right)&=\overline{N}_0\left(R,\frac{1}{F-c_1}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{F-c_1}\right)\\&\leq 2\overline{N}_0\left(R,\frac{1}{f-1}\right)+\sum_{i=1}^{n-3}N_0\left(R,\frac{1}{f-a_i}\right)\\&\leq(n-1)T_0(R,f)+S(R),\\\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_2}\right)&=\overline{N}_0\left(R,\frac{1}{F-c_2}\right)+\overline{N}_0^{(2}\left(R,\frac{1}{F-c_2}\right)\\&\leq 2\overline{N}_0\left(R,\frac{1}{f}\right)+\sum_{j=1}^{2}N_0\left(R,\frac{1}{f-b_j}\right)\\&\leq 4T_0(R,f)+S(R),\end{aligned} \tag{50}$$

where a_i ($i=1,\ldots,n-3$) and b_j ($j=1,2$) are the zeros of $Q_1(w)$ and $Q_2(w)$ in $\mathbb{A}$, respectively.

From (47), we have

$$\overline{N}_0\left(R,\frac{1}{F'}\right)\leq\overline{N}_0\left(R,\frac{1}{f}\right)+\overline{N}_0\left(R,\frac{1}{f-1}\right)+\overline{N}\left(R,\frac{1}{f'}\right). \tag{51}$$

From Lemma 19, we have $T_0(R,F) = nT_0(R,f) + S(R)$. Thus, combining (50) and (51), by Lemmas 21 and 20 and $n \ge 11$, we have

$$\begin{aligned}\limsup_{R\to\infty, R\notin E}&\left(\left(3\overline{N}_0(R,F) + \sum_{j=1}^{2}\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_j}\right)\right.\right.\\ &\left.\left.+\overline{N}_0\left(R,\frac{1}{F'}\right)\right)\times\left(T_0(R,F)\right)^{-1}\right)\\ \le \limsup_{R\to\infty, R\notin E}&\frac{4\overline{N}_0(R,f) + (n+6)T_0(R,f)}{nT_0(R,f)} < 2.\end{aligned} \tag{52}$$

Similarly, we can obtain

$$\begin{aligned}\limsup_{R\to\infty, R\notin E}&\left(\left(3\overline{N}_0(R,G) + \sum_{j=1}^{2}\overline{N}_0^{(2)}\left(R,\frac{1}{G-c_j}\right)\right.\right.\\ &\left.\left.+\overline{N}_0\left(R,\frac{1}{G'}\right)\right)\times\left(T_0(R,G)\right)^{-1}\right)\\ \le \limsup_{R\to\infty, R\notin E}&\frac{4\overline{N}_0(R,g) + (n+6)T_0(R,g)}{nT_0(R,g)} < 2.\end{aligned} \tag{53}$$

Thus, by Lemma 21, we have

$$\frac{F''}{F'} - 2\frac{F'}{F} \equiv \frac{G''}{G'} - 2\frac{G'}{G}. \tag{54}$$

From the previous equality, by integration, we can get

$$F \equiv \frac{aG+b}{cG+d}, \tag{55}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \ne 0$. Since $E^{\mathbb{A}}(f,S)$ is nonempty and $E^{\mathbb{A}}(f,S) = E^{\mathbb{A}}(g,S)$, we have $b = 0$, $a \ne 0$. Hence,

$$F \equiv \frac{aG}{cG+d} \equiv \frac{G}{AG+B}, \tag{56}$$

where $A = c/a$, $B = d/a \ne 0$.

Two cases will be considered as follows.

Case 1 ($A \ne 0$). From the definition of $P_1(w)$ and (56), we can see that every zero of $P_1(g) + B/A$ in $\mathbb{A}$ has a multiplicity of at least n. Here, three following subcases will be discussed.

Subcase 1 ($B/A = -c_1$). From (48), we have

$$P_1(g) + \frac{B}{A} = (g-1)^3(g-a_1)(g-a_2)\cdots(g-a_{n-3}), \tag{57}$$

where $a_i \ne 0, 1$ are distinct values. It follows that

$$\begin{aligned}\Theta_0(a_i,f) &= 1 - \limsup_{R\to\infty}\frac{\overline{N}_0(R,a)}{T_0(R,f)}\\ &\ge 1 - \limsup_{R\to\infty}\frac{\overline{N}_0(R,a)}{N_0(R,f)} \ge \frac{1}{2}.\end{aligned} \tag{58}$$

We can see that $P_1(g) + B/A$ has $n-2$ values satisfying the above inequality. Thus, from Lemma 15 and $n \ge 11$, we can get a contradiction.

Subcase 2 ($B/A = -c_2$). From (48), we have

$$P_1(g) + \frac{B}{A} = g^{n-2}(g-b_1)(g-b_2), \tag{59}$$

where $b_1 \ne b_2$, $b_i \ne 0, 1$ $(i = 1, 2)$. It follows that every zero of g in $\mathbb{A}$ has a multiplicity of at least 2 and every zero of $g - b_i$ $(i = 1, 2)$ in $\mathbb{A}$ has a multiplicity of at least n. Then, by Lemma 15, we have

$$\begin{aligned}T_0(R,g) &\le \overline{N}_0\left(R,\frac{1}{g}\right) + \overline{N}_0\left(R,\frac{1}{g-b_1}\right)\\ &\quad + \overline{N}_0\left(R,\frac{1}{g-b_2}\right) + S(R)\\ &\le \frac{1}{2}N_0\left(R,\frac{1}{g}\right) + \frac{1}{n}N_0\left(R,\frac{1}{g-b_1}\right)\\ &\quad + \frac{1}{n}N_0\left(R,\frac{1}{g-b_1}\right) + S(R)\\ &\le \left(\frac{1}{2}+\frac{2}{n}\right)T_0(R,g) + S(R).\end{aligned} \tag{60}$$

Since g is an admissible function in $\mathbb{A}$ and $n \ge 11$, we can get a contradiction.

Subcase 3 ($B/A \ne -c_1, -c_2$). By using the same argument as in Subcases 1 or 2, we can get a contradiction.

Case 2 ($A = 0$). If $B \ne 1$, from (56); we have $F = G/B$; that is,

$$P_1(f) = \frac{1}{B}P_1(g). \tag{61}$$

From (49) and (61), we have

$$\begin{aligned}P_1(f) - \frac{c_2}{B} &= \frac{1}{B}\left(P_1(g) - c_2\right)\\ &= \frac{1}{B}g^{n-2}(g-b_1)(g-b_2).\end{aligned} \tag{62}$$

Since $c_2/B \ne c_2$, from (47), it follows that $P_1(f) - c_2/B$ has at least $n-2$ distinct zeros $e_1, e_2, \ldots, e_{n-2}$. Then, by applying Lemma 15, we have

$$\begin{aligned}&(n-4)T_0(R,f)\\ &\quad\le \sum_{i=1}^{n-2}\overline{N}_0\left(R,\frac{1}{f-e_i}\right) + S(R)\\ &\quad\le \overline{N}_0\left(R,\frac{1}{g}\right) + \overline{N}_0\left(R,\frac{1}{g-b_1}\right) + \overline{N}_0\left(R,\frac{1}{g-b_2}\right) + S(R)\\ &\quad\le 3T_0(R,g) + S(R).\end{aligned} \tag{63}$$

By applying Lemma 21 to (61), and from (63), since $n \geq 11$ and f is an admissible function in $\mathbb{A}$, we can get a contradiction.

Thus, we have $A = 0$ and $B = 1$; that is, $P_1(f) \equiv P_1(g)$. Noting the form of $P_1(w)$; we can get that $P_1(f) \equiv P_1(g)$, that is,

$$\begin{aligned}\frac{(n-1)(n-2)}{2} f^n - n(n-2) f^{n-1} + \frac{n(n-1)}{2} f^{n-2} &\equiv \frac{(n-1)(n-2)}{2} g^n - n(n-2) g^{n-1} \\ &\quad + \frac{n(n-1)}{2} g^{n-2}.\end{aligned} \tag{64}$$

Since f, g are admissible functions in $\mathbb{A}$, then it follows by Lemma 18 that $f \equiv g$.

Therefore, the proof of Theorem 5 is completed.

3.2. The Proof of Theorem 7. Since $\Theta_0(\infty, f) > 3/4$ and $\Theta_0(\infty, g) > 3/4$, it follows that

$$\limsup_{R\to\infty} \frac{\overline{N}_0(R,f)}{T_0(R,f)} < \frac{1}{4}, \qquad \limsup_{R\to\infty} \frac{\overline{N}_0(R,g)}{T_0(R,g)} < \frac{1}{4}. \tag{65}$$

By applying (65), from (52) and (53), and since $n \geq 7$, we can get

$$\begin{aligned}&\limsup_{R\to\infty, R\notin E} \left(\left(3\overline{N}_0(R,F) + \sum_{j=1}^{2} \overline{N}_0^{(2)}\left(R, \frac{1}{F-c_j}\right) + \overline{N}_0\left(R, \frac{1}{F'}\right)\right) \times (T_0(R,F))^{-1}\right) \\ &\quad \leq \limsup_{R\to\infty, R\notin E} \frac{4\overline{N}_0(R,f) + (n+6)T_0(R,f)}{nT_0(R,f)} < 2, \\ &\limsup_{R\to\infty, R\notin E} \left(\left(3\overline{N}_0(R,G) + \sum_{j=1}^{2} \overline{N}_0^{(2)}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0\left(R, \frac{1}{G'}\right)\right) \times (T_0(R,G))^{-1}\right) \\ &\quad \leq \limsup_{R\to\infty, R\notin E} \frac{4\overline{N}_0(R,g) + (n+6)T_0(R,g)}{nT_0(R,g)} < 2.\end{aligned} \tag{66}$$

Then, from Lemma 21, we have $F \equiv (aG+b)/(cG+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Thus, by using the same argument as that in Theorem 5, we can prove the conclusion of Theorem 7.

4. Proofs of Theorems 9 and 11

4.1. The Proof of Theorem 9. Since $E_1^{\mathbb{A}}(f, S) = E_1^{\mathbb{A}}(g, S)$, we have $E_1^{\mathbb{A}}(F, 0) = E_1^{\mathbb{A}}(G, 0)$. From (47)–(49), we can get

$$\overline{N}_0^{(2}\left(R, \frac{1}{F}\right) = \sum_{i=1}^{n} \overline{N}_0\left(R, \frac{1}{f-d_i}\right) \leq \overline{N}\left(R, \frac{1}{f'}\right), \tag{67}$$

where d_i $(i = 1, \ldots, n)$ are the distinct zeros of $P_1(w)$. And from (51) and (67), by Lemma 20, we have

$$\begin{aligned}&\overline{N}_0\left(R, \frac{1}{F'}\right) + 2\overline{N}_0^{(2}\left(R, \frac{1}{F}\right) \\ &\quad \leq \overline{N}_0\left(R, \frac{1}{f}\right) + \overline{N}_0\left(R, \frac{1}{f-1}\right) + 3N_0\left(R, \frac{1}{f}\right) \\ &\qquad + 3\overline{N}_0(R,f) \\ &\quad \leq 5T_0(R,f) + 3\overline{N}_0(R,f) + S(R).\end{aligned} \tag{68}$$

Then, from (50) and (68), since $T_0(R,F) = nT_0(R,f) + S(R)$ and $n \geq 15$, we have

$$\begin{aligned}&\limsup_{R\to\infty, R\notin E} \left(\left(3\overline{N}_0(R,F) + \sum_{j=1}^{2} \overline{N}_0^{(2)}\left(R, \frac{1}{F-c_j}\right) + \overline{N}_0\left(R, \frac{1}{F'}\right) + 2\overline{N}_0^{(2}\left(R, \frac{1}{F}\right)\right) \times (T_0(R,F))^{-1}\right) \\ &\quad \leq \limsup_{R\to\infty, R\notin E} \frac{6\overline{N}_0(R,f) + (n+8)T_0(R,f)}{nT_0(R,f)} < 2.\end{aligned} \tag{69}$$

Similarly, we can get

$$\begin{aligned}&\limsup_{R\to\infty, R\notin E} \left(\left(3\overline{N}_0(R,G) + \sum_{j=1}^{2} \overline{N}_0^{(2)}\left(R, \frac{1}{G-c_j}\right) + \overline{N}_0\left(R, \frac{1}{G'}\right) + 2\overline{N}_0^{(2}\left(R, \frac{1}{G}\right)\right) \times (T_0(R,G))^{-1}\right) \\ &\quad \leq \limsup_{R\to\infty, R\notin E} \frac{6\overline{N}_0(R,g) + (n+8)T_0(R,g)}{nT_0(R,g)} < 2.\end{aligned} \tag{70}$$

Thus, by Lemma 22, we have

$$F \equiv \frac{aG+b}{cG+d}, \tag{71}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. By using arguments similar to those in the proof of Theorem 5, we can get that $f \equiv g$.

Therefore, this completes the proof of Theorem 9.

4.2. The Proof of Theorem 11. Since $\Theta_0(\infty, f) > 5/6$ and $\Theta_0(\infty, g) > 5/6$, it follows that

$$\limsup_{R\to\infty}\frac{\overline{N}_0(R,f)}{T_0(R,f)}<\frac{1}{6},\qquad \limsup_{R\to\infty}\frac{\overline{N}_0(R,g)}{T_0(R,g)}<\frac{1}{6}. \tag{72}$$

By applying (72), from (69) and (70), since $n \geq 7$, we can get

$$\begin{aligned}
&\limsup_{R\to\infty, R\notin E}\left(\left(3\overline{N}_0(R,F)+\sum_{j=1}^{2}\overline{N}_0^{(2)}\left(R,\frac{1}{F-c_j}\right)\right.\right.\\
&\qquad\left.+\overline{N}_0\left(R,\frac{1}{F'}\right)+2\overline{N}_0^{(2}\left(R,\frac{1}{F}\right)\right)\\
&\qquad\left.\times\left(T_0(R,F)\right)^{-1}\right)\\
&\quad\leq\limsup_{R\to\infty, R\notin E}\frac{6\overline{N}_0(R,f)+(n+8)T_0(R,f)}{nT_0(R,f)}<2,\\
&\limsup_{R\to\infty, R\notin E}\left(\left(3\overline{N}_0(R,G)+\sum_{j=1}^{2}\overline{N}_0^{(2)}\left(R,\frac{1}{G-c_j}\right)\right.\right.\\
&\qquad\left.+\overline{N}_0\left(R,\frac{1}{G'}\right)+2\overline{N}_0^{(2}\left(R,\frac{1}{G}\right)\right)\\
&\qquad\left.\times\left(T_0(R,G)\right)^{-1}\right)\\
&\quad\leq\limsup_{R\to\infty, R\notin E}\frac{6\overline{N}_0(R,g)+(n+8)T_0(R,g)}{nT_0(R,g)}<2.
\end{aligned}\tag{73}$$

Then, from Lemma 22, we have $F \equiv (aG+b)/(cG+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Thus, by using the same argument as in Theorem 5, we can prove the conclusion of Theorem 11.

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