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Research Article

Necessary Conditions for Existence Results of Some Integral System

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In this paper, we give some necessary conditions for the existence of positive solutions for integral systems.

1. Introduction

In this paper, we study the necessary condition for the existence of positive solutions for the following integral system:

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N} |x-y|^{\lambda} v(y)^p dy \quad \text{in } \mathbb{R}^N, \\ v(x) &= \int_{\mathbb{R}^N} |x-y|^{\lambda} u(y)^q dy \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1}$$

where λ, p, and q are real parameters.

As for one single equation

$$u(x) = \int_{\mathbb{R}^N} |x-y|^{\lambda} u(y)^p dy \quad \text{in } \mathbb{R}^N, \tag{2}$$

there are a lot of results of this problem. If $\lambda = \alpha - N$ with $0 < \alpha < N$, then problem (2) is equivalent to the following differential equation:

$$(-\Delta)^{\alpha/2} u(x) = u(x)^p \quad \text{in } \mathbb{R}^N. \tag{3}$$

This problem has been widely studied in the past few years. For example, in order to answer a question raised by Lieb in [1], the authors studied the symmetric property and the uniqueness of solutions for problem (2) in [2]. Later, they studied the integral system (1) in [3]. Also, after the work of [2], Li studied the general form of (2) in [4]. For the case $\lambda < 0$, he obtained similar results to [2] but with less regularity requirement. For the case $\lambda > 0$, he shows that if problem (2) has a nonnegative solution in $\mathbb{R}^N$ and $\lambda(p+1) + 2N \geq 0$, then $q = -1 - 2N/\lambda$. The main ingredients in these papers are the moving plane method and moving sphere method based on the maximum principle of integral forms. This method has been widely used in other works. For example, inspired by these works, the author studied the Liouville-type theorems for problems (1) and (2) with general nonlinearities in [5, 6]. For further results of this type of integral equations, see [7–18], and so forth. We note that all these results concern the cases $\lambda < 0$ and $p > 0$. A natural question is whether similar results hold for $\lambda > 0$ or $p < 0$. We note that the case $\lambda < 0$ and $p > 0$ is quite different from the case $\lambda > 0$ or $p < 0$. Generally speaking, the moving plane method or the moving sphere method does not work in the latter case, so we have to look for other methods. In a recent paper [19], the author give a sufficient and necessary condition for the existence of positive solutions for problem (2) with $\lambda > 0$. Based on some integral estimates, the author proved that problem (2) possesses a positive solution if and only if $\lambda p = -(\lambda + 2N)$. Inspired by the work of [19], we first study the integral system (1) with $\lambda > 0$. Our main result is the following theorem.

Theorem 1. *Suppose that $\lambda > 0$ and problem (1) possesses a C^1 positive solution; then*

$$\frac{1}{1+p} + \frac{1}{1+q} = -\frac{\lambda}{N}. \tag{4}$$

As for $\lambda < 0$, we have the following nonexistence result.

Theorem 2. *If $-N < \lambda < 0$, then problem* (1) *possesses no C^1 positive solution provided that $p < 0$ or $q < 0$.*

This paper is organized as follows. We prove Theorem 1 in Section 2. The proof of Theorem 2 is completed in Section 3.

2. Proof of Theorem 1

We first claim that $u(x) \in L^{q+1}(\mathbb{R}^N)$ and $v(x) \in L^{p+1}(\mathbb{R}^N)$.

In fact, we infer from

$$\begin{aligned} v(0) &= \int_{\mathbb{R}^N} |y|^{\lambda} u(y)^q \, dy \\ &\geq \int_{\mathbb{R}^N \setminus B_1(0)} |y|^{\lambda} u(y)^q \, dy \\ &\geq \int_{\mathbb{R}^N \setminus B_1(0)} u(y)^q \, dy \end{aligned} \tag{5}$$

that $u(x) \in L^q(\mathbb{R}^N)$. Also, it follows from (1) that

$$\begin{aligned} |x|^{\lambda} u\left(\frac{x}{|x|^2}\right) &= \int_{\mathbb{R}^N} |x|^{\lambda} \left| \frac{x}{|x|^2} - y \right|^{\lambda} v(y)^p dy \\ &= \int_{\mathbb{R}^N} |y|^{\lambda} \left| x - \frac{y}{|y|^2} \right|^{\lambda} v^p(y) \, dy. \end{aligned} \tag{6}$$

Now taking limit in (6) by letting $x \to 0$, we obtain

$$\lim_{|x| \to 0} \left[|x|^{\lambda} u\left(\frac{x}{|x|^2}\right) \right] = \int_{\mathbb{R}^N} v^p(y) \, dy < \infty. \tag{7}$$

We point out that we can take the limit under the integral sign because of the dominated convergence theorem. In fact, we note that when $\lambda > 0$ and $|x| \leq 1$, we have

$$\left| x - \frac{y}{|y|^2} \right|^{\lambda} \leq \left(|x| + \frac{1}{|y|} \right)^{\lambda} \leq \left(1 + \frac{1}{|y|} \right)^{\lambda}. \tag{8}$$

It is easy to check that $|y|^{\lambda}(1 + 1/|y|)^{\lambda} v^p(y) \in L^1(\mathbb{R}^N)$.

It follows from (7) that there exist $R > 0$ and $C > 0$ such that

$$C^{-1}|x|^{\lambda} \leq u(x) \leq C|x|^{\lambda} \tag{9}$$

for $|x| \geq R$. Finally, we have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} u^{q+1}(x) \, dx &= \int_{\mathbb{R}^N \setminus B_R(0)} u^q u \, dx \\ &\leq C \int_{\mathbb{R}^N} |x|^{\lambda} u^q \, dx \\ &= C v(0), \end{aligned} \tag{10}$$

which further implies that $u(x) \in L^{q+1}(\mathbb{R}^N)$. Similarly, we have $v(x) \in L^{p+1}(\mathbb{R}^N)$.

Next, we can prove as in [19] that

$$\begin{aligned} \nabla u(x) &= \lambda \int_{\mathbb{R}^N} |x - y|^{\lambda - 2} (x - y) v^p(y) \, dy, \\ \nabla v(x) &= \lambda \int_{\mathbb{R}^N} |x - y|^{\lambda - 2} (x - y) u^q(y) \, dy \end{aligned} \tag{11}$$

in the sense of distribution. Hence, we infer from (11) that

$$\begin{aligned} \nabla u^{1+q}(x) = {} & (1 + q) u^q(x) \lambda \\ & \cdot \int_{\mathbb{R}^N} |x - y|^{\lambda - 2} (x - y) v^p(y) \, dy, \\ \nabla v^{1+p}(x) = {} & (1 + p) v^p(x) \lambda \\ & \cdot \int_{\mathbb{R}^N} |x - y|^{\lambda - 2} (x - y) u^q(y) \, dy. \end{aligned} \tag{12}$$

Now we choose a cut-off function $\eta \in C^{\infty}([0, +\infty))$ satisfying $0 \leq \eta \leq 1$, $0 \leq |\eta'| \leq 2$, $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. For any $R > 0$, if we multiply (12) by $\eta(|x|/R)x$ and integrate over $\mathbb{R}^N$, then we get

$$\begin{aligned} & \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \langle x, \nabla v(x) \rangle \, dx \\ & = \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p \int_{\mathbb{R}^N} \lambda |x - y|^{\lambda - 2} \langle x, x - y \rangle u(y)^q dy \, dx, \end{aligned} \tag{13}$$

$$\begin{aligned} & \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) u^q(x) \langle x, \nabla u(x) \rangle \, dx \\ & = \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) u^q(x) \int_{\mathbb{R}^N} \lambda |x - y|^{\lambda - 2} \langle x, x - y \rangle v(y)^p dy \, dx. \end{aligned} \tag{14}$$

While the left-hand side of (13) equals

$$\begin{aligned} & \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \langle x, \nabla v(x) \rangle \, dx \\ & = \frac{1}{1 + p} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) \left\langle x, \nabla v^{1+p}(x) \right\rangle dx \\ & = -\frac{N}{1 + p} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^{1+p}(x) \, dx \\ & \quad - \frac{1}{1 + p} \int_{\mathbb{R}^N} \left\langle \nabla\left(\eta\left(\frac{|x|}{R}\right)\right), x \right\rangle v^{1+p} dx, \end{aligned} \tag{15}$$

it follows from

$$\left\langle \nabla\left(\eta\left(\frac{|x|}{R}\right)\right), x \right\rangle \leq \frac{2|x|}{R} \tag{16}$$

that

$$\begin{aligned} & \int_{\mathbb{R}^N} \left\langle \nabla\left(\eta\left(\frac{|x|}{R}\right)\right), x \right\rangle v^{1+p} \, dx \\ & \leq 4 \int_{R \leq |x| \leq 2R} v^{1+p} \, dx \longrightarrow 0 \end{aligned} \tag{17}$$

as $R \to \infty$ by $v \in L^{1+p}(\mathbb{R}^N)$. Thus we conclude that

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \langle x, \nabla v(x)\rangle \, dx = -\frac{N}{1+p} \int_{\mathbb{R}^N} v^{1+p}(x) \, dx. \tag{18}$$

While the right-hand side of (13) equals

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \cdot \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x, x-y\rangle \, u(y)^q dy\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \cdot \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x-y, x-y\rangle u(y)^q dy\, dx \\
&\quad + \frac{\lambda}{2} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \cdot \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x+y, x-y\rangle u(y)^q dy\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^{1+p}(x)\, dx \\
&\quad + \frac{\lambda}{2} \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \cdot \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x+y, x-y\rangle u(y)^q dy\, dx,
\end{aligned} \tag{19}$$

it can be checked as in [19] that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x+y, x-y\rangle u(y)^q v(x)^p dy\, dx < \infty. \tag{20}$$

Hence, by letting $R \to \infty$ in (19) we get

$$\begin{aligned}
&\lim_{R\to\infty} \lambda \int_{\mathbb{R}^N} \eta\left(\frac{|x|}{R}\right) v^p(x) \cdot \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \langle x, x-y\rangle \, u(y)^q dy\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{1+p}(x)\, dx + \frac{\lambda}{2} \cdot \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \cdot \langle x+y, x-y\rangle \, u^q(y) \, v^p(x)\, dy\, dx.
\end{aligned} \tag{21}$$

We infer from (13), (18), and (21) that

$$\begin{aligned}
&-\frac{N}{1+p} \int_{\mathbb{R}^N} v^{1+p}(x)\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} v^{1+p}(x)\, dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \cdot \langle x+y, x-y\rangle \, u^q(y)\, v^p(x)\, dy\, dx.
\end{aligned} \tag{22}$$

Similarly, we can prove that

$$\begin{aligned}
&-\frac{N}{1+p} \int_{\mathbb{R}^N} u^{1+q}(x)\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} u^{1+q}(x)\, dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \cdot \langle x+y, x-y\rangle \, u^q(x)\, v^p(y)\, dy\, dx \\
&= \frac{\lambda}{2} \int_{\mathbb{R}^N} u^{1+q}(x)\, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda-2} \cdot \langle x+y, x-y\rangle \, u^q(y)\, v^p(x)\, dx\, dy.
\end{aligned} \tag{23}$$

The above two equations imply that

$$-\frac{N}{1+p} \int_{\mathbb{R}^N} u^{1+q}(x)\, dx - \frac{N}{1+p} \int_{\mathbb{R}^N} v^{1+p}(x)\, dx = \frac{\lambda}{2}\left[\int_{\mathbb{R}^N} u^{1+q}(x)\, dx + \int_{\mathbb{R}^N} v^{1+p}(x)\, dx\right]. \tag{24}$$

On the other hand, since

$$v(x)^{1+p} = v(x)^p \int_{\mathbb{R}^N} |x-y|^{\lambda} u^q(y)\, dy, \tag{25}$$

we have

$$\int_{\mathbb{R}^N} v^{1+p}(x)\, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda} u^q(y)\, v(x)^p dy\, dx \tag{26}$$

by taking into account that $v(x) \in L^{1+p}(\mathbb{R}^N)$. Similarly, we have

$$\int_{\mathbb{R}^N} u^{1+q}(x)\, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^{\lambda} u^q(y)\, v(x)^p dy\, dx. \tag{27}$$

Then it follows from (26) and (27) that

$$\int_{\mathbb{R}^N} u^{1+q}(x)\,dx = \int_{\mathbb{R}^N} v^{1+p}(x)\,dx. \tag{28}$$

Finally, we infer from (24) and (28) that

$$\frac{1}{1+p} + \frac{1}{1+q} = -\frac{\lambda}{N}. \tag{29}$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

We assume that $q < 0$ without loss of generality. First, we note that by Lemma 3.11.3 in [20], we have, for all $r > 0$,

$$\begin{aligned}
&\frac{1}{\omega_N r^N}\int_{B_r(0)} u(x)\,dx \\
&\quad= \int_{\mathbb{R}^N}\left\{\frac{1}{\omega_N r^N}\int_{B_r(0)} |x-y|^{\lambda}\,dx\right\} v^p(y)\,dy \\
&\quad\le C\int_{\mathbb{R}^N} |y|^{\lambda} v^p(y)\,dy = Cu(0).
\end{aligned} \tag{30}$$

Similarly, we have

$$\frac{1}{\omega_N r^N}\int_{B_r(0)} v(x)\,dx \le Cv(0) \tag{31}$$

for any $r > 0$.

If we choose $\alpha = q/(q-1)$, $\beta = 1-q$, and $\delta = (q-1)/q$, then we can infer from the Holder inequality that

$$\begin{aligned}
1 &= \frac{1}{\omega_N r^N}\int_{B_r(0)} u(x)^{-\alpha} u(x)^{\alpha} dx \\
&\le \left(\frac{1}{\omega_N r^N}\int_{B_r(0)} u(x)^{-\alpha\beta} dx\right)^{1/\beta} \\
&\quad\cdot\left(\frac{1}{\omega_N r^N}\int_{B_r(0)} u(x)^{\alpha\delta} dx\right)^{1/\delta} \\
&= \left(\frac{1}{\omega_N r^N}\int_{B_r(0)} u(x)^{q} dx\right)^{1/(1-q)} \\
&\quad\cdot (Cu(0))^{q/(q-1)}.
\end{aligned} \tag{32}$$

That is,

$$(Cu(0))^q \le \frac{1}{\omega_N r^N}\int_{B_r(0)} u^q(x)\,dx. \tag{33}$$

Since $\lambda < 0$, so if $|x| < r$, then we have $r^{\lambda} < |x|^{\lambda}$. Multiplying both sides of (33) by $\omega_N r^{N+\lambda}$, we get

$$\begin{aligned}
&C^q \omega_N r^{N+\lambda} u(0)^q \\
&\quad\le \omega_N r^{N+\lambda}\frac{1}{\omega_N r^N}\int_{B_r(0)} u^q(x)\,dx \\
&\quad= r^{\lambda}\int_{B_r(0)} u^q(x)\,dx \\
&\quad\le \int_{B_r(0)} |x|^{\lambda} u^q(x)\,dx = v(0).
\end{aligned} \tag{34}$$

Since $-N < \lambda < 0$, we have $N + \lambda > 0$. Hence the left-hand side of (34) goes to infinity as $r \to \infty$, which is a contradiction. This completes the proof.

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