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Research Article

Existence of Nontrivial Solutions and High Energy Solutions for a Class of Quasilinear Schrödinger Equations via the Dual-Perturbation Method

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We study the quasilinear Schrödinger equation of the form $-\Delta u+V(x)u-\Delta(u^2)u = h(x,u)$, $x \in R^N$. Under appropriate assumptions on $V(x)$ and $h(x,u)$, existence results of nontrivial solutions and high energy solutions are obtained by the dual-perturbation method.

1. Introduction and Preliminaries

In this paper we consider the quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - \Delta\left(u^2\right)u = h(x,u), \quad x \in R^N, \tag{1}$$

where $h \in C(R^N \times R, R)$ and $V \in C(R^N, R)$. Solutions of (1) are standing waves of the following quasilinear Schrödinger equation:

$$\begin{aligned} i\psi_t + \Delta\psi - V(x)\psi + k\Delta\left(\alpha\left(|\psi|^2\right)\right)\alpha'\left(|\psi|^2\right)\psi \\ + g(x,\psi) = 0, \quad x \in R^N, \end{aligned} \tag{2}$$

where $V(x)$ is a given potential, k is a real constant, and α and g are real functions. The quasilinear Schrödinger equations (2) are derived as models of several physical phenomena; for example, see [1–5]. Several methods can be used to solve (1). For instance, the existence of a positive ground state solution has been proved in [6, 7] by using a constrained minimization argument; the problem is transformed to a semilinear one in [8–11] by a change of variables (dual approach); Nehari method is used to get the existence results of ground state solutions in [12, 13].

Recently, some new methods have been applied to these equations. In [14], the authors prove that the critical points are L^∞ functions by the Moser's iteration; then the existence of multibump type solutions is constructed for this class of quasilinear Schrödinger equations. In [15], by analysing the behavior of the solutions for subcritical case, the authors pass to the limit as the exponent approaches to the critical exponent in order to establish the existence of both one-sign and nodal ground state solutions. Another new method which works for these equations is perturbations. In [16] 4-Laplacian perturbations are led into these equations; then high energy solutions are obtained on bounded smooth domain.

In this paper, the perturbation, combined with dual approach, is applied to search the existence of nontrivial solution and sequence of high energy solutions of (1) on the whole space R^N. For simplicity we call this method the dual-perturbation method.

We need the following several notations. Let $C_c^\infty(R^N)$ be the collection of smooth functions with compact support. Let

$$H^1(R) := \left\{u \in L^2\left(R^N\right) : |\nabla u| \in L^2\left(R^N\right)\right\}, \tag{3}$$

with the inner product

$$\langle u, v\rangle_{H^1} = \int_{R^N} [\nabla u \cdot \nabla v + uv]\, dx \tag{4}$$

and the norm

$$\|u\|_{H^1} = \langle u, u\rangle_{H^1}^{1/2}. \quad (5)$$

Let the following assumption (V) hold:

(V) $V \in C(R^N, R)$ satisfies $\inf_{x\in R^N} V(x) \geq a_0 > 0$ and $\lim_{|x|\to\infty} V(x) = +\infty$.

Set

$$E := \left\{u \in H^1\left(R^N\right) : \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty\right\} \quad (6)$$

with the inner product

$$\langle u, v\rangle_E = \int_{R^N} [\nabla u \cdot \nabla v + V(x) uv]\, dx \quad (7)$$

and the norm

$$\|u\|_E = \langle u, u\rangle_E^{1/2}. \quad (8)$$

Then both $H^1(R^N)$ and E are Hilbert spaces.

By the continuity of the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [2, 2^*]$ we know that, for each $s \in [2, 2^*]$, there exists constant $a_s > 0$ such that

$$\|u\|_s \leq a_s \|u\|_E, \quad \forall u \in E, \quad (9)$$

where $\|\cdot\|_s$ denotes the L^s-norm. In the following, we use C or C_i to denote various positive constants. Moreover, we need the following assumptions:

(h_1) there exist $4 < p < 2(2^*)$ if $N \geq 3$ and $4 < p < \infty$ if $N = 2$ such that

$$|h(x, s)| \leq C\left(1 + |s|^{p-1}\right), \quad \forall s \in R, \quad (10)$$

(h_2) $\lim_{s\to 0} h(x, s)/s = 0$ uniformly in $x \in R^N$,

(h_3) there exist $\mu > 4$ and $r > 0$ such that

$$\begin{gathered} c_0 := \inf_{x\in R^N, |s|=r} H(x, s) > 0, \\ \mu H(x, s) \leq h(x, s)\, s \end{gathered} \quad (11)$$

for all $x \in R^N$ and $|s| \geq r$, where $H(x, s) = \int_0^s h(x, t)dt$.

By Lemma 3.4 in [17] we know that, under the assumption (V), the embedding $E \hookrightarrow L^s(R^N)$ is compact for each $2 \leq s < 2^*$.

Equation (1) is the Euler-Lagrange equation of the energy functional

$$\begin{aligned} J(u) = {} & \frac{1}{2}\int_{R^N} \left[\left(1 + 2u^2\right)|\nabla u|^2 + V(x) u^2\right] dx \\ & - \int_{R^N} H(x, u)\, dx, \end{aligned} \quad (12)$$

where $H(x, u) = \int_0^u h(x, t)dt$. Due to the presence of the term $\int_{R^N} u^2|\nabla u|^2 dx$, $J(u)$ is not well defined in E. To overcome this difficulty, a dual approach is used in [9, 10]. Following the idea from these papers, let f be defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}} \quad (13)$$

on $[0, +\infty)$, $f(0) = 0$ and $f(-t) = -f(t)$ on $(-\infty, 0]$. Then f has the following properties:

(f_1) f is uniquely defined C^∞ function and invertible;

(f_2) $0 < f'(t) \leq 1$ for all $t \in R$;

(f_3) $|f(t)| \leq |t|$ for all $t \in R$;

(f_4) $\lim_{t\to 0} f(t)/t = 1$;

(f_5) $\lim_{t\to+\infty} f(t)/\sqrt{t} = 2^{1/4}$, $\lim_{t\to-\infty} f(t)/\sqrt{|t|} = -2^{1/4}$;

(f_6) $(1/2)f(t) \leq tf'(t) \leq f(t)$ for all $t \geq 0$ and $f(t) \leq tf'(t) \leq (1/2)f(t)$ for all $t \leq 0$;

(f_7) $|f(t)| \leq 2^{1/4}\sqrt{|t|}$ for all $t \in R$;

(f_8) the function $f^2(t)$ is strictly convex;

(f_9) there exists a positive C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{1/2}, & |t| \geq 1; \end{cases} \quad (14)$$

(f_{10}) there exist positive constants C_1 and C_2 such that

$$|t| \leq C_1|f(t)| + C_2|f(t)|^2 \quad (15)$$

for all $t \in R$;

(f_{11}) $|f(t)f'(t)| \leq 1/\sqrt{2}$ for all $t \in R$;

(f_{12}) for each $\xi > 0$, there exists $C(\xi) > 0$ such that $f^2(\xi t) \leq C(\xi) f^2(t)$.

The properties (f_1)–(f_{11}) have been proved in [8–11]. It suffices to prove (f_{12}).

Indeed, by (f_1), (f_4), and (f_5), there exist $\delta > 0$ and $M > 0$ such that, for $|t| \leq \delta$,

$$\frac{1}{2}t^2 \leq f^2(t) \leq \frac{3}{2}t^2, \quad (16)$$

and for $|t| \geq M$,

$$\frac{\sqrt{2}}{2}|t| \leq f^2(t) \leq \frac{3\sqrt{2}}{2}|t|. \quad (17)$$

Since there exists a $C_0 > 0$ such that $f^2(2t) \leq C_0 f^2(t)$ (see [10]), we can assume that $0 < \xi < 1$. For $|t| \leq \delta$, we have $|\xi t| \leq \delta$, and hence

$$f^2(\xi t) \leq \frac{3}{2}\xi^2 t^2 \leq 3\xi^2 f^2(t); \quad (18)$$

for $|t| \geq M/\xi > M$, one has $|\xi t| \geq M$, and hence

$$f^2(\xi t) \leq \frac{3\sqrt{2}}{2}\xi|t| \leq 3\xi f^2(t); \quad (19)$$

and for $\delta \le |t| \le M/\xi$, there exist $m(\xi) > 0$ and $M(\xi) > 0$ such that $f^2(\xi t) \le M(\xi)$ and $f^2(t) \ge m(\xi)$. Then we have

$$f^2(\xi t) \le M(\xi) \le \frac{M(\xi)}{m(\xi)} f^2(t). \tag{20}$$

Hence $f^2(\xi t) \le C(\xi) f^2(t)$, where $C(\xi) = \max\{3\xi^2, M(\xi)/m(\xi)\}$.

After the change of variable, $J(u)$ can be reduced to

$$\begin{aligned} I(v) := J(f(v)) = \frac{1}{2}\int_{R^N}\left(|\nabla v|^2 + V(x) f^2(v)\right)dx \\ - \int_{R^N} H(x, f(v))\,dx. \end{aligned} \tag{21}$$

From [8, 9, 11] we know that if $v \in E$ is a critical point of I, that is,

$$\begin{aligned} \left\langle I'(v), \varphi\right\rangle = \int_{R^N} \nabla v \nabla \varphi dx + \int_{R^N} V(x) f(v) f'(v)\varphi dx \\ - \int_{R^N} h(x, f(v)) f'(v)\varphi dx = 0 \end{aligned} \tag{22}$$

for all $\varphi \in E$, then $u := f(v)$ is a weak solution of (1). Particularly, if $v \in H^1(R^N) \cap C^2(R^N)$ is a critical point of I, then $u := f(v)$ is a classical solution of (1).

A sequence $\{u_n\} \subset E$ is called a Cerami sequence of J if $\{J(u_n)\}$ is bounded and $(1 + \|u_n\|)J'(u_n) \to 0$ in E^*. We say that J satisfies the Cerami condition if every Cerami sequence possesses a convergent subsequence.

2. Some Lemmas

Consider the following perturbation functional I_θ defined by

$$I_\theta(v) = I(v) + \frac{\theta}{2}\int_{R^N} V(x) v^2 dx, \tag{23}$$

where $\theta \in (0, 1]$. We have the following lemmas.

Lemma 1. *If assumptions (V), (h_1), and (h_2) hold, then the functional I_θ is well defined on E and $I_\theta \in C^1(E, R)$.*

Proof. By conditions (h_1) and (h_2), the properties (f_2), (f_3), (f_7), and (f_{11}) imply that there exists $\delta > 0$ such that

$$\begin{aligned} \left|h(x, f(v)) f'(v)\right| &\le \left|f(v)\right| f'(v) \le |v| \quad \text{for } |v| < \delta, \\ \left|h(x, f(v)) f'(v)\right| &\le C\left|f(v)\right|^{p-1} f'(v) \\ &\le C\left|f(v)\right|^{p-2} \le C|v|^{(p-2)/2} \quad \text{for } |v| \ge \delta. \end{aligned} \tag{24}$$

Hence

$$\left|h(x, f(v)) f'(v)\right| \le C\left(|v| + |v|^{(p/2)-1}\right), \tag{25}$$

$$\left|H(x, f(v))\right| \le C\left(|v|^2 + |v|^{p/2}\right) \tag{26}$$

for all $v \in R$. By (26) and the continuity of the embedding $E \hookrightarrow L^s(R^N)$ ($s \in [2, 2^*]$),

$$\int_{R^N} H(x, f(v))\,dx < +\infty, \quad \forall v \in E. \tag{27}$$

Hence I_θ is well defined in E.

Now, we prove that $I_\theta \in C^1(E, R)$. It suffices to prove that

$$\begin{aligned} \Psi_1(v) &:= \int_{R^N} H(x, f(v))\,dx \in C^1(E, R), \\ \Psi_2(v) &:= \int_{R^N} V(x) f^2(v)\,dx \in C^1(E, R). \end{aligned} \tag{28}$$

For any $v, \phi \in E$ and $0 < |t| < 1$, by the mean value theorem, (25) and (f_2)-(f_3), we have

$$\begin{aligned} &\frac{\left|H(x, f(v + t\phi)) - H(x, f(v))\right|}{|t|} \\ &\quad \le \int_0^1 \left|h(f(x, v + st\phi)) f'(v + st\phi)\phi\right| ds \\ &\quad \le C\left[|v||\phi| + |\phi|^2 + |v|^{(p-2)/2}|\phi| + |\phi|^{p/2}\right], \\ &\frac{\left|V(x) f^2(v + t\phi) - V(x) f^2(v)\right|}{|t|} \\ &\quad \le 2\int_0^1 V(x)\left|f(v + st\phi) f'(v + st\phi)\phi\right| ds \\ &\quad \le 2V(x)\int_0^1 |v + st\phi||\phi|\,ds \\ &\quad \le 2V(x)\left[|v||\phi| + |\phi|^2\right]. \end{aligned} \tag{29}$$

The Hölder inequality implies that

$$\begin{aligned} C\left[|v||\phi| + |\phi|^2 + |v|^{(p-2)/2}|\phi| + |\phi|^{p/2}\right] \in L^1\left(R^N\right), \\ 2V(x)\left[|v||\phi| + |\phi|^2\right] \in L^1\left(R^N\right). \end{aligned} \tag{30}$$

Hence, by the Lebesgue theorem, we have

$$\begin{aligned} \left\langle \Psi_1'(v), \phi\right\rangle &= \int_{R^N} h(x, f(v)) f'(v)\phi dx, \\ \left\langle \Psi_2'(v), \phi\right\rangle &= 2\int_{R^N} V(x) f(v) f'(v)\phi dx \end{aligned} \tag{31}$$

for all $\phi \in E$. Now, we show that $\Psi_i'(\cdot) : E \to E^*$, $i = 1, 2$, are continuous. Indeed, if $v_n \to v$ in E, then $v_n \to v$ in $L^s(R^N)$ for all $s \in [2, 2^*]$.

On the space $L^{p_1}(R^N) \cap L^{p_2}(R^N)$, we define the norm

$$\|v\|_{p_1 \wedge p_2} = \|v\|_{p_1} + \|v\|_{p_2}. \tag{32}$$

Then

$$v_n \longrightarrow v \quad \text{in } L^2\left(R^N\right) \cap L^{(p/2)-1}\left(R^N\right). \tag{33}$$

Moreover, on the space $L^{p_1}(R^N) + L^{p_2}(R^N)$, we define the norm

$$\|v\|_{p_1 \vee p_2} = \inf \left\{ \|u\|_{p_1} + \|w\|_{p_2} : v = u + w,\ u \in L^{p_1}\left(R^N\right), w \in L^{p_2}\left(R^N\right) \right\}. \tag{34}$$

By (25), we have

$$\begin{aligned} \left|h\left(x, f(v)\right) f'(v)\right| &\le C\left(|v| + |v|^{(p/2)-1}\right) \\ &\le C\left(|v|^{2/2} + |v|^{q/r}\right), \end{aligned} \tag{35}$$

where $q = p/2$ and $r = p/(p-2)$. Then Theorem A.4 in [18] implies

$$\begin{aligned} h\left(x, f\left(v_n\right)\right) f'\left(v_n\right) - h\left(x, f(v)\right) f'(v) &\longrightarrow 0 \\ &\text{in } L^2\left(R^N\right) + L^r\left(R^N\right) \end{aligned} \tag{36}$$

as $n \to +\infty$. If $h(x, f(v_n))f'(v_n) - h(x, f(v))f'(v) = y_n + z_n$ with $y_n \in L^2(R^N)$ and $z_n \in L^r(R^N)$, one has

$$\begin{aligned} &\left|\int_{R^N} \left[h\left(x, f\left(v_n\right)\right) f'\left(v_n\right) - h\left(x, f(v)\right) f'(v)\right] \phi dx\right| \\ &\quad \le \int_{R^N} \left|y_n\right| |\phi| + \left|z_n\right| |\phi| \, dx \\ &\quad \le C\left(\|y_n\|_2 + \|z_n\|_r\right) \|\phi\|_E. \end{aligned} \tag{37}$$

Hence

$$\begin{aligned} &\left|\int_{R^N} \left[h\left(x, f\left(v_n\right)\right) f'\left(v_n\right) - h\left(x, f(v)\right) f'(v)\right] \phi dx\right| \\ &\quad \le C\left\|h\left(x, f\left(v_n\right)\right) f'\left(v_n\right) - h\left(x, f(v)\right) f'(v)\right\|_{2\vee r} \|\phi\|_E, \end{aligned} \tag{38}$$

and hence

$$\left\|\Psi_1'\left(v_n\right) - \Psi_1'(v)\right\| \longrightarrow 0 \tag{39}$$

as $n \to \infty$. Therefore, $\Psi_1 \in C^1(E, R)$.

Define

$$L_V^s\left(R^N\right) = \left\{u : R^N \longrightarrow R : u \text{ is measurable and } \int_{R^N} V(x) u^s dx < \infty\right\} \tag{40}$$

with the norm $\|u\|_{L_V^s} = (\int_{R^N} V(x)u^s dx)^{1/s}$. On the space $L_V^{p_1}(R^N) \cap L_V^{p_2}(R^N)$, we define the norm

$$\|v\|_{p_1 \wedge p_2} = \|v\|_{L_V^{p_1}} + \|v\|_{L_V^{p_2}}. \tag{41}$$

On the space $L_V^{p_1}(R^N) + L_V^{p_2}(R^N)$, we define the norm

$$\|v\|_{p_1 \vee p_2} = \inf \left\{ \|v\|_{p_1 \vee p_2} = \|u\|_{L_V^{p_1}} + \|w\|_{L_V^{p_2}} : v = u + w, u \in L_V^{p_1}\left(R^N\right),\ w \in L_V^{p_2}\left(R^N\right) \right\}. \tag{42}$$

From $v_n \to v$ in E, one has $v_n, v \in L_V^2(R^N)$ and

$$v_n \longrightarrow v \quad \text{in } L_V^2\left(R^N\right) \cap L_V^2\left(R^N\right) \tag{43}$$

as $n \to \infty$. Since $|f(v)f'(v)| \le |v|$, by the following Lemma 2, we have

$$f\left(v_n\right) f'\left(v_n\right) \longrightarrow f(v) f'(v) \qquad \text{in } L_V^2\left(R^N\right) + L_V^2\left(R^N\right). \tag{44}$$

If $f(v_n)f'(v_n) - f(v)f'(v) = y_n + z_n$ with $y_n \in L_V^2(R^N)$ and $z_n \in L_V^2(R^N)$, one has

$$\begin{aligned} &\left|\int_{R^N} V(x) \left[f\left(v_n\right) f'\left(v_n\right) - f(v) f'(v)\right] \phi dx\right| \\ &\quad \le \int_{R^N} V(x) \left|y_n\right| |\phi| + V(x) \left|z_n\right| |\phi| \, dx \\ &\quad \le \left(\|y_n\|_{L_V^2} + \|z_n\|_{L_V^2}\right) \|\phi\|_E. \end{aligned} \tag{45}$$

Hence

$$\begin{aligned} &\left|\int_{R^N} V(x) \left[f\left(v_n\right) f'\left(v_n\right) - f(v) f'(v)\right] \phi dx\right| \\ &\quad \le \left\|f(v_n)f'(v_n) - f(v) f'(v)\right\|_{2\vee 2} \|\phi\|_E, \end{aligned} \tag{46}$$

and hence

$$\left\|\Psi_2'\left(v_n\right) - \Psi_2'(v)\right\| \longrightarrow 0 \tag{47}$$

as $n \to \infty$. Therefore, $\Psi_2 \in C^1(E, R)$. This completes the proof. □

Lemma 2. *Assume that* $1 \le p, q, r, s < +\infty$, $g \in C(R^N \times R)$ *and*

$$\left|g(x, v)\right| \le C\left(|v|^{p/r} + |v|^{q/s}\right). \tag{48}$$

Then, for every $v \in L_V^p(R^N) \cap L_V^q(R^N)$, $g(\cdot, v) \in L_V^r(R^N) + L_V^s(R^N)$, *and the operator*

$$\begin{aligned} A : L_V^p\left(R^N\right) \cap L_V^q\left(R^N\right) \longrightarrow L_V^r\left(R^N\right) + L_V^s\left(R^N\right) : v \longmapsto g(x, v) \end{aligned} \tag{49}$$

is continuous.

Proof. Let $\eta(s)$ be a smooth cut-off function such that $\eta(s) = 1$ for $|s| \le 1$ and $\eta(s) = 0$ for $|s| \ge 2$. Define

$$\begin{aligned} g_1(x, v) &:= \eta(v) g(x, v), \\ g_2(x, v) &:= (1 - \eta(v)) g(x, v). \end{aligned} \tag{50}$$

We can assume that $p/r \le q/s$. Hence

$$\left|g_1(x, v)\right| \le C|v|^{p/r}, \qquad \left|g_2(x, v)\right| \le C|v|^{q/s} \tag{51}$$

for all $(x, v) \in R^N \times R$. Assume $v_n \to v$ in $L_V^p(R^N) \cap L_V^q(R^N)$. Then $v_n \to v$ in $L_V^p(R^N)$ and $g(\cdot, v_n) \to g(\cdot, v)$ in $L_V^r(R^N)$. As in the proof of Lemma A.1 in [18], there exists a subsequence $\{w_n\}$ of $\{v_n\}$ and $\alpha \in L_V^p(R^N)$ such that $w_n(x) \to v(x)$ and $|v(x)|, |w_n(x)| \le \alpha(x)$ for a.e. $x \in R^N$. Hence, from (51), one has

$$|g_1(x, w_n) - g_1(x, v)|^r \le 2^r C |\alpha(x)|^p \tag{52}$$

a.e. on R^N. It follows from the Lebesgue theorem that $g_1(\cdot, w_n) \to g_1(\cdot, v)$ in $L_V^r(R^N)$. Consequently, $g_1(\cdot, v_n) \to g_1(\cdot, v)$ in $L_V^r(R^N)$. Similarly, we can prove $g_2(\cdot, v_n) \to g_2(\cdot, v)$ in L_V^s. Since

$$\begin{aligned} \|g(\cdot, v_n) - g(\cdot, v)\|_{r\vee s} \le{}& \|g_1(\cdot, v_n) - g_1(\cdot, v)\|_{L_V^r} \\ &+ \|g_2(\cdot, v_n) - g_2(\cdot, v)\|_{L_V^s}, \end{aligned} \tag{53}$$

it follows that $g(\cdot, v_n) \to g(\cdot, v)$ in $L_V^r + L_V^s$. This completes the proof. □

Lemma 3. *Let (V), (h_1), and (h_2) hold. Then every bounded sequence $\{v_n\} \subset E$ with $I'_\theta(v_n) \to 0$ possesses a convergent subsequence.*

Proof. Since $\{v_n\} \subset E$ is bounded, then, by the compactness of the embedding $E \hookrightarrow L^s(R^N)$ $(2 \le s < 2^*)$, passing to a subsequence, one has $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(R^N)$ for all $2 \le s < 2^*$, and $v_n(x) \to v(x)$ for a.e. $x \in R^N$. By (25)

$$\begin{aligned} &\left| \int_{R^N} h(x, f(v_n)) f'(v_n)(v - v_n)\, dx \right| \\ &\quad \le \int_{R^N} C\left(|v_n| + |v_n|^{(p/2)-1} \right) |v_n - v|\, dx \\ &\quad \le C\left(\|v_n\|_2 \|v_n - v\|_2 + \|v_n\|_{p/2}^{(p/2)-1} \|v_n - v\|_{p/2} \right) \\ &\quad \le C\left(\|v_n\|_E \|v_n - v\|_2 + \|v_n\|_E^{(p/2)-1} \|v_n - v\|_{p/2} \right) \longrightarrow 0 \end{aligned} \tag{54}$$

as $n \to \infty$. Similarly, $\int_{R^N} h(x, f(v)) f'(v)(v - v_n)dx \to 0$ as $n \to \infty$. Hence, by the property of (f_8), we have

$$\begin{aligned} &\langle I'_\theta(v_n) - I'_\theta(v), v_n - v \rangle \\ &\quad = \int_{R^N} |\nabla(v_n - v)|^2 dx + \theta \int_{R^N} V(x) |v_n - v|^2 dx \\ &\qquad + \int_{R^N} V(x) \left[f(v_n) f'(v_n) - f(v) f'(v) \right] \\ &\qquad\qquad \times (v_n - v)\, dx \\ &\qquad - \int_{R^N} \left[h(x, f(v_n)) f'(v_n) \right. \\ &\qquad\qquad \left. - h(x, f(v)) f'(v) \right] (v_n - v)\, dx \\ &\quad \ge \theta \|v_n - v\|_E^2 - o_n(1), \end{aligned} \tag{55}$$

where $o_n(1) \to 0$ as $n \to \infty$. This shows that $\|v_n - v\|_E^2 \to 0$ as $n \to \infty$. This completes the proof. □

The following Lemma 4 has been proved in [10] (see Proposition 2.1(3) in [10]).

Lemma 4. *If $v_n(x) \to v(x)$ a.e. in R^N and $\lim_{n\to\infty} \int_{R^N} V(x) f^2(v_n) dx = \int_{R^N} V(x) f^2(v) dx$, then $\int_{R^N} V(x) f^2(v_n - v) dx \to 0$ as $n \to \infty$.*

3. Main Results

Theorem 5. *Assume conditions (V), (h_1)–(h_3) hold. Let $\{\theta_n\} \subset (0, 1]$ be such that $\theta_n \to 0$. Let $v_n \in E$ be a critical point of I_{θ_n} with $I_{\theta_n}(v_n) \le c$ for some constant c independent of n. Then, up to subsequence, one has $v_n \to v$ in E, $I_{\theta_n}(v_n) \to I(v)$ and v is a critical point of I.*

Proof. By (h_2), for $0 < \varepsilon_0 < (1/4)(1/2 - 1/\mu)a_0$, there exists $\delta_0 > 0$ such that

$$\left| \frac{1}{\mu} s h(x, s) - H(x, s) \right| \le \varepsilon_0 s^2, \quad \forall s \in [-\delta_0, \delta_0]. \tag{56}$$

By (h_1), for $\delta_0 \le |s| \le r$ (r is the constant appearing in condition (h_3)), we have

$$\left| \frac{1}{\mu} s h(x, s) - H(x, s) \right| \le 2C \left(\frac{1}{\delta_0^2} + r^{p-2} \right) s^2, \tag{57}$$

where C is the constant appearing in condition (h_1). Hence

$$\begin{aligned} \left| \frac{1}{\mu} s h(x, s) - H(x, s) \right| \le \varepsilon_0 s^2 + 2C \left(\frac{1}{\delta_0^2} + r^{p-2} \right) s^2, \\ \forall s \in [-r, r]. \end{aligned} \tag{58}$$

Since $\lim_{|x|\to\infty} V(x) = +\infty$, there exists $\rho_0 > 0$ such that

$$\frac{1}{4}\left(\frac{1}{2} - \frac{1}{\mu} \right) V(x) > 2C \left(\frac{1}{\delta_0^2} + r^{p-2} \right) \tag{59}$$

for all $|x| \ge \rho_0$. Hence

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{R^N} V(x) f^2(v)\, dx \\ &\quad + \int_{\{x : |f(v)| \le r\}} \left[\frac{1}{\mu} f(v) h(x, f(v)) \right. \\ &\qquad\qquad \left. - H(x, f(v)) \right] dx \\ &\quad \ge \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{R^N} V(x) f^2(v)\, dx \\ &\qquad - 2C \left(\frac{1}{\delta_0^2} + r^{p-2} \right) r^2 \left| B_{\rho_0} \right|. \end{aligned} \tag{60}$$

Since v_n is a critical point of I_{θ_n},

$$\begin{aligned}\left\langle I'_{\theta_n}(v_n), \phi \right\rangle = & \int_{R^N} \nabla v_n \nabla \phi dx \\ & + \int_{R^N} V(x) f(v_n) f'(v_n) \phi dx \\ & + \theta_n \int_{R^N} V(x) v_n \phi dx \\ & - \int_{R^N} h(x, f(v_n)) f'(v_n) \phi dx = 0\end{aligned} \tag{61}$$

for all $\phi \in E$. Consequently, taking $\phi = f(v_n)/f'(v_n) \in E$, by (h_3) and (f_6) we have

$$\begin{aligned}c \geq I_{\theta_n}(v_n) &= I_{\theta_n}(v_n) - \frac{1}{\mu}\left\langle I'_{\theta_n}(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \\ &= \int_{R^N} \left[\frac{1}{2} - \frac{1}{\mu}\left(1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right)\right] |\nabla v_n|^2 dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{R^N} V(x) f^2(v_n) dx \\ &\quad + \frac{\theta_n}{2} \int_{R^N} V(x) v_n^2 dx \\ &\quad - \frac{\theta_n}{\mu} \int_{R^N} V(x) \frac{v_n f(v_n)}{f'(v_n)} dx \\ &\quad + \int_{R^N} \left[\frac{1}{\mu} h(x, f(v_n)) f(v_n) - H(x, f(v_n))\right] dx \\ &\geq \left(\frac{1}{2} - \frac{2}{\mu}\right) \int_{R^N} |\nabla v_n|^2 dx + \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{R^N} V(x) f^2(v_n) dx \\ &\quad + \left(\frac{1}{2} - \frac{2}{\mu}\right) \theta_n \int_{R^N} V(x) v_n^2 dx \\ &\quad + \int_{\{x:|f(v_n)|\leq r\}} \left[\frac{1}{\mu} h(x, f(v_n)) f(v_n) \right. \\ &\qquad \left. - H(x, f(v_n))\right] dx \\ &\geq \left(\frac{1}{2} - \frac{2}{\mu}\right) \int_{R^N} |\nabla v_n|^2 dx \\ &\quad + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{R^N} V(x) f^2(v_n) dx \\ &\quad + \left(\frac{1}{2} - \frac{2}{\mu}\right) \theta_n \int_{R^N} V(x) v_n^2 dx \\ &\quad - 2C\left(\frac{1}{\delta_0^2} + r^{p-2}\right) r^2 |B_{\rho_0}| \\ &\geq \left(\frac{1}{2} - \frac{2}{\mu}\right) \int_{R^N} |\nabla v_n|^2 dx \\ &\quad + \left(\frac{1}{4} - \frac{1}{2\mu}\right) \int_{R^N} V(x) f^2(v_n) dx \\ &\quad + \left(\frac{1}{2} - \frac{2}{\mu}\right) \theta_n \int_{R^N} V(x) v_n^2 dx - C_1,\end{aligned} \tag{62}$$

and hence

$$\begin{aligned}\int_{R^N} |\nabla v_n|^2 dx + \int_{R^N} V(x) f^2(v_n) dx \\ + \theta_n \int_{R^N} V(x) v_n^2 dx \leq C\end{aligned} \tag{63}$$

for some constant C independent of n. By the boundedness of $\int_{R^N} |\nabla v_n|^2 dx$, there exists $C_2 > 0$ such that

$$\begin{aligned}2 \int_{\mathbb{R}^N} f^2(v_n) |\nabla f(v_n)|^2 dx &\leq \int_{\mathbb{R}^N} \left[1 + 2f^2(v_n)\right] |\nabla f(v_n)|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leq C_2\end{aligned} \tag{64}$$

for all n. Hence, by the Sobolev embedding theorem, one has

$$\|f(v_n)\|_{2(2^*)}^4 = \left\|f^2(v_n)\right\|_{2^*}^2 \leq C_3 \left\|\nabla f^2(v_n)\right\|_2^2 \leq C. \tag{65}$$

Next, we prove that $f(v_n) \in L^\infty(R^N)$ and $\|f(v_n)\|_{L^\infty} \leq C$, where the positive constant C is independent of n. Setting $T > 2$, $r > 0$, define $\widetilde{v}_n^T = b(v_n)$, where $b : R \to R$ is a smooth function satisfying $b(s) = s$ for $|s| \leq T-1$, $b(-s) = -b(s)$; $b'(s) = 0$ for $s \geq T$, and $b'(s)$ is decreasing in $[T-1, T]$.

This means that $\widetilde{v}_n^T = v_n$, for $|v_n| \leq T-1$; $|\widetilde{v}_n^T| = |b(v_n)| \leq |v_n|$, for $T-1 \leq |v_n| \leq T$; $|\widetilde{v}_n^T| = C_T > 0$, for $|v_n| \geq T$, where $T-1 \leq C_T \leq T$.

Let $\phi = (f(v_n)/f'(v_n))|f(\widetilde{v}_n^T)|^{2r}$; then $\phi \in E$. By (61) $\langle I'(v_n), \phi \rangle = 0$. Hence

$$\begin{aligned}I_1 + I_2 + I_3 + I_4 + I_5 \\ = \int_{\mathbb{R}^N} h(x, f(v_n)) f(v_n) \left|f\left(\widetilde{v}_n^T\right)\right|^{2r} dx,\end{aligned} \tag{66}$$

where

$$\begin{aligned}
I_5 &:= \int_{R^N} V(x) f^2(v_n) \left|f(\widetilde{v}_n^T)\right|^{2r} dx \\
&\quad + \theta_n \int_{R^N} V(x) \frac{v_n f(v_n)}{f'(v_n)} \left|f(\widetilde{v}_n^T)\right|^{2r} dx \\
&\geq \int_{R^N} V(x) f^2(v_n) \left|f(\widetilde{v}_n^T)\right|^{2r} dx, \\
I_1 &:= \int_{\{x:|v_n|\geq T\}} \left[1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right] \left|f(\widetilde{v}_n^T)\right|^{2r} |\nabla v_n|^2 dx \\
&\geq \int_{\{x:|v_n|\geq T\}} \left|f(\widetilde{v}_n^T)\right|^{2r} |\nabla v_n|^2 dx \\
&= \int_{\{x:|v_n|\geq T\}} \left[1+2f^2(v_n)\right] \\
&\qquad \times |\nabla f(v_n)|^2 \left|f(\widetilde{v}_n^T)\right|^{2r} dx \\
&\geq 2\int_{\{x:|v_n|\geq T\}} f^2(v_n) |\nabla f(v_n)|^2 \left|f(\widetilde{v}_n^T)\right|^{2r} dx \\
&= \frac{1}{2} \int_{\{x:|v_n|\geq T\}} \left|\nabla \left[f^2(v_n) f^r(\widetilde{v}_n^T)\right]\right|^2 dx,
\end{aligned} \tag{67}$$

$$\begin{aligned}
I_2 &:= \int_{\{x:|v_n|\leq T-1\}} \left[2r+1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right] \\
&\qquad \times \left|f(\widetilde{v}_n^T)\right|^{2r} |\nabla v_n|^2 dx \\
&\geq \int_{\{x:|v_n|\leq T-1\}} \frac{2f^2(v_n)}{1+2f^2(v_n)} \left|f(\widetilde{v}_n^T)\right|^{2r} |\nabla v_n|^2 dx \\
&\geq \int_{\{x:|v_n|\leq T-1\}} |f(v_n)|^{2r+2} |\nabla f(v_n)|^2 dx \\
&= \frac{1}{(r+2)^2} \int_{\{x:|v_n|\leq T-1\}} \left|\nabla f^{r+2}(v_n)\right|^2 dx \\
&= \frac{1}{(r+2)^2} \int_{\{x:|v_n|\leq T-1\}} \left|\nabla \left[f^2(v_n) f^r(\widetilde{v}_n^T)\right]\right|^2 dx, \\
I_3 &:= \int_{\{x:T-1\leq|v_n|\leq T\}} \left[1+\frac{2f^2(v_n)}{1+2f^2(v_n)}\right] f^{2r}(\widetilde{v}_n^T) |\nabla v_n|^2 dx \\
&\geq \int_{\{x:T-1\leq|v_n|\leq T\}} \frac{2f^2(v_n)}{1+2f^2(v_n)} f^{2r}(\widetilde{v}_n^T) |\nabla v_n|^2 dx \\
&= \frac{1}{2} \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^r(\widetilde{v}_n^T) \nabla f^2(v_n)\right]^2 dx, \\
I_4 &:= 2r \int_{\{x:T-1\leq|v_n|\leq T\}} f^{2r-1}(\widetilde{v}_n^T) f'(\widetilde{v}_n^T) \\
&\qquad \times b'(v_n) \frac{f(v_n)}{f'(v_n)} |\nabla v_n|^2 dx.
\end{aligned} \tag{68}$$

For $T-1 \leq |v_n| \leq T$, $|\widetilde{v}_n^T| = |b(v_n)| \leq |v_n|$. By the properties of f and b, the mean value theorem implies

$$\begin{aligned}
|f(b(v_n))| &\geq f'(b(v_n)) b'(v_n) |v_n| \\
&\geq \frac{1}{\sqrt{2}} f'(b(v_n)) b'(v_n) f^2(v_n).
\end{aligned} \tag{69}$$

Hence

$$\begin{aligned}
I_4 &= 2r \int_{\{x:T-1\leq|v_n|\leq T\}} f^{2r-1}(\widetilde{v}_n^T) f'(\widetilde{v}_n^T) \\
&\qquad \times b'(v_n) \frac{f(v_n)}{f'(v_n)} |\nabla v_n|^2 dx \\
&= 2r \int_{\{x:T-1\leq|v_n|\leq T\}} f^{2r-1}(b(v_n)) f'(b(v_n)) \\
&\qquad \times b'(v_n) f(v_n) \\
&\qquad \times \sqrt{1+2f^2(v_n)} |\nabla v_n|^2 dx \\
&\geq 2r \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^{r-1}(b(v_n)) f'(b(v_n)) b'(v_n)\right]^2 \\
&\qquad \times f(v_n) v_n \sqrt{2f^2(v_n)} |\nabla v_n|^2 dx \\
&\geq 2r \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^{r-1}(b(v_n)) f'(b(v_n)) b'(v_n)\right]^2 \\
&\qquad \times f^4(v_n) |\nabla v_n|^2 dx \\
&= 2r \int_{\{x:T-1\leq|v_n|\leq T\}} f^4(v_n) \\
&\qquad \times \left[f^{r-1}(b(v_n)) \nabla f(b(v_n))\right]^2 dx \\
&= \frac{2}{r} \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^2(v_n) \nabla f^r(\widetilde{v}_n^T)\right]^2 dx.
\end{aligned} \tag{70}$$

Consequently,

$$\begin{aligned}
I_3 + I_4 &= \frac{1}{2} \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^r(\widetilde{v}_n^T) \nabla f^2(v_n)\right]^2 dx \\
&\quad + \frac{2}{r} \int_{\{x:T-1\leq|v_n|\leq T\}} \left[f^2(v_n) \nabla f^r(\widetilde{v}_n^T)\right]^2 dx \\
&\geq \frac{1}{(r+2)^2} \int_{\{x:T-1\leq|v_n|\leq T\}} 2\left[f^r(\widetilde{v}_n^T) \nabla f^2(v_n)\right]^2 \\
&\qquad + 2\left[f^2(v_n) \nabla f^r(\widetilde{v}_n^T)\right]^2 dx \\
&\geq \frac{1}{(r+2)^2} \int_{\{x:T-1\leq|v_n|\leq T\}} \left|\nabla \left[f^2(v_n) f^r(\widetilde{v}_n^T)\right]\right|^2 dx.
\end{aligned} \tag{71}$$

Combining (67) and (68), we have

$$I_1 + I_2 + I_3 + I_4 \geq \frac{1}{(r+2)^2} \int_{\mathbb{R}^N} \left|\nabla \left[f^2(v_n) f^r(\tilde{v}_n^T) \right]\right|^2 dx. \tag{72}$$

For any $\varepsilon > 0$, by (h_1) and (h_2), there exists $C(\varepsilon) > 0$ such that

$$|h(x,s)| \leq \varepsilon |s| + C(\varepsilon) |s|^{p-1}. \tag{73}$$

Combining (66), (72), and (73), one has

$$\begin{aligned} &\frac{1}{(r+2)^2} \int_{R^N} \left|\nabla \left[f^2(v_n) f^r(\tilde{v}_n^T) \right]\right|^2 dx \\ &\quad \leq C(\varepsilon) \int_{R^N} |f(v_n)|^p \left| f(\tilde{v}_n^T)\right|^{2r} dx. \end{aligned} \tag{74}$$

By the Hölder inequality and (65),

$$\begin{aligned} &\int_{R^N} |f(v_n)|^p \left| f(\tilde{v}_n^T)\right|^{2r} dx \\ &= \int_{R^N} |f(v_n)|^{p-4} \left| f(\tilde{v}_n^T)\right|^{2r} f^4(v_n)\, dx \\ &\leq \left(\int_{R^N} |f(v_n)|^{(p-4)\cdot(4N/(p-4)(N-2))} dx \right)^{(p-4)(N-2)/4N} \\ &\quad \cdot \left(\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^{2r} \times f^4(v_n) \right]^{4N/(4N-(p-4)(N-2))} dx \right)^{(4N-(p-4)(N-2))/4N} \\ &= \left(\int_{R^N} |f(v_n)|^{22^*} dx \right)^{((p-4)(N-2))/4N} \\ &\quad \cdot \left(\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^{r} \times f^2(v_n) \right]^{8N/(4N-(p-4)(N-2))} dx \right)^{(4N-(p-4)(N-2))/4N} \\ &\leq C \left(\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^{r} \times f^2(v_n) \right]^{8N/(4N-(p-4)(N-2))} dx \right)^{(4N-(p-4)(N-2))/4N}. \end{aligned} \tag{75}$$

Moreover,

$$\begin{aligned} &\frac{1}{(r+2)^2} \int_{R^N} \left|\nabla \left[f^2(v_n) f^r(\tilde{v}_n^T) \right]\right|^2 dx \\ &\quad \geq \frac{C}{(r+2)^2} \left(\int_{R^N} \left[f^2(v_n) \left| f(\tilde{v}_n^T)\right|^r \right]^{2^*} dx \right)^{2/2^*}. \end{aligned} \tag{76}$$

Hence

$$\begin{aligned} &\left(\int_{R^N} \left[f^2(v_n) \left| f(\tilde{v}_n^T)\right|^r \right]^{2^*} dx \right)^{2/2^*} \\ &\leq C(r+2)^2 \\ &\quad \times \left(\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^{r} \times f^2(v_n) \right]^{8N/(4N-(p-4)(N-2))} dx \right)^{(4N-(p-4)(N-2))/4N}. \end{aligned} \tag{77}$$

Since $4 < p < 2(2^*)$, $d = 2^*/(8N/(4N-(p-4)(N-2))) = 2^*/2 - p/4 + 1 > 1$. Set $q = 8N/(4N-(p-4)(N-2))$. Then

$$\begin{aligned} &\left(\int_{R^N} \left[f^2(v_n) \left| f(\tilde{v}_n^T)\right|^r \right]^{qd} dx \right)^{1/qd(r+2)} \\ &\quad \leq \left[C(r+2)^2 \right]^{1/2(r+2)} \\ &\quad\quad \times \left(\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^r f^2(v_n) \right]^q dx \right)^{1/q(r+2)}. \end{aligned} \tag{78}$$

Take $r = r_0$ such that $(2+r_0)q = 2(2^*)$. Since $|\tilde{v}_n^T| = |b(v_n)| \leq |v_n|$, $|f(\tilde{v}_n^T)| \leq |f(v_n)|$. Hence, from (65), we have

$$\int_{R^N} \left[\left| f(\tilde{v}_n^T)\right|^{r_0} f^2(v_n) \right]^q dx \leq \int_{R^N} |f(v_n)|^{(2+r_0)q} dx < C. \tag{79}$$

Since $f(\tilde{v}_n^T) \to f(v_n)$ as $T \to +\infty$, taking $T \to +\infty$ in (78) with $r = r_0$, we have

$$\begin{aligned} &\left(\int_{R^N} |f(v_n)|^{(2+r_0)qd} dx \right)^{1/qd(r_0+2)} \\ &\quad \leq \left[C(r_0+2)^2 \right]^{1/2(r_0+2)} \\ &\quad\quad \times \left(\int_{R^N} |f(v_n)|^{(2+r_0)q} dx \right)^{1/q(r_0+2)}. \end{aligned} \tag{80}$$

Set $2 + r_1 = (2+r_0)d$. Then

$$\begin{aligned} &\left(\int_{R^N} |f(v_n)|^{(2+r_1)q} dx \right)^{1/q(r_1+2)} \\ &\quad \leq \left[C(r_0+2)^2 \right]^{1/2(r_0+2)} \\ &\quad\quad \times \left(\int_{R^N} |f(v_n)|^{(2+r_0)q} dx \right)^{1/q(r_0+2)}. \end{aligned} \tag{81}$$

Inductively, we have

$$\begin{aligned}\left(\int_{R^N}|f(v_n)|^{(2+r_{k+1})q}dx\right)^{1/q(r_{k+1}+2)} &\le \left[C(r_k+2)^2\right]^{1/2(r_k+2)} \\ &\quad\times\left(\int_{R^N}|f(v_n)|^{(2+r_k)q}dx\right)^{1/q(r_k+2)} \\ &\le \prod_{i=0}^{k}\left[C(r_i+2)^2\right]^{1/2(r_i+2)} \\ &\quad\times\left(\int_{R^N}|f(v_n)|^{(2+r_0)q}dx\right)^{1/q(r_0+2)},\end{aligned} \tag{82}$$

where $(2+r_i)=d^i(2+r_0)$ $(i=0,1,\ldots,k)$, and

$$\begin{aligned}\prod_{i=0}^{k}\left[C(r_i+2)^2\right]^{\frac{1}{2(r_i+2)}} &= \exp\left\{\sum_{i=0}^{k}\frac{\ln\sqrt{C}d^i(r_0+2)}{d^i(r_0+2)}\right\} \\ &= \exp\left\{\sum_{i=0}^{k}\left[\frac{\ln\sqrt{C}(r_0+2)}{d^i(r_0+2)}+\frac{i\ln d}{d^i(r_0+2)}\right]\right\}\end{aligned} \tag{83}$$

is convergent as $k\to\infty$. Let $C_k=\prod_{i=0}^{k}[C(r_i+2)^2]^{1/2(r_i+2)}$. Then $C_k\to C_\infty>0$ as $k\to\infty$. Hence

$$\|f(v_n)\|_{L^{(2+r_0)qd^{k+1}}}\le C_k\|f(v_n)\|_{L^{2(2^*)}}. \tag{84}$$

Let $k\to\infty$; by (65), we have

$$\|f(v_n)\|_{L^\infty}\le C_\infty\|f(v_n)\|_{L^{2(2^*)}}\le C,\qquad \|f(v)\|_{L^\infty}\le C. \tag{85}$$

Hence, by (f_9) and (85), we have

$$\begin{aligned}\int_{R^N}V(x)v_n^2dx &= \int_{\{x:|v_n(x)|\le 1\}}V(x)v_n^2dx \\ &\quad+\int_{\{x:|v_n(x)|>1\}}V(x)v_n^2dx \\ &\le \frac{1}{C}\int_{\{x:|v_n(x)|\le 1\}}V(x)f^2(v_n)dx \\ &\quad+\frac{1}{C}\int_{\{x:|v_n(x)|>1\}}V(x)f^4(v_n)dx \\ &\le \frac{1}{C}\int_{\{x:|v_n(x)|\le 1\}}V(x)f^2(v_n)dx \\ &\quad+C\int_{\{x:|v_n(x)|\ge 1\}}V(x)f^2(v_n)dx \\ &\le C\int_{R^N}V(x)f^2(v_n)dx.\end{aligned} \tag{86}$$

By (63) we know that $\int_{R^N}V(x)v_n^2dx$ is bounded, and hence $\{v_n\}$ is bounded in E. Up to subsequence, one has $v_n\rightharpoonup v$ in E, $v_n\to v$ in $L^s(R^N)$ for $s\in[2,2^*)$, and $v_n(x)\to v(x)$ a.e. $x\in R^N$.

Now, we show that v is a critical point of I. For any $\psi\in C_0^\infty(R^N)$ with $\psi\ge 0$, by (85), we know that $\phi=\psi\exp(-f(v_n))\in E$. Take $\phi=\psi\exp(-f(v_n))$ as the test function in (61); we have

$$\begin{aligned}0 &= \int_{R^N}\exp(-f(v_n))\nabla v_n\cdot\nabla\psi dx \\ &\quad-\int_{R^N}|\nabla v_n|^2\psi\exp(-f(v_n))f'(v_n)dx \\ &\quad+\theta_n\int_{R^N}V(x)v_n\psi\exp(-f(v_n))dx \\ &\quad+\int_{R^N}V(x)f(v_n)f'(v_n)\psi\exp(-f(v_n))dx \\ &\quad-\int_{R^N}h(x,f(v_n))f'(v_n)\psi\exp(-f(v_n))dx.\end{aligned} \tag{87}$$

By $|\nabla(v_n-v)|^2\psi\exp(-f(v_n))f'(v_n)\ge 0$, one has

$$\begin{aligned}\limsup_{n\to\infty}\int_{R^N}-|\nabla v_n|^2\psi\exp(-f(v_n))f'(v_n)dx \\ \le -\int_{R^N}|\nabla v|^2\psi\exp(-f(v))f'(v)dx.\end{aligned} \tag{88}$$

Since $\theta_n\to 0$, by (63)

$$\theta_n\int_{R^N}V(x)v_n\psi\exp(-f(v_n))dx\longrightarrow 0 \tag{89}$$

as $n\to\infty$. Moreover, notice that $v_n\rightharpoonup v$ in E, $v_n\to v$ in $L^s(R^N)$ for $s\in[2,2^*)$, and $v_n(x)\to v(x)$ a.e. $x\in R^N$; by Hölder inequality and Lebesgue theorem, we have

$$\begin{aligned}&\int_{R^N}\exp(-f(v_n))\nabla v_n\cdot\nabla\psi dx \\ &\quad\longrightarrow\int_{R^N}\exp(-f(v))\nabla v\cdot\nabla\psi dx, \\ &\int_{R^N}V(x)f(v_n)f'(v_n)\psi\exp(-f(v_n))dx \\ &\quad\longrightarrow\int_{R^N}V(x)f(v)f'(v)\psi\exp(-f(v))dx, \\ &\int_{R^N}h(x,f(v_n))f'(v_n)\psi\exp(-f(v_n))dx \\ &\quad\longrightarrow\int_{R^N}h(x,f(v))f'(v)\psi\exp(-f(v))dx.\end{aligned} \tag{90}$$

Hence, from (87), we have

$$\begin{aligned}0 \le & \int_{R^N} \exp(-f(v)) \nabla v \cdot \nabla \psi dx \\ & - \int_{R^N} |\nabla v|^2 \psi \exp(-f(v)) f'(v) dx \\ & + \theta_n \int_{R^N} V(x) v \psi \exp(-f(v)) dx \\ & + \int_{R^N} V(x) f(v) f'(v) \psi \exp(-f(v)) dx \\ & - \int_{R^N} h(x, f(v)) f'(v) \psi \exp(-f(v)) dx \\ = & \int_{R^N} \nabla v \cdot \nabla (\psi \exp(-f(v))) dx \\ & + \int_{R^N} V(x) f(v) f'(v) \psi \exp(-f(v)) dx \\ & - \int_{R^N} h(x, f(v)) f'(v) \psi \exp(-f(v)) dx.\end{aligned} \tag{91}$$

For any $\varphi \in E$ with $\varphi \ge 0$, by (85) we know that $\zeta := \varphi \exp(f(v)) \in E$. By Theorem 2.8 in [19], there exists a sequence $\{\psi_n\} \subset C_0^\infty(R^N)$ such that $\psi_n \ge 0$ and $\psi_n \to \zeta$ and $\psi_n(x) \to \zeta(x)$ for a.e. $x \in R^N$. Take $\psi = \psi_n$ in (91), and let $n \to \infty$; we have

$$\begin{aligned}0 \le & \int_{R^N} \nabla v \cdot \nabla \varphi dx + \int_{R^N} V(x) f(v) f'(v) \varphi dx \\ & - \int_{R^N} h(x, f(v)) f'(v) \varphi dx.\end{aligned} \tag{92}$$

The opposite inequality can be obtained by taking $\phi = \psi \exp(f(v_n))$ and $\zeta = \varphi \exp(-f(v))$. Consequently,

$$\begin{aligned}& \int_{R^N} \nabla v \cdot \nabla \varphi dx + \int_{R^N} V(x) f(v) f'(v) \varphi dx \\ & - \int_{R^N} h(x, f(v)) f'(v) \varphi dx = 0, \quad \forall \varphi \in E.\end{aligned} \tag{93}$$

This shows that $v \in E$ is a critical point of I, and by taking $\varphi = f(v)/f'(v) \in E$, one has

$$\begin{aligned}& \int_{R^N} \left[1 + \frac{2f^2(v)}{1 + 2f^2(v)}\right] |\nabla v|^2 dx + \int_{R^N} V(x) f^2(v) dx \\ & - \int_{R^N} h(x, f(v)) f(v) dx = 0.\end{aligned} \tag{94}$$

Finally, taking $\phi = f(v_n)/f'(v_n)$ as the test function in (61), we have

$$\begin{aligned}& \int_{R^N} \left[1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right] |\nabla v_n|^2 dx + \theta_n \int_{R^N} V(x) \frac{v_n f(v_n)}{f'(v_n)} dx \\ & + \int_{R^N} V(x) f^2(v_n) dx \\ & - \int_{R^N} h(x, f(v_n)) f(v_n) dx = 0.\end{aligned} \tag{95}$$

Since

$$\begin{aligned}& \int_{R^N} h(x, f(v_n)) f(v_n) dx \longrightarrow \int_{R^N} h(x, f(v)) f(v) \mathrm{d}x, \\ & \int_{R^N} \left[1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right] |\nabla (v_n - v)|^2 dx \ge 0,\end{aligned} \tag{96}$$

by Fatou's Lemma, (63), (94), (95), up to subsequence, one has

$$\begin{aligned}& \theta_n \int_{R^N} V(x) \frac{v_n f(v_n)}{f'(v_n)} dx \longrightarrow 0, \\ & \int_{R^N} |\nabla v_n|^2 dx \longrightarrow \int_{R^N} |\nabla v|^2 dx, \\ & \int_{R^N} \frac{2f^2(v_n)}{1 + 2f^2(v_n)} |\nabla v_n|^2 dx \longrightarrow \int_{R^N} \frac{2f^2(v)}{1 + 2f^2(v)} |\nabla v|^2 dx,\end{aligned} \tag{97}$$

$$\int_{R^N} V(x) f^2(v_n) dx \longrightarrow \int_{R^N} V(x) f^2(v) dx. \tag{98}$$

Hence $I_{\theta_n}(v_n) \to I(v)$ as $n \to \infty$. Set $w_n := v_n - v \in E$. By (f_8), (f_{12}), and (85), one has

$$\begin{aligned}f^2(w_n) = f^2\left(2 \cdot \frac{w_n}{2}\right) & \le C\left[\frac{1}{2} f^2(v_n) + \frac{1}{2} f^2(v)\right] \\ & \le C\left[f^2(v_n) + f^2(v)\right] \le C.\end{aligned} \tag{99}$$

Consequently, by (f_9), (98), and Lemma 4, one has

$$\begin{aligned}& \int_{R^N} V(x) |w_n|^2 dx \\ & = \int_{\{x:|w_n| \le 1\}} V(x) |w_n|^2 dx \\ & \quad + \int_{\{x:|w_n| \ge 1\}} V(x) |w_n|^2 dx \\ & \le C \int_{\{x:|w_n| \le 1\}} V(x) f^2(w_n) dx \\ & \quad + C \int_{\{x:|w_n| \ge 1\}} V(x) f^4(w_n) dx\end{aligned}$$

$$\begin{aligned}&\le C\int_{\{x:|w_n|\le 1\}} V(x) f^2(w_n)\,dx\\&\quad + C\int_{\{x:|w_n|\ge 1\}} V(x) f^2(w_n)\,dx\\&= C\int_{R^N} V(x) f^2(w_n)\,dx \longrightarrow 0\end{aligned} \tag{100}$$

as $n \to \infty$. Therefore, $v_n \to v$ in E. This completes the proof. □

Theorem 6. *Assume conditions* (V), (h_1)–(h_3) *hold; then* (1) *has a weak solution.*

Proof. First, we prove that, for each $\theta \in (0,1]$, I_θ satisfy the Cerami condition. Indeed, let $\{v_n\} \subset E$ be an arbitrary Cerami sequence of I_θ. Set $\phi = f(v_n)/f'(v_n)$. Then $\|\phi\|_E \le C\|v_n\|_E$. Similar to the proof of (63), we can prove that $\{v_n\}$ is bounded in E. Hence, by Lemma 3, the sequence $\{v_n\}$ possesses a convergent subsequence in E. This shows that I_θ satisfy the Cerami condition.

Next, for any $\varepsilon > 0$, by (h_1), (h_2), (f_3), and (f_7), there exists $C(\varepsilon) > 0$ such that

$$H(x, f(t)) \le \varepsilon t^2 + C(\varepsilon)|t|^{p/2} \tag{101}$$

for all $(x,t) \in R^N \times R$. For small $0 < \rho \ll 1$, set

$$S_\rho = \{v \in E : \|v\|_E = \rho\}. \tag{102}$$

Then, from (101), for $v \in S_\rho$,

$$\begin{aligned}I_\theta(v) &= \frac{1}{2}\int_{R^N}\left[|\nabla v|^2 + V(x) f^2(v)\right]dx\\&\quad + \frac{\theta}{2}\int_{R^N} V(x) v^2 dx - \int_{R^N} H(x, f(v))\,dx\\&\ge \frac{\theta}{2}\int_{R^N}\left[|\nabla v|^2 + V(x) v^2\right]dx\\&\quad - \varepsilon\int_{R^N} v^2 dx - C(\varepsilon)\int_{R^N}|v|^{p/2}dx\\&\ge \frac{\theta}{2}\|v\|_E^2 - \varepsilon a_2^2\|v\|_E^2 - C(\varepsilon) a_{p/2}^{p/2}\|v\|_E^{p/2}\\&\ge \rho^2\left(\frac{\theta}{4} - C\rho^{(p-4)/2}\right) \ge \delta > 0\end{aligned} \tag{103}$$

for small $\varepsilon > 0$ and $\rho > 0$. Moreover, by (h_3), for any $(x,z) \in R^N \times R$ with $|z| \ge r$, one has

$$H(x,z) \ge c_0|z|^\mu. \tag{104}$$

Since $\mu > 4$, there exists a constant $4 < \alpha < \min\{\mu, 2(2^*)\}$. Hence, by (f_5), we have

$$\lim_{|t|\to\infty}\frac{H(x,f(t))}{|t|^{\alpha/2}} = \lim_{|t|\to\infty}\frac{H(x,f(t))}{|f(t)|^\alpha}\cdot\frac{|f(t)|^\alpha}{|t|^{\alpha/2}} = +\infty \tag{105}$$

uniformly in $x \in R^N$. Consequently, there exist constants $\tau > 1$ such that

$$H(x, f(t)) \ge |t|^{\alpha/2}, \quad \forall |t| \ge \tau, \tag{106}$$

for all $x \in R^N$. For any finite-dimensional subspace $\widetilde{E} \subset E$, by the equivalency of all norms in the finite-dimensional space, there is a constant $a > 0$ such that

$$\|v\|_{\alpha/2} \ge a\|v\|_E, \quad \forall v \in \widetilde{E}. \tag{107}$$

By (h_1), (h_2), and (106), there exists a positive constant $C > 0$ such that

$$H(x, f(t)) \ge |t|^{\alpha/2} - Ct^2, \quad \forall (x,t) \in R^N \times R. \tag{108}$$

Since $4 < \alpha < 2(2^*)$, by (f_3), (107), and (108), we have

$$\begin{aligned}I_\theta(v) &= \frac{1}{2}\int_{R^N}\left[|\nabla v|^2 + V(x) f^2(v)\right]dx\\&\quad + \frac{\theta}{2}\int_{R^N} V(x) v^2 dx - \int_{R^N} H(x, f(v))\,dx\\&\le \|v\|_E^2 - \|v\|_{\alpha/2}^{\alpha/2} + C\|v\|_2^2\\&\le C\|v\|_E^2 - a^{\alpha/2}\|v\|_E^{\alpha/2}\end{aligned} \tag{109}$$

for all $v \in \widetilde{E}$. Hence there exists a large $R > 0$ such that $I_\theta < 0$ on $\widetilde{E}\setminus B_R$. Set a fixed $e \in \widetilde{E}$ with $\|e\|_E = 1$. For any fixed $T > \rho$, define the path $h_T : [0,1] \mapsto \widetilde{E} \subset E$ by $h_T(t) = tTe$. Then for large $T > 0$, by (109), one has

$$\begin{aligned}&I_\theta(h_T(1)) \le CT^2 - a^{\alpha/2}T^{\alpha/2} < 0,\\&\|h_T(1)\|_E = T > \rho,\\&\sup_{t\in[0,1]} I_\theta(h_T(t)) \le CT^2 < +\infty.\end{aligned} \tag{110}$$

Hence by Theorem 2.2 with the Cerami condition in [20], I_θ possesses a critical value

$$\begin{aligned}&c_\theta := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\theta(\gamma(t)) \ge \delta > 0,\\&c_\theta \le \sup_{t\in[0,1]} I_\theta(h_T(t)) \le CT^2,\end{aligned} \tag{111}$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = h_T(1)\}. \tag{112}$$

Consequently, by Theorem 5, we know that (1) has a weak solution. This completes the proof of Theorem 6. □

Remark 7. Let $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$. Set

$$\begin{aligned}I^\pm(u) &= \frac{1}{2}\int_{R^N}\left[|\nabla v|^2 + V(x) f^2(v)\right]dx\\&\quad - \int_{R^N} H(x, f(v^\pm))\,dx,\\I_\theta^\pm(v) &= \frac{1}{2}\theta\int_{R^N} V(x) v^2 dx + I^\pm(v)\end{aligned} \tag{113}$$

instead of $I(u)$ and $I_\theta(u)$, respectively. Then, under the conditions of Theorem 6, we can obtain the existence of a positive solution and a negative solution for (1).

Theorem 8. *Assume conditions* (V), (h_1)–(h_3) *hold. If* $h(x,s)$ *is odd in* s, *then* (1) *has a sequence* $\{v_m\}$ *of solutions such that* $I(v_m) \to +\infty$.

Proof. Consider the eigenvalue of the operate $L = -\Delta + V$. By assumption (V) and the compactness of the embedding $E \hookrightarrow L^2(R^N)$, we know that the spectrum $\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_n \cdots\}$ of L with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \tag{114}$$

and $\lambda_n \to +\infty$ as $n \to +\infty$ (see page 3820 in [21]). Let φ_n be the eigenfunction corresponding to λ_n. By regularity argument we know that $\varphi_n \in E$. Set $E_n = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Then we can decompose the space E as $E = E_n \oplus W_n$ for $n = 1, 2, \ldots$, where W_n is orthogonal to E_n in E. For $\rho > 0$, set

$$Q_\rho = \left\{ v \in E : \int_{R^N} \left[|\nabla v|^2 + V(x) f^2(v) \right] dx \le \rho^2 \right\}. \tag{115}$$

By (109) there exists $r_n > 0$ independent of θ such that

$$I_\theta(v) < 0, \quad \forall v \in \overline{E_n \setminus Q_{r_n}}. \tag{116}$$

Set

$$\begin{gathered} D_n = E_n \cap Q_{r_n}, \\ G_n = \left\{ \varphi \in C(D_n, E) : \varphi \text{ is odd and } \varphi|_{\partial Q_{r_n} \cap E_n} = id \right\}, \\ \Gamma_j = \left\{ \varphi\left(\overline{D_n \setminus Q_{r_n}}\right) : \varphi \in G_n,\ n \ge j, \right. \\ \left. A = -A \subset E_n \cap Q_{r_n} \text{ is closed and } \gamma(A) \le n - j \right\}, \end{gathered} \tag{117}$$

where $\gamma(\cdot)$ is the genus. Let

$$c_j(\theta) = \inf_{B \in \Gamma_j} \sup_{v \in B} I_\theta, \quad j = 1, 2 \ldots. \tag{118}$$

We have the following three facts (we refer the reader to [16] for their proofs).

Fact (1). For each $B \in \Gamma_j$, if $0 < \rho < r_n$ for all $n \ge j$, then $B \cap \partial Q_\rho \cap W_{j-1} \ne \emptyset$.

Fact (2). There exist constants $\alpha_j \le \beta_j$ such that $c_j(\theta) \in [\alpha_j, \beta_j]$ and $\alpha_j \to +\infty$ as $j \to +\infty$.

Fact (3). $c_j(\theta)$, $j = 1, 2, \ldots$ are critical values of I_θ.

Consequently, Theorem 8 follows from Theorem 5 and the above Facts (2)-(3). This completes the proof. □

Corollary 9. *If the following conditions* (h_4) *and* (h_5) *are used in place of* (h_3); *then the conclusions of Theorem 5, Theorem 6, and Theorem 8 hold:*

(h_4) $\lim_{|s| \to +\infty} \inf H(x,s) > 0$ *uniformly in* $x \in \mathbb{R}^N$,

(h_5) *there exist* $\mu > 4$ *and* $\tau > 0$ *such that*

$$\mu H(x,s) \le h(x,s)\, s \tag{119}$$

for all $x \in \mathbb{R}^N$ *and* $|s| \ge \tau$.

Proof. By (h_4), there are constants $\lambda > 0$ and $r_1 > 0$ such that whenever $|s| \ge r_1$, one has

$$H(x,s) > \lambda, \quad \forall x \in \mathbb{R}^N. \tag{120}$$

Set $r = \max\{\tau, r_1\}$. Then, by (h_5),

$$\begin{gathered} c_0 := \inf_{x \in \mathbb{R}^N, |s| = r} H(x,s) \ge \lambda > 0, \\ \mu H(x,s) \le h(x,s)\, s \end{gathered} \tag{121}$$

for all $x \in \mathbb{R}^N$ and $|s| \ge r$. Therefore, condition (h_3) holds. This completes the proof. □

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