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Research Article

Bernstein-Type Inequality for Widely Dependent Sequence and Its Application to Nonparametric Regression Models

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We present the Bernstein-type inequality for widely dependent random variables. By using the Bernstein-type inequality and the truncated method, we further study the strong consistency of estimator of fixed design regression model under widely dependent random variables, which generalizes the corresponding one of independent random variables. As an application, the strong consistency for the nearest neighbor estimator is obtained.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. It is well known that the Bernstein-type inequality for the partial sum $\sum_{i=1}^{n} X_i$ plays an important role in probability limit theory and mathematical statistics. The main purpose of the paper is to present the Bernstein-type inequality, by which, we will further investigate the strong consistency for the estimator of nonparametric regression models based on widely dependent random variables.

1.1. Brief Review. Consider the following fixed design regression model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where x_{ni} are known fixed design points from A, where $A \subset \mathbb{R}^p$ is a given compact set for some $p \geq 1$, $g(\cdot)$ is an unknown regression function defined on A, and ε_{ni} are random errors. Assume that for each $n \geq 1$, $(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. As an estimator of $g(\cdot)$, the following weighted regression estimator will be considered:

$$g_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^p, \tag{2}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \ldots, x_{nn})$, $i = 1, 2, \ldots, n$ are the weight functions.

The above estimator was first proposed by Georgiev [1] and subsequently has been studied by many authors. For instance, when ε_{ni} are assumed to be independent, consistency and asymptotic normality have been studied by Georgiev and Greblicki [2], Georgiev [3] and Müller [4] among others. Results for the case when ε_{ni} are dependent have also been studied by various authors in recent years. Fan [5] extended the work of Georgiev [3] and Müller [4] in the estimation of the regression model to the case where it forms an L_q-mixingale sequence for some $1 \leq q \leq 2$. Roussas [6] discussed strong consistency and quadratic mean consistency for $g_n(x)$ under mixing conditions. Roussas et al. [7] established asymptotic normality of $g_n(x)$ assuming that the errors are from a strictly stationary stochastic process and satisfying the strong mixing condition. Tran et al. [8] discussed again asymptotic normality of $g_n(x)$ assuming that the errors form a linear time series, more precisely, a weakly stationary linear process based on a martingale difference sequence. Hu et al. [9] studied the asymptotic normality for double array sum of linear time series. Hu et al. [10] gave the mean consistency, complete consistency, and asymptotic normality of regression models with linear process errors. Liang and Jing [11] presented some asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences. Yang et al. [12] generalized the results of Liang and Jing [11] for negatively associated sequences to the case of negatively orthant dependent sequences and obtained the strong consistency for the estimator of the nonparametric regression models based on negatively orthant dependent errors. Wang et al. [13] studied the complete consistency of the estimator of nonparametric regression models based on $\widetilde{\rho}$-mixing sequences, and so forth. The main purpose of this paper is to investigate the strong consistency for the estimator of the nonparametric regression models based on widely dependent random variables, which contains independent random variables, negatively associated random variables, negatively orthant dependent random variables, extended negatively orthant dependent random variables, and some positively dependent random variables as specials cases. For more details about the strong consistency for the estimator of $g(\cdot)$, Ren and Chen [14] obtained the strong consistency for the least squares estimator of β and the nonparametric estimator of $g(\cdot)$ based on negatively associated samples, Baek and Liang [15] studied the strong consistency for the weighted least squares estimator of β and nonparametric estimator of $g(\cdot)$ in a semi-parametric model under negatively associated samples, which extended the corresponding one on independent random error settings, Liang et al. [16] also studied the strong consistency in a in semiparametric model for a linear process with negatively associated innovations and established the convergence rate, they also pointed out that their results on nonparametric estimator of $g(\cdot)$ can attain the optimal convergence rate, and so forth.

1.2. Concepts of Wide Dependence. In this section, we will present some wide dependence structures introduced in Wang et al. [17].

Definition 1. For the random variables $\{\varepsilon_n,\ n \geq 1\}$, if there exists a finite real sequence $\{g_U(n),\ n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(\varepsilon_1 > x_1, \varepsilon_2 > x_2, \ldots, \varepsilon_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(\varepsilon_i > x_i), \tag{3}$$

then we say that the random variables $\{\varepsilon_n,\ n \geq 1\}$ are widely upper orthant dependent (WUOD); if there exists a finite real sequence $\{g_L(n),\ n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(\varepsilon_1 \leq x_1, \varepsilon_2 \leq x_2, \ldots, \varepsilon_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(\varepsilon_i \leq x_i), \tag{4}$$

then we say that the $\{\varepsilon_n,\ n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{\varepsilon_n,\ n \geq 1\}$ are widely orthant dependent (WOD).

WUOD, WLOD, and WOD random variables are called by a joint name wide dependent (WD) random variables, and $g_U(n)$, $g_L(n)$, $n \geq 1$, are called dominating coefficients.

For examples of WD random variables with various dominating coefficients, we refer the reader to Wang et al. [17]. These examples show that WD random variables contain some common negatively dependent random variables, some positively dependent random variables, and some others. For details about WD random variables, one can refer to Wang et al. [17], Wang and Cheng [18], Wang et al. [19], Chen et al. [20], and so forth.

In what follows, denote $g(n) = \max\{g_U(n), g_L(n)\}$. Recall that when $g_L(n) = g_U(n) = 1$ for any $n \geq 1$ in (3) and (4), the random variables $\{\varepsilon_n, n \geq 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively. If they are both NUOD and NLOD, then we say that the random variables $\{\varepsilon_n, n \geq 1\}$ are negatively orthant dependent (NOD) (see, e.g., Ebrahimi and Ghosh [21], Block et al. [22], Joag-Dev and Proschan [23], Wang et al. [24–26], Wu [27, 28], Wu and Jiang [29], or Wu and Chen [30]).

If both (3) and (4) hold when $g_L(n) = g_U(n) = M$ for some constant M, the random variables $\{X_n, n \geq 1\}$ are called extended negatively upper orthant dependent (ENUOD) and extended negatively lower orthant dependent (ENLOD), respectively. If they are both ENUOD and ENLOD, then we say that the random variables $\{\varepsilon_n, n \geq 1\}$ are extended negatively orthant dependent (ENOD) (see, e.g., Liu [31]). The concept of general extended negative dependence was proposed by Liu [31, 32] and further promoted by Chen et al. [33, 34], Shen [35], Wang and Chen [18], S. J. Wang and W. S. Wang [36] and Wang et al. [37], and so forth.

Wang et al. [17] obtained the following properties for WD random variables, which will be used to prove the main results of the paper.

Proposition 2. *(1) Let $\{\varepsilon_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n)$, $n \geq 1$ $(g_U(n), n \geq 1)$. If $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(\varepsilon_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n)$, $n \geq 1$ $(g_U(n), n \geq 1)$; if $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(\varepsilon_n), n \geq 1\}$ are WUOD (WLOD) with dominating coefficients $g_L(n)$, $n \geq 1$ $(g_U(n), n \geq 1)$.*

(2) If $\{\varepsilon_n, n \geq 1\}$ are nonnegative and WUOD with dominating coefficients $g_U(n)$, $n \geq 1$, then for each $n \geq 1$,

$$E\prod_{i=1}^{n} \varepsilon_i \leq g_U(n) \prod_{i=1}^{n} E\varepsilon_i. \tag{5}$$

In particular, if $\{\varepsilon_n, n \geq 1\}$ are WUOD with dominating coefficients $g_U(n)$, $n \geq 1$, then for each $n \geq 1$ and any $s > 0$,

$$E\exp\left\{s\sum_{i=1}^{n} \varepsilon_i\right\} \leq g_U(n) \prod_{i=1}^{n} E\exp\{s\varepsilon_i\}. \tag{6}$$

By Proposition 2, we can get the following corollary immediately.

Corollary 3. (1) *Let $\{\varepsilon_n, n \geq 1\}$ be WD. If $\{f_n(\cdot), n \geq 1\}$ are nondecreasing (or nonincreasing), then $\{f_n(\varepsilon_n), n \geq 1\}$ are still WD.*

(2) *If $\{X_n, n \geq 1\}$ are WD, then for each $n \geq 1$ and any $s \in \mathbb{R}$,*

$$E\exp\left\{s\sum_{i=1}^{n} \varepsilon_i\right\} \leq g(n) \prod_{i=1}^{n} E\exp\{s\varepsilon_i\}. \tag{7}$$

In this paper, we will present the Bernstein-type inequality for WD random variables. By using the Bernstein-type inequality, we will further investigate the strong consistency for the estimator of nonparametric regression models based on WD errors.

This work is organized as follows: the Bernstein-type inequality for WD random variables is provided in Section 2 and strong consistency for the estimator of nonparametric regression models based on WD errors is investigated in Section 3.

Throughout the paper, C denotes a positive constant not depending on n, which may be different in various places. $a_n = O(b_n)$ represents $a_n \le Cb_n$ for all $n \ge 1$. Let $\lceil x \rceil$ denote the integer part of x and $I(A)$ be the indicator function of the set A. Denote $x^+ = xI(x \ge 0)$ and $x^- = -xI(x < 0)$. Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of WD random variables. Denote $S_n = \sum_{i=1}^n \varepsilon_i$. In the sequel, we will use the following different assumptions in different situations:

$$\lim_{n\to\infty} g(n)\, e^{-an^c} = 0, \tag{8}$$

$$\lim_{n\to\infty} g(n)\, e^{-d\log^{3/2} n} = 0, \tag{9}$$

where a, c, and d are finite positive constants.

2. Bernstein-Type Inequality for WD Random Variables

In this section, we will present the Bernstein-type inequality for WD random variables, which will be used to prove the strong consistency for estimator of the nonparametric regression model based on WD random variables.

Theorem 4. *Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of WD random variables with $E\varepsilon_i = 0$ and $|\varepsilon_i| \le b$ for each $i \ge 1$, where b is a positive constant. Denote $\sigma_i^2 = E\varepsilon_i^2$ and $B_n^2 = \sum_{i=1}^n \sigma_i^2$ for each $n \ge 1$. Then for any $\varepsilon > 0$,*

$$P(S_n \ge \varepsilon) \le g_U(n) \exp\left\{-\frac{\varepsilon^2}{2B_n^2 + (2/3)\, b\varepsilon}\right\}, \tag{10}$$

$$P(|S_n| \ge \varepsilon) \le 2g(n) \exp\left\{-\frac{\varepsilon^2}{2B_n^2 + (2/3)\, b\varepsilon}\right\}. \tag{11}$$

Proof. For any $t > 0$, by Taylor's expansion, $EX_i = 0$ and the inequality $1 + x \le e^x$ for $x \in \mathbb{R}$, we can get that for $i = 1, 2, \ldots, n$,

$$\begin{aligned} E\exp\{t\varepsilon_i\} &= 1 + \sum_{j=2}^{\infty} \frac{E(t\varepsilon_i)^j}{j!} \le 1 + \sum_{j=2}^{\infty} \frac{t^j E|\varepsilon_i|^j}{j!} \\ &= 1 + \frac{t^2\sigma_i^2}{2} \sum_{j=2}^{\infty} \frac{t^{j-2} E|\varepsilon_i|^j}{(1/2)\,\sigma_i^2\, j!} \\ &\doteq 1 + \frac{t^2\sigma_i^2}{2} F_i(t) \le \exp\left\{\frac{t^2\sigma_i^2}{2} F_i(t)\right\}, \end{aligned} \tag{12}$$

where

$$F_i(t) = \sum_{j=2}^{\infty} \frac{t^{j-2} E|\varepsilon_i|^j}{(1/2)\,\sigma_i^2\, j!}, \quad i = 1, 2, \ldots, n. \tag{13}$$

Denote $C = b/3$ and $M_n = b\varepsilon/3B_n^2 + 1$. Choosing $t > 0$ such that $tC < 1$ and

$$tC \le \frac{M_n - 1}{M_n} = \frac{C\varepsilon}{C\varepsilon + B_n^2}. \tag{14}$$

It is easy to check that for $i = 1, 2, \ldots, n$ and $j \ge 2$,

$$E|\varepsilon_i|^j \le \sigma_i^2 b^{j-2} \le \frac{1}{2}\sigma_i^2 C^{j-2} j!, \tag{15}$$

which implies that for $i = 1, 2, \ldots, n$,

$$F_i(t) = \sum_{j=2}^{\infty} \frac{t^{j-2} E|\varepsilon_i|^j}{(1/2)\,\sigma_i^2\, j!} \le \sum_{j=2}^{\infty} (tC)^{j-2} = (1 - tC)^{-1} \le M_n. \tag{16}$$

By Markov's inequality, Corollary 3, (12), and (16), we can get

$$\begin{aligned} P(S_n \ge \varepsilon) &\le e^{-t\varepsilon} E\exp\{tS_n\} \le e^{-t\varepsilon} g_U(n) \prod_{i=1}^{n} E\exp\{t\varepsilon_i\} \\ &\le g_U(n) \exp\left\{-t\varepsilon + \frac{t^2 B_n^2}{2} M_n\right\}. \end{aligned} \tag{17}$$

Taking $t = \varepsilon/B_n^2 M_n = \varepsilon/(C\varepsilon + B_n^2)$. It is easily seen that $tC < 1$ and $tC = C\varepsilon/(C\varepsilon + B_n^2)$. Substituting $t = \varepsilon/B_n^2 M_n$ into the right-hand side of (17), we can obtain (10) immediately. By (10), we have

$$P(S_n \le -\varepsilon) = P(-S_n \ge \varepsilon) \le g_L(n) \exp\left\{-\frac{\varepsilon^2}{2B_n^2 + (2/3)\, b\varepsilon}\right\}, \tag{18}$$

since $\{-\varepsilon_n, n \ge 1\}$ is still a sequence of WD random variables. The desired result (11) follows from (10) and (18) immediately. □

By Theorem 4, we can get the following complete convergence for WD random variables immediately.

Corollary 5. *Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of WD random variables with $E\varepsilon_i = 0$ and $|\varepsilon_i| \le b$ for each $i \ge 1$, where b is a positive constant. Assume that $\sum_{i=1}^\infty E\varepsilon_i^2 < \infty$. $r > 0$. Let the dominating coefficients $g_U(n)$, $g_L(n)$, $n \ge 1$ satisfy* (8) *with any finite positive constant a and $c = r$. Then*

$$n^{-r} S_n \longrightarrow 0, \quad \textit{completely, as } n \longrightarrow \infty. \tag{19}$$

Proof. For any $\varepsilon > 0$, it follows from (11) that

$$\begin{aligned} &\sum_{n=1}^{\infty} P(|S_n| \ge n^r\varepsilon) \\ &\quad \le 2\sum_{n=1}^{\infty} g(n) \exp\left\{-\frac{n^{2r}\varepsilon^2}{2\sum_{i=1}^{\infty} EX_i^2 + (2/3)\, bn^r\varepsilon}\right\} \\ &\quad \le C_1 \sum_{n=1}^{\infty} [\exp(-C)]^{n^r} < \infty, \end{aligned} \tag{20}$$

which implies (19). Here C and C_1 are positive constants not depending on n. □

3. The Strong Consistency for the Estimator of Nonparametric Regression Models Based on WD Errors

Unless otherwise specified, we assume throughout the paper that $g_n(x)$ is defined by (2). For any function $g(x)$, we use $c(g)$ to denote all continuity points of the function g on A. The norm $\|x\|$ is the Euclidean norm. For any fixed design point $x \in A$, the following assumptions on weight functions $W_{ni}(x)$ will be used:

(A_1) $|\sum_{i=1}^{n} W_{ni}(x) - 1| = O(n^{-1/4})$;

(A_2) $\sum_{i=1}^{n} |W_{ni}(x)| \le C$ for all $n \ge 1$ and $\max_{1\le i\le n}|W_{ni}(x)| = O(n^{-1/2}\log^{-3/2} n)$;

(A_3) $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > \sigma n^{-1/4}) = O(n^{-1/4})$ for some $\sigma > 0$.

Theorem 6. *Let $\{\varepsilon_n, n \ge 1\}$ be a sequence of mean zero WD random variables such that $\sup_{n\ge1} E\varepsilon_n^2 < \infty$. Suppose that the conditions (A_1)–(A_3) hold true and (9) holds for any positive constant d. Assume that $g(x)$ satisfies a local Lipschitz condition around the point x. Then for any $x \in A$,*

$$g_n(x) \longrightarrow g(x), \quad \text{as } n \longrightarrow \infty, \text{ a.s.} \tag{21}$$

Proof. For $x \in A$, we have by (1) and (2) that

$$\begin{aligned} &|Eg_n(x) - g(x)| \\ &\quad \le \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I\left(\|x_{ni} - x\| \le \sigma n^{-1/4}\right) \\ &\qquad + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I\left(\|x_{ni} - x\| > \sigma n^{-1/4}\right) \\ &\qquad + |g(x)| \cdot \left|\sum_{i=1}^{n} W_{ni}(x) - 1\right|. \end{aligned} \tag{22}$$

By (22), the conditions (A_1)–(A_3) and the assumption on $g(x)$, we have

$$|Eg_n(x) - g(x)| = O\left(n^{-1/4}\right), \quad x \in A. \tag{23}$$

Hence, to prove (21), we only need to show that

$$g_n(x) - Eg_n(x) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \text{ a.s.} \tag{24}$$

For fixed design point $x \in A$, without loss of generality, we assume that $W_{ni}(x) > 0$ in what follows (otherwise, we use $W_{ni}^{+}(x)$ and $W_{ni}^{-}(x)$ instead of $W_{ni}(x)$, respectively, and note that $W_{ni}(x) = W_{ni}^{+}(x) - W_{ni}^{-}(x)$). Let

$$\begin{aligned} \varepsilon_{1,i}^{(n)} &= -i^{1/2} I\left(\varepsilon_{ni} < -i^{1/2}\right) + \varepsilon_{ni} I\left(|\varepsilon_{ni}| \le i^{1/2}\right) \\ &\quad + i^{1/2} I\left(\varepsilon_{ni} > i^{1/2}\right), \\ \varepsilon_{2,i}^{(n)} &= \left(\varepsilon_{ni} - i^{1/2}\right) I\left(\varepsilon_{ni} > i^{1/2}\right), \\ \varepsilon_{3,i}^{(n)} &= \left(\varepsilon_{ni} + i^{1/2}\right) I\left(\varepsilon_{ni} < -i^{1/2}\right), \\ \varepsilon_{1,i} &= -i^{1/2} I\left(\varepsilon_{i} < -i^{1/2}\right) + \varepsilon_{i} I\left(|\varepsilon_{i}| \le i^{1/2}\right) \\ &\quad + i^{1/2} I\left(\varepsilon_{i} > i^{1/2}\right), \\ \varepsilon_{2,i} &= \left(\varepsilon_{i} - i^{1/2}\right) I\left(\varepsilon_{i} > i^{1/2}\right), \\ \varepsilon_{3,i} &= \left(\varepsilon_{i} + i^{1/2}\right) I\left(\varepsilon_{i} < -i^{1/2}\right). \end{aligned} \tag{25}$$

Since $E\varepsilon_{ni} = E\varepsilon_i = 0$ for each n, it is easy to see that

$$\begin{aligned} g_n(x) - Eg_n(x) &= \sum_{i=1}^{n} W_{ni}(x) \varepsilon_{ni} \\ &= \sum_{i=1}^{n} W_{ni}(x) \left[\varepsilon_{1,i}^{(n)} - E\varepsilon_{1,i}^{(n)}\right] \\ &\quad + \sum_{i=1}^{n} W_{ni}(x) \left[\varepsilon_{2,i}^{(n)} - E\varepsilon_{2,i}^{(n)}\right] \\ &\quad + \sum_{i=1}^{n} W_{ni}(x) \left[\varepsilon_{3,i}^{(n)} - E\varepsilon_{3,i}^{(n)}\right] \\ &=: T_{n1} + T_{n2} + T_{n3}. \end{aligned} \tag{26}$$

By the condition (A_2), we can see that

$$\begin{aligned} \max_{1\le i\le n} \left|W_{ni}(x)\left(\varepsilon_{1,i} - E\varepsilon_{1,i}\right)\right| &\le 2n^{1/2} \max_{1\le i\le n} |W_{ni}(x)| \\ &\le C \log^{-3/2} n, \\ \sum_{i=1}^{n} \operatorname{Var}\left[W_{ni}(x)\left(\varepsilon_{1,i} - E\varepsilon_{1,i}\right)\right] &\le \sum_{i=1}^{n} W_{ni}^2(x) E\varepsilon_i^2 \\ &\le C \max_{1\le i\le n} |W_{n,i}(x)| \sum_{i=1}^{n} |W_{ni}(x)| \\ &\le C n^{-1/2} \log^{-3/2} n. \end{aligned} \tag{27}$$

For fixed $x \in A$ and n, since $(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn})$ have the same distribution as $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $\{W_{ni}(x)(\varepsilon_{1,i} - E\varepsilon_{1,i}),$

$1 \le i \le n\}$ are WD with mean zero, we have by applying Theorem 4 that for every $\epsilon > 0$,

$$\begin{aligned} P\left(\left|T_{n1}\right| \ge \epsilon\right) &= P\left(\left|\sum_{i=1}^{n} W_{ni}(x)\left[\varepsilon_{1,i}^{(n)} - E\varepsilon_{1,i}^{(n)}\right]\right| \ge \epsilon\right) \\ &= P\left(\left|\sum_{i=1}^{n} W_{ni}(x)\left[\varepsilon_{1,i} - E\varepsilon_{1,i}\right]\right| \ge \epsilon\right) \\ &\le 2g(n)\exp\left\{-\frac{\epsilon^2}{Cn^{-1/2}\log^{-3/2}n + C\epsilon\log^{-3/2}n}\right\} \\ &\le 2g(n)\exp\left\{-C\log^{3/2}n\right\} \le Cn^{-2}, \\ &\qquad \text{for } n \text{ large enough,} \end{aligned} \tag{28}$$

which implies

$$T_{n1} = \sum_{i=1}^{n} W_{ni}(x)\left[\varepsilon_{1,i}^{(n)} - E\varepsilon_{1,i}^{(n)}\right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \text{ a.s.} \tag{29}$$

by Borel-Cantelli lemma.

Next, we will estimate T_{n2} and T_{n3}. It can be checked by $\sup_{n\ge1} E\varepsilon_n^2 < \infty$ that

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{E\varepsilon_{2,i}^{(n)}}{i^{1/2}\log^{5/4}(2i)} &= \sum_{i=1}^{\infty} \frac{E\varepsilon_{2,i}}{i^{1/2}\log^{5/4}(2i)} \le \sum_{i=1}^{\infty} \frac{E\left[\varepsilon_i I\left(\varepsilon_i > i^{1/2}\right)\right]}{i^{1/2}\log^{5/4}(2i)} \\ &\le \sum_{i=1}^{\infty} \frac{E\varepsilon_i^2}{i\log^{5/4}(2i)} < \infty, \end{aligned} \tag{30}$$

which implies

$$\sum_{i=1}^{\infty} \frac{\varepsilon_{2,i}^{(n)}}{i^{1/2}\log^{5/4}(2i)} < \infty, \quad \text{a.s.} \tag{31}$$

Consequently, by Kronecker's lemma, we have that

$$\frac{1}{n^{1/2}\log^{5/4}(2n)} \sum_{i=1}^{n} \varepsilon_{2,i}^{(n)} \longrightarrow 0, \quad \text{a.s.} \tag{32}$$

Thus, by the condition (A_2), it is easy to see that

$$\begin{aligned} \left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{2,i}^{(n)}\right| &\le \max_{1\le i\le n}\left|W_{ni}(x)\right| \sum_{i=1}^{n} \varepsilon_{2,i}^{(n)} \\ &\le Cn^{-1/2}\log^{-3/2} n \sum_{i=1}^{n} \varepsilon_{2,i}^{(n)} \\ &= o\left(\log^{-1/4} n\right), \quad \text{a.s.} \end{aligned} \tag{33}$$

By $\sup_{n\ge1} E\varepsilon_n^2 < \infty$ and (A_2) again, we have

$$\begin{aligned} \left|\sum_{i=1}^{n} W_{n,i}(x)E\varepsilon_{2,i}^{(n)}\right| &= \left|\sum_{i=1}^{n} W_{n,i}(x)E\varepsilon_{2,i}\right| \\ &\le \max_{1\le i\le n}\left|W_{ni}(x)\right| \sum_{i=1}^{n} E\left[\left|\varepsilon_i\right| I\left(\left|\varepsilon_i\right| \ge i^{1/2}\right)\right] \\ &\le Cn^{-1/2}\log^{-3/2} n \sum_{i=1}^{n} i^{-1/2} E\left[\varepsilon_i^2 I\left(\left|\varepsilon_i\right| \ge i^{1/2}\right)\right] \\ &= O\left(\log^{-3/2} n\right). \end{aligned} \tag{34}$$

Combining (33) and (34), it follows that

$$\left|T_{n2}\right| = \left|\sum_{i=1}^{n} W_{ni}(x)\left[\varepsilon_{2,i}^{(n)} - E\varepsilon_{2,i}^{(n)}\right]\right| = o\left(\log^{-1/4} n\right), \quad \text{a.s.} \tag{35}$$

Likewise, by $\sup_{n\ge1} E\varepsilon_n^2 < \infty$, we can see that

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{E\left|\varepsilon_{3,i}^{(n)}\right|}{i^{1/2}\log^{5/4}(2i)} &= \sum_{i=1}^{\infty} \frac{E\left|\varepsilon_{3,i}\right|}{i^{1/2}\log^{5/4}(2i)} \\ &\le \sum_{i=1}^{\infty} \frac{-E\left[\varepsilon_i I\left(\varepsilon_i < -i^{1/2}\right)\right]}{i^{1/2}\log^{5/4}(2i)} \\ &\le \sum_{i=1}^{\infty} \frac{E\varepsilon_i^2}{i\log^{5/4}(2i)} < \infty, \end{aligned} \tag{36}$$

which implies

$$\sum_{i=1}^{\infty} \frac{\left|\varepsilon_{3,i}^{(n)}\right|}{i^{1/2}\log^{5/4}(2i)} < \infty, \quad \text{a.s.} \tag{37}$$

Hence, by Kronecker's lemma,

$$\frac{1}{n^{1/2}\log^{5/4}(2n)} \sum_{i=1}^{n} \left|\varepsilon_{3,i}^{(n)}\right| \longrightarrow 0, \quad \text{a.s.} \tag{38}$$

Consequently, we have by (A_2) that

$$\begin{aligned} \left|\sum_{i=1}^{n} W_{ni}(x)\varepsilon_{3,i}^{(n)}\right| &\le \max_{1\le i\le n}\left|W_{ni}(x)\right| \sum_{i=1}^{n} \left|\varepsilon_{3,i}^{(n)}\right| \\ &= o\left(\log^{-1/4} n\right), \quad \text{a.s.} \end{aligned} \tag{39}$$

On the other hand, by (A_2) and $\sup_{n\geq1}E\varepsilon_n^2<\infty$ again, we can see that

$$\begin{aligned}\left|\sum_{i=1}^{n}W_{ni}(x)E\varepsilon_{3,i}^{(n)}\right| &= \left|\sum_{i=1}^{n}W_{ni}(x)E\varepsilon_{3,i}\right|\\ &\leq \max_{1\leq i\leq n}\left|W_{ni}(x)\right|\sum_{i=1}^{n}E\left[\left|\varepsilon_i\right|I\left(\left|\varepsilon_i\right|>i^{1/2}\right)\right]\\ &\leq Cn^{-1/2}\log^{-3/2}n\sum_{i=1}^{n}i^{-1/2}E\left[\varepsilon_i^2I\left(\left|\varepsilon_i\right|>i^{1/2}\right)\right]\\ &= O\left(\log^{-3/2}n\right).\end{aligned}\tag{40}$$

From the statements above, we have

$$\left|T_{n3}\right| = \left|\sum_{i=1}^{n}W_{n,i}(x)\left[\varepsilon_{3,i}^{(n)}-E\varepsilon_{3,i}^{(n)}\right]\right| = o\left(\log^{-1/4}n\right),\quad \text{a.s.}\tag{41}$$

Therefore, (24) follows from (26), (29), (35), and (41) immediately. This completes the proof of the theorem. □

Theorem 7. *Let $\{\varepsilon_n, n\geq1\}$ be a sequence of mean zero WD random variables such that $\sup_{n\geq1}E\varepsilon_n^4<\infty$. Suppose that the conditions (A_1)–(A_3) hold true and (9) holds for any positive constant d. Assume that $g(x)$ satisfies a local Lipschitz condition around the point x. Then for any $x\in A$,*

$$g_n(x)-g(x)=O\left(n^{-1/4}\right),\quad \text{a.s.}\tag{42}$$

Proof. According to (23), we can see that in order to prove (42), we only need to show that

$$\left|g_n(x)-Eg_n(x)\right|=O\left(n^{-1/4}\right),\quad \text{a.s.}\tag{43}$$

We still assume that $W_{ni}(x)>0$ in what follows. The proof is similar to that of Theorem 6. We use the same notations $\varepsilon_{q,i}^{(n)}$, $\varepsilon_{q,i}$ and T_{nq} for $q=1,2,3$ as those in Theorem 6, where $i^{1/2}$ is replaced by $i^{1/4}$. Obviously $\sup_{n\geq1}E\varepsilon_n^4<\infty$ implies $\sup_{n\geq1}E\varepsilon_n^2<\infty$. It follows by (A_2) that

$$\begin{aligned}\max_{1\leq i\leq n}\left|W_{ni}(x)\left(\varepsilon_{1,i}-E\varepsilon_{1,i}\right)\right| &\leq 2n^{1/4}\max_{1\leq i\leq n}\left|W_{n,i}(x)\right|\\ &\leq Cn^{-1/4}\log^{-3/2}n,\\ \sum_{i=1}^{n}\operatorname{Var}\left[W_{ni}(x)\left(\varepsilon_{1,i}-E\varepsilon_{1,i}\right)\right] &\leq \sum_{i=1}^{n}W_{n,i}^2(x)E\varepsilon_i^2\\ &\leq Cn^{-1/2}\log^{-3/2}n.\end{aligned}\tag{44}$$

By applying Theorem 4 and (9), we can see that for every $\epsilon>0$,

$$\begin{aligned}&P\left(\left|T_{n1}\right|\geq\epsilon n^{-1/4}\right)\\ &\quad= P\left(\sum_{i=1}^{n}W_{ni}(x)\left[\varepsilon_{1,i}^{(n)}-E\varepsilon_{1,i}^{(n)}\right]\geq\epsilon n^{-1/4}\right)\\ &\quad= P\left(\sum_{i=1}^{n}W_{ni}(x)\left[\varepsilon_{1,i}-E\varepsilon_{1,i}\right]\geq\epsilon n^{-1/4}\right)\\ &\quad\leq 2g(n)\exp\left\{-\frac{\epsilon^2n^{-1/2}}{Cn^{-1/2}\log^{-3/2}n+C\epsilon n^{-1/2}\log^{-3/2}n}\right\}\\ &\quad\leq 2g(n)\exp\left\{-C\log^{3/2}n\right\}\leq Cn^{-2},\quad \text{for } n \text{ large enough,}\end{aligned}\tag{45}$$

which implies by Borel-Cantelli lemma that

$$n^{1/4}T_{n1}\longrightarrow0,\quad \text{a.s.}\tag{46}$$

Meanwhile, it can be checked by $\sup_{n\geq1}E\varepsilon_n^4<\infty$ that

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{E\varepsilon_{2,i}^{(n)}}{i^{1/4}\log^{3/2}(2i)} = \sum_{i=1}^{\infty}\frac{E\varepsilon_{2,i}}{i^{1/4}\log^{3/2}(2i)} &\leq \sum_{i=1}^{\infty}\frac{E\left[\varepsilon_iI\left(\varepsilon_i>i^{1/4}\right)\right]}{i^{1/4}\log^{3/2}(2i)}\\ &\leq \sum_{i=1}^{\infty}\frac{E\varepsilon_i^4}{i\log^{3/2}(2i)}<\infty,\end{aligned}\tag{47}$$

which implies

$$\sum_{i=1}^{\infty}\frac{\varepsilon_{2,i}^{(n)}}{i^{1/4}\log^{3/2}(2i)}<\infty,\quad \text{a.s.}\tag{48}$$

Then, we have by Kronecker's lemma that

$$\frac{1}{n^{1/4}\log^{3/2}(2n)}\sum_{i=1}^{n}\varepsilon_{2,i}^{(n)}\longrightarrow0,\quad \text{a.s.}\tag{49}$$

Consequently, it follows by (A_2) that

$$\left|\sum_{i=1}^{n}W_{ni}(x)\varepsilon_{2,i}^{(n)}\right|\leq\max_{1\leq i\leq n}\left|W_{ni}(x)\right|\sum_{i=1}^{n}\varepsilon_{2,i}^{(n)}=o\left(n^{-1/4}\right),\quad \text{a.s.,}\tag{50}$$

$$\begin{aligned}\left|\sum_{i=1}^{n}W_{ni}(x)E\varepsilon_{2,i}^{(n)}\right| &= \left|\sum_{i=1}^{n}W_{ni}(x)E\varepsilon_{2,i}\right|\\ &\leq \max_{1\leq i\leq n}\left|W_{ni}(x)\right|\sum_{i=1}^{n}E\left[\left|\varepsilon_i\right|I\left(\left|\varepsilon_i\right|>i^{1/4}\right)\right]\\ &\leq Cn^{-1/2}\log^{-3/2}n\sum_{i=1}^{n}i^{-3/4}E\left[\varepsilon_i^4I\left(\left|\varepsilon_i\right|>i^{1/4}\right)\right]\\ &= O\left(n^{-1/4}\log^{-3/2}n\right).\end{aligned}\tag{51}$$

On the other hand, it can be checked that

$$\sum_{i=1}^{\infty}\frac{E\left|\varepsilon_{3,i}^{(n)}\right|}{i^{1/4}\log^{3/2}(2i)}=\sum_{i=1}^{\infty}\frac{E\left|\varepsilon_{3,i}\right|}{i^{1/4}\log^{3/2}(2i)}\le\sum_{i=1}^{\infty}\frac{-E\left[\varepsilon_i I\left(\varepsilon_i<-i^{1/4}\right)\right]}{i^{1/4}\log^{3/2}(2i)}\le\sum_{i=1}^{\infty}\frac{E\varepsilon_i^4}{i\log^{3/2}(2i)}<\infty,\tag{52}$$

which implies

$$\sum_{i=1}^{\infty}\frac{\left|\varepsilon_{3,i}^{(n)}\right|}{i^{1/4}\log^{3/2}(2i)}<\infty,\quad\text{a.s.}\tag{53}$$

So, by Kronecker's lemma,

$$\frac{1}{n^{1/4}\log^{3/2}(2n)}\sum_{i=1}^{n}\left|\varepsilon_{3,i}^{(n)}\right|\longrightarrow 0,\quad\text{a.s.}\tag{54}$$

Consequently, we have by (A_2) that

$$\left|\sum_{i=1}^{n}W_{ni}(x)\,\varepsilon_{3,i}^{(n)}\right|\le\max_{1\le i\le n}\left|W_{ni}(x)\right|\sum_{i=1}^{n}\left|\varepsilon_{3,i}^{(n)}\right|=o\left(n^{-1/4}\right),\quad\text{a.s.,}\tag{55}$$

$$\begin{aligned}\left|\sum_{i=1}^{n}W_{ni}(x)\,E\varepsilon_{3,i}^{(n)}\right|&=\left|\sum_{i=1}^{n}W_{ni}(x)\,E\varepsilon_{3,i}\right|\\&\le\max_{1\le i\le n}\left|W_{ni}(x)\right|\sum_{i=1}^{n}E\left[\left|\varepsilon_i\right|I\left(\left|\varepsilon_i\right|>i^{1/4}\right)\right]\\&\le Cn^{-1/2}\log^{-3/2}n\sum_{i=1}^{n}i^{-3/4}E\left[\varepsilon_i^4 I\left(\left|\varepsilon_i\right|>i^{1/4}\right)\right]\\&=O\left(n^{-1/4}\log^{-3/2}n\right).\end{aligned}\tag{56}$$

Finally, similar to the proof of (21), we can get (43) immediately by (46)–(56). This completes the proof of the theorem. □

As an application of Theorems 6 and 7, we give the strong consistency for the nearest neighbor estimator of $g(x)$. Without loss of generality, put $A=[0,1]$, taking $x_{ni}=i/n$, $i=1,2,\ldots,n$. For any $x\in A$, we rewrite $|x_{n1}-x|,|x_{n2}-x|,\ldots,|x_{nn}-x|$ as follows:

$$\left|x_{R_1(x)}^{(n)}-x\right|\le\left|x_{R_2(x)}^{(n)}-x\right|\le\cdots\le\left|x_{R_n(x)}^{(n)}-x\right|,\tag{57}$$

if $|x_{ni}-x|=|x_{nj}-x|$, then $|x_{ni}-x|$ is permuted before $|x_{nj}-x|$ when $x_{ni}<x_{nj}$.

Let $1\le k_n\le n$, the nearest neighbor weight function estimator of $g(x)$ in model (1) is defined as follows:

$$\widetilde{g}_n(x)=\sum_{i=1}^{n}\widetilde{W}_{ni}(x)\,Y_{ni},\tag{58}$$

where

$$\widetilde{W}_{ni}(x)=\begin{cases}\dfrac{1}{k_n}, & \text{if } \left|x_{ni}-x\right|\le\left|x_{R_{k_n}(x)}^{(n)}-x\right|,\\ 0, & \text{otherwise.}\end{cases}\tag{59}$$

Based on the notations above, we can get the following result by using Theorems 6 and 7.

Corollary 8. *Let $\{\varepsilon_n,n\ge 1\}$ be a sequence of mean zero WD random variables and (9) holds for any positive constant d. Assume that $g(x)$ satisfies a local Lipschitz condition around the point x. Denote $k_n=\lceil n^{5/8}\rceil$.*

(i) *If $\sup_{n\ge1}E\varepsilon_n^2<\infty$, then (21) holds for any $x\in A$.*

(ii) *If $\sup_{n\ge1}E\varepsilon_n^4<\infty$, then (42) holds for any $x\in A$.*

Proof. It suffices to show that the conditions (A_1)–(A_3) are satisfied. For any $x\in[0,1]$, it follows from the definitions of $R_i(x)$ and $\widetilde{W}_{ni}(x)$ that

$$\begin{gathered}\sum_{i=1}^{n}\widetilde{W}_{ni}(x)=\sum_{i=1}^{n}\widetilde{W}_{nR_i(x)}(x)=\sum_{i=1}^{k_n}\frac{1}{k_n}=1,\\ \max_{1\le i\le n}\widetilde{W}_{ni}(x)=\frac{1}{k_n}\le Cn^{-5/8},\quad \widetilde{W}_{ni}(x)\ge 0,\end{gathered}$$

$$\begin{aligned}&\sum_{i=1}^{n}\left|\widetilde{W}_{ni}(x)\right|\cdot\left|g\left(x_{ni}\right)-g(x)\right|I\left(\left|x_{ni}-x\right|>\sigma n^{-1/4}\right)\\&\quad\le C\sum_{i=1}^{n}\frac{\left(x_{ni}-x\right)^2\left|\widetilde{W}_{ni}(x)\right|}{\sigma^2n^{-1/2}}=C\sum_{i=1}^{k_n}\frac{\left(x_{R_i(x)}^{(n)}-x\right)^2n^{1/2}}{k_n\sigma^2}\\&\quad\le C\sum_{i=1}^{k_n}\frac{(i/n)^2n^{1/2}}{k_n\sigma^2}\le C\left(\frac{k_n}{n\sigma}\right)^2n^{1/2}\le Cn^{-1/4},\quad\forall a>0.\end{aligned}\tag{60}$$

Hence, conditions (A_1)–(A_3) are satisfied. By Theorems 6 and 7, we can get (i) and (ii) immediately. This completes the proof of the corollary. □

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