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Research Article

Dynamics of a Multigroup SIR Epidemic Model with Nonlinear Incidence and Stochastic Perturbation

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We introduce stochasticity into a multigroup SIR model with nonlinear incidence. We prove that when the intensity of white noise is small, the solution of stochastic system converges weakly to a singular measure (i.e., a distribution) if $\mathscr{R}_0 \leq 1$ and there exists an invariant distribution which is ergodic if $\mathscr{R}_0 > 1$. This is the same situation as the corresponding deterministic case. When the intensity of white noise is large, white noise controls this system. This means that the disease will extinct exponentially regardless of the magnitude of $\mathscr{R}_0$.

1. Introduction

Considering different contact patterns, distinct number of sexual partners, or different geography, and so forth, individual hosts are often divided into groups in modeling epidemic diseases. Multigroup models have been proposed in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations. One of the earliest work on multigroup models is the seminal paper by Lajmanovich and Yorke [1] on a class of SIS multigroup models for the transmission dynamics of Gonorrhea. From then on, much research has been done on various forms of multigroup models, see, for example, [2–6]. It is well known that the global stability of the endemic equilibrium of multigroup models is a very challenging problem. Recently, Guo et al. [7, 8] and Li and Shuai [9] proposed a graph-theoretic approach to the method of global Lyapunov functions and completely solved this problem for some multigroup epidemic models. Subsequently, this approach or ideas of [7–9] were applied to the investigation into the dynamics of several classes of multigroup epidemic models (e.g., [10–13]).

Now we consider a deterministic multigroup SIR (susceptible, infective, and recovered) epidemic model with nonlinear incidence:

$$\begin{aligned}
\dot{S}_k &= \Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k \phi_j\left(I_j\right) - d_k S_k, \\
\dot{I}_k &= \sum_{j=1}^{n} \beta_{kj} S_k \phi_j\left(I_j\right) - \left(\epsilon_k + \gamma_k\right) I_k, \\
\dot{R}_k &= \gamma_k I_k - \delta_k R_k, \qquad k = 1, 2, \ldots, n.
\end{aligned} \tag{1}$$

Here $S_k(t)$, $I_k(t)$, and $R_k(t)$ denote the susceptible, infective, and recovered population at time t in the kth group, respectively, $k = 1, 2, \ldots, n$. Suppose the death rates of S_k, I_k, R_k in the kth group are different. The parameters in the model (1) are assumed to be positive and summarized in the following list:

Λ_k: influx of individuals into the kth group;

β_{kj}: transmission coefficient between compartments S_k and I_j;

d_k: death rate of S compartment in the kth group;

ϵ_k: death rate of I compartment in the kth group;

δ_k: death rate of R compartment in the kth group;

γ_k: recovery rate of infectious individuals in the kth group.

Considering that the death rates of infective and recovered population are usually no less than the susceptible's, we assume $d_k \leq \min\{\epsilon_k, \delta_k\}$ for all k.

Throughout this paper, we consider the following basic assumptions on functions $\phi_j(x)$, $x \in (0, +\infty)$, $j = 1, 2, \ldots, n$:

(C_1) $\phi_j(x)$ is a C^1-function on $(0, +\infty)$, $\phi_j(x) > 0$ for $x > 0$ and $\phi_j(0) = 0$;

(C_2) there exist positive constants c_j such that $\phi_j(x) \le c_j x$, $x \in (0, +\infty)$;

(C_3) $\lim_{x \to 0^+}(\phi_j(x)/x) = c_j$;

(C_4) $[\phi_j(x) - \phi_j(y)][\phi_j(x)/x - \phi_j(y)/y] \le 0$, for any $x > 0$ and $y > 0$.

It is easy to check that classes of $\phi_j(x)$ satisfying (C_1)–(C_3) include common incidence functions such as $\phi_j(I_j) = I_j$ [7], $\phi_j(I_j) = I_j/(1 + \alpha_j I_j)$ [14], $\phi_j(I_j) = I_j/(1 + \alpha I_j^2)$ [15]. Furthermore, $\phi_j(I_j) = I_j$, $\phi_j(I_j) = I_j/(1 + \alpha_j I_j)$ satisfy (C_4).

In Sun [13], the author considered a general multigroup SIR models with nonlinear incidence by using the same method as in [9]. Noting that model (1) is a special case of multigroup SIR models appeared in [13], according to the results of [13], the following results hold for system (1).

Proposition 1. *Assume that $B = (\beta_{kj})$ is irreducible and $\phi_j(x)$ satisfies (C_1)–(C_3).*

(1) *If $\mathscr{R}_0 \le 1$, then for system (1), $P_0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_n^0, 0, 0)$ is the unique equilibrium and it is globally asymptotically stable in Γ;*

(2) *if $\mathscr{R}_0 > 1$ and (C_4) is satisfied, then there exists an endemic equilibrium $P^* = (S_1^*, I_1^*, R_1^*, \ldots, S_n^*, I_n^*, R_n^*)$ for system (1) and P^* is globally asymptotically stable in Γ,*

where $S_k^0 = \Lambda_k/d_k$, $k = 1, 2, \ldots, n$, $\mathscr{R}_0 = \rho(M_0)$ (the spectral radius of M_0), $M_0 = M(S_1^0, S_2^0, \ldots, S_n^0) = (\beta_{kj} S_k^0 c_j/(\epsilon_k + \gamma_k))_{n \times n}$, and $\Gamma = \{(S_1, I_1, R_1, \ldots, S_n, I_n, R_n) \in \mathbb{R}_+^{3n} : S_k \le \Lambda_k/d_k, S_k + I_k \le \Lambda_k/d_k, k = 1, 2, \ldots, n\}$.

The aim of this paper is to evaluate the effect of stochastic parameter perturbation on system (1). Considering the existence of environmental noise, we introduce randomness into the model (1) by replacing the parameters d_k, ϵ_k, and δ_k by

$$\begin{aligned} d_k &\longrightarrow d_k + \alpha_k \dot{B}_{1k}(t), \\ \epsilon_k &\longrightarrow \epsilon_k + \beta_k \dot{B}_{2k}(t), \\ \delta_k &\longrightarrow \delta_k + \sigma_k \dot{B}_{3k}(t), \end{aligned} \tag{2}$$

where $B_{1k}(t)$, $B_{2k}(t)$, $B_{3k}(t)$, $k = 1, 2, \ldots, n$ are mutual independent standard Brownian motions with $B_{1k}(0) = 0$, $B_{2k}(0) = 0$, $B_{3k}(0) = 0$, and the intensities of white noises $\alpha_k^2 > 0$, $\beta_k^2 > 0$, $\sigma_k^2 > 0$, respectively. Then the stochastic version corresponding to the deterministic model (1) takes the following form:

$$\begin{aligned} dS_k(t) &= \left[\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k(t) \phi_j\left(I_j(t)\right) - d_k S_k(t)\right] dt \\ &\quad - \alpha_k S_k(t)\, dB_{1k}(t), \\ dI_k(t) &= \left[\sum_{j=1}^{n} \beta_{kj} S_k(t) \phi_j\left(I_j(t)\right) - (\epsilon_k + \gamma_k) I_k(t)\right] dt \\ &\quad - \beta_k I_k(t)\, dB_{2k}(t), \\ dR_k(t) &= \left[\gamma_k I_k(t) - \delta_k R_k(t)\right] dt - \sigma_k R_k(t)\, dB_{3k}(t), \\ &\qquad k = 1, 2, \ldots, n. \end{aligned} \tag{3}$$

System (3) with bilinear incidence has been researched by Ji et al. [16]. They proved that if $\mathscr{R}_0 \le 1$, the solution of the model is fluctuating around a steady state, which is the disease-free equilibrium of the corresponding deterministic model; while if $\mathscr{R}_0 > 1$, there is a stationary distribution, which means the disease will prevail.

In this paper, we will prove that when the intensity of white noise is small, the solution of system (3) converges weakly to a singular measure (i.e., a distribution) if $\mathscr{R}_0 \le 1$ and there exists an invariant distribution which is ergodic if $\mathscr{R}_0 > 1$. This is the same situation as in Proposition 1. When the intensity of white noise is large, the disease will extinct exponentially regardless of the magnitude of $\mathscr{R}_0$. Compared to the results of Ji et al. [16] our results can provide a deep insight into the dynamics of corresponding multigroup model.

It is worth mentioning that by now there are many excellent works about stochastic single-group SIR model. Beretta et al. [17] considered a stochastic SIR model with time delays and obtained asymptotic mean square stability conditions for positive equilibrium. In this paper, the authors assumed that stochastic perturbations are of white noise type, which are directly proportional to distances $S(t)$, $I(t)$, $R(t)$ from values of S^*, I^*, R^*, influence on the $\dot{S}(t)$, $\dot{I}(t)$, $\dot{R}(t)$, respectively. Tornatore et al. [18] discuss a single-group case of (3). They proved that $0 < \beta < \min\{\lambda + \mu - \sigma^2/2, 2\mu\}$ is a sufficient condition for the asymptotic stability of the disease-free equilibrium. And only by computer simulations they showed that if $\min\{\lambda + \mu - \sigma^2/2, 2\mu\} < \beta < \lambda + \mu + \sigma^2/2$, the disease-free equilibrium is stable and the disease does not occur; if $(\lambda + \mu)(R_0 - 1) > \sigma^2/2$, the disease-free equilibrium is unstable. Ji et al. [19] considered the same model as in [18]. They deduce the globally asymptotical stability and exponential mean square stability of the disease-free equilibrium under some conditions and investigate the asymptotic behavior of the solution around the endemic equilibrium of the deterministic model.

The rest of this paper is organized as follows. In Section 2, we show there is a unique nonnegative solution of system (3). In Section 3, if $\mathscr{R}_0 \le 1$ combined with small or large enough intensity of white noise, we show that the solution converges

weakly to a singular measure. Section 4 focuses on the persistence of the disease. By choosing appropriate Lyapunov function, we show that there is a stationary distribution for system (3) and it is ergodic, if $\mathscr{R}_0 > 1$. Finally, for the self-contained, we present an Appendix which contains some results used in the previous sections.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Denote

$$\begin{aligned} \mathbb{R}_+^{3n} &= \left\{x \in \mathbb{R}^{3n} : x_k > 0 \ \forall 1 \leq k \leq 3n\right\}, \\ \overline{\mathbb{R}}_+^{3n} &= \left\{x \in \mathbb{R}^{3n} : x_k \geq 0 \ \forall 1 \leq k \leq 3n\right\}. \end{aligned} \tag{4}$$

In general, consider a d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)\, dt + g(x(t), t)\, dB(t) \quad \text{on } t \geq t_0 \tag{5}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^d$. $B(t)$ denotes d-dimensional standard Brownian motion defined on the above probability space. Define the differential operator L associated with (5) by

$$\begin{aligned} L = \frac{\partial}{\partial t} + \sum_{k=1}^{d} f_k(x,t) \frac{\partial}{\partial x_k} \\ + \frac{1}{2} \sum_{k,j=1}^{d} \left[g^T(x,t)\, g(x,t)\right]_{kj} \frac{\partial^2}{\partial x_k \partial x_j}. \end{aligned} \tag{6}$$

If L acts on a function $V \in C^{2,1}(R^d \times \overline{\mathbb{R}}_+; \overline{\mathbb{R}}_+)$, then

$$\begin{aligned} LV(x,t) = V_t(x,t) + V_x(x,t) f(x,t) \\ + \frac{1}{2} \operatorname{trace}\left[g^T(x,t) V_{xx}(x,t) g(x,t)\right], \end{aligned} \tag{7}$$

where $V_t = \partial V/\partial t$, $V_x = (\partial V/\partial x_1, \ldots, \partial V/\partial x_d)$, $V_{xx} = (\partial^2 V/\partial x_k \partial x_j)_{d\times d}$. By Itô's formula, if $x(t) \in R^d$, then

$$\begin{aligned} dV(x(t),t) = LV(x(t),t)\, dt \\ + V_x(x(t),t)\, g(x(t),t)\, dB(t). \end{aligned} \tag{8}$$

Consider (5), assume $f(0,t) = 0$ and $g(0,t) = 0$ for all $t \geq t_0$. So $x(t) \equiv 0$ is a solution of (5), called the trivial solution or equilibrium position.

2. Existence and Uniqueness of the Nonnegative Solution

For a population model, one is interested in whether the solution is nonnegative and global. Hence in this section we show that the solution of system (3) is global and nonnegative. By making the change of variables and Lyapunov analysis method (see [20]), we will show global existence and uniqueness of the positive solution. From now on, we denote the solution $(S_1(t), I_1(t), R_1(t), \ldots, S_n(t), I_n(t), R_n(t))$ of system (3) as $Y(t)$.

Theorem 2. *If assumption (C_1) holds and $B = (\beta_{kj})_{n\times n}$ is irreducible, then for any initial value $Y(0) \in R_+^{3n}$, there is a unique solution $Y(t)$ of system (3) on $t \geq 0$, and the solution will remain in R_+^{3n} with probability 1.*

Proof. First consider system

$$\begin{aligned} du_k(t) = \left[\Lambda_k e^{-u_k(t)} - \sum_{j=1}^{n} \beta_{kj} \phi_j\left(e^{v_j(t)}\right) - d_k - \frac{\alpha_k^2}{2}\right] dt \\ - \alpha_k dB_{1k}(t), \\ dv_k(t) = \left[\sum_{j=1}^{n} \beta_{kj} e^{u_k(t) - v_k(t)} \phi_j\left(e^{v_j(t)}\right) - (\epsilon_k + \gamma_k) - \frac{\beta_k^2}{2}\right] dt \\ - \beta_k dB_{2k}(t), \\ dw_k(t) = \left[\gamma_k e^{v_k(t)} - \delta_k w_k(t)\right] dt - \sigma_k w_k(t)\, dB_{3k}(t), \\ k = 1, 2, \ldots, n. \end{aligned} \tag{9}$$

With initial value $u_k(0) = \ln S_k(0)$, $v_k(0) = \ln I_k(0)$, $w_k(0) = R_k(0)$, $1 \leq k \leq n$. Noting that (C_1) holds, the coefficients of system (9) satisfy the local Lipschitz condition; then there is a unique local solution $(u_1(t), v_1(t), w_1(t), \ldots, u_n(t), v_n(t), w_n(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Therefore, by Itô's formula, it is easy to see $(e^{u_1(t)}, e^{v_1(t)}, w_1(t), \ldots, e^{u_n(t)}, e^{v_n(t)}, w_n(t))$ is the unique positive local solution to system (3) with initial value $Y(0) \in R_+^{3n}$.

Next, we will prove that this solution is global. The following proof is almost the same as in the proof of Theorem 3.1 in [16]. We do not alter any words except replacing I_j by $\phi_j(I_j)$ and therefore we can obtain that $LV \leq \widetilde{M}$. Here we omit the details. □

3. Exponential Stability of Infectious Disease

It is clear that $P_0 = (\Lambda_1/d_1, 0, 0, \Lambda_2/d_2, 0, 0, \ldots, \Lambda_n/d_n, 0, 0)$ is the disease-free equilibrium of system (1) but not (3). For system (1), which has been mentioned in Introduction, P_0 is globally stable if $\mathscr{R}_0 \leq 1$, which means the disease will die out after some period of time. Hence, it is interesting to study the disease-free equilibrium for controlling infectious disease. Although there is none of disease-free equilibrium of system (3), in this section, we can still present sufficient conditions for the disease to extinct exponentially.

Theorem 3. *Assume $(C_1), (C_2)$ hold and $B = (\beta_{kj})_{n\times n}$ is irreducible. If $2d_k > \alpha_k^2$, $k = 1, 2, \ldots, n$, then the solution $Y(t)$ of system (3) with initial value $Y(0) \in R_+^{3n}$ has the property*

$\lim_{t\to+\infty}(\log(\sum_{k=1}^{n}a_kI_k)/t)\le c$, *a.s. If* $c<0$, *the disease will extinct exponentially almost surely. Here*

$$M_0=M\left(S^0\right)=\left(\frac{\beta_{kj}S_k^0c_j}{\epsilon_k+\gamma_k}\right)_{n\times n},\quad S_k^0=\frac{\Lambda_k}{d_k},$$

$$a_k=\frac{\omega_k}{(\epsilon_k+\gamma_k)},\quad k=1,2,\ldots,n,$$

$(\omega_1,\omega_2,\ldots,\omega_n)$ *is a left eigenvector of* M_0 *corresponding to* $\rho(M_0)$,

$$\begin{aligned}c:=&\max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\sum_{k=1}^{n}\frac{a_k\alpha_kS_k^0}{\sqrt{2d_k-\alpha_k^2}}\\&+\min_{1\le k\le n}\{\epsilon_k+\gamma_k\}(\mathscr{R}_0-1)\mathbf{1}_{\mathscr{R}_0\le1}\\&+\max_{1\le k\le n}\{\epsilon_k+\gamma_k\}(\mathscr{R}_0-1)\mathbf{1}_{\mathscr{R}_0\ge1}-\frac{1}{2\sum_{k=1}^{n}(1/\beta_k^2)}.\end{aligned}\tag{10}$$

Proof. Let X_k, $1\le k\le n$ be the solution of the equation:

$$\begin{aligned}dX_k(t)&=\left[\Lambda_k-d_kX_k(t)\right]dt-\alpha_kX_k(t)\,dB_{1k}(t),\\X_k(0)&=S_k(0).\end{aligned}\tag{11}$$

By comparison theorem for stochastic equations, we have

$$S_k(t)\le X_k(t),\quad \text{a.s.}\tag{12}$$

It is easy to know that $X_k(t)$ has the following property:

$$\lim_{t\to\infty}\frac{\int_0^tX_kdt}{t}=S_k^0\quad\text{a.s.},\tag{13}$$

$$\lim_{t\to\infty}\frac{\int_0^t\left|X_k-S_k^0\right|dt}{t}\le\frac{\alpha_kS_k^0}{\sqrt{2d_k-\alpha_k^2}},\tag{14}$$

$$\text{if } 2d_k>\alpha_k^2,\ k=1,2,\ldots,n.$$

As for the proof of the above two properties, the reader may refer to the proof of Theorems 3.1 and 4.1 in [21] for details.

Since $B=(\beta_{kj})_{n\times n}$ is irreducible, $\beta_{kj}\ge0$, $k,j=1,2,\ldots,n$ and $S_k^0>0$, $\epsilon_k+\gamma_k>0$, $c_k>0$, $k=1,2,\ldots,n$, then M_0 is also nonnegative and irreducible. Hence by Lemma A.1, there is a left eigenvector of M_0 corresponding to $\rho(M_0)$, which is denoted as $(\omega_1,\omega_2,\ldots,\omega_n)$ and $\omega_k>0$, $k=1,2,\ldots,n$; that is,

$$(\omega_1,\omega_2,\ldots,\omega_n)\,\rho(M_0)=(\omega_1,\omega_2,\ldots,\omega_n)\,M_0.\tag{15}$$

Let $a_k=\omega_k/(\epsilon_k+\gamma_k)$, $k=1,2,\ldots,n$. Define C^2-function $V:R_+^n\to R_+$ as follows:

$$V(I_1,I_2,\ldots,I_n)=\sum_{k=1}^{n}a_kI_k.\tag{16}$$

By Itô's formula, we compute

$$d(\log V)=L(\log V)\,dt-\frac{1}{V}\sum_{k=1}^{n}a_k\beta_kI_k\,dB_{2k},\tag{17}$$

where

$$\begin{aligned}L(\log V)&=\frac{1}{V}\sum_{k=1}^{n}a_k\left[\sum_{j=1}^{n}\beta_{kj}S_k\phi_j(I_j)-(\epsilon_k+\gamma_k)I_k\right]\\&\quad-\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2V^2}\\&\le\frac{1}{V}\sum_{k=1}^{n}a_k\left[\sum_{j=1}^{n}\beta_{kj}X_kc_jI_j-(\epsilon_k+\gamma_k)I_k\right]\\&\quad-\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2V^2}\\&=\frac{1}{V}\sum_{k=1}^{n}a_k\left[\sum_{j=1}^{n}\beta_{kj}c_j\left(X_k-S_k^0\right)I_j\right.\\&\qquad\left.+\sum_{j=1}^{n}\beta_{kj}c_jS_k^0I_j-(\epsilon_k+\gamma_k)I_k\right]\\&\quad-\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2V^2}\end{aligned}\tag{18}$$

according to (C_2) and (12). Let

$$\begin{aligned}H_1&=\frac{1}{V}\sum_{k=1}^{n}a_k\sum_{j=1}^{n}\beta_{kj}c_j\left(X_k-S_k^0\right)I_j,\\H_2&=\frac{1}{V}\sum_{k=1}^{n}a_k\left[\sum_{j=1}^{n}\beta_{kj}c_jS_k^0I_j-(\epsilon_k+\gamma_k)I_k\right],\\H_3&=-\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2V^2}.\end{aligned}\tag{19}$$

Then

$$\begin{aligned}H_1&\le\frac{1}{V}\sum_{k=1}^{n}a_k\sum_{j=1}^{n}\beta_{kj}c_j\left|X_k-S_k^0\right|I_j\\&=\frac{1}{V}\sum_{k=1}^{n}\left(a_k\left|X_k-S_k^0\right|\sum_{j=1}^{n}\beta_{kj}c_jI_j\right)\\&\le\max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\frac{1}{V}\sum_{k=1}^{n}\left(a_k\left|X_k-S_k^0\right|\sum_{j=1}^{n}a_jI_j\right)\\&=\max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\sum_{k=1}^{n}\left(a_k\left|X_k-S_k^0\right|\right).\end{aligned}\tag{20}$$

In view of the definition of a_k and (15), we have

$$\begin{aligned}
H_2 &= \frac{1}{V}\left[\sum_{k=1}^{n}\sum_{j=1}^{n} a_k\beta_{kj}c_jS_k^0I_j \right.\\
&\qquad \left. -\sum_{k=1}^{n} a_k\left(\epsilon_k+\gamma_k\right)I_k\right]\\
&= \frac{1}{V}\left[\sum_{k=1}^{n}\sum_{j=1}^{n}\omega_k\frac{\beta_{kj}c_jS_k^0}{\epsilon_k+\gamma_k}I_j - \sum_{k=1}^{n}\omega_kI_k\right]\\
&= \frac{1}{V}\left(\omega_1,\omega_2,\ldots,\omega_n\right)\\
&\quad \times\left(M_0(I_1,I_2,\ldots,I_n)^T - \left(I_1,I_2,\ldots,I_n\right)^T\right)\\
&= \frac{1}{V}\left(\mathscr{R}_0-1\right)\sum_{k=1}^{n}\omega_kI_k\\
&= \frac{1}{V}\left(\mathscr{R}_0-1\right)\sum_{k=1}^{n}\left(\epsilon_k+\gamma_k\right)a_kI_k\\
&\le \min_{1\le k\le n}\left\{\epsilon_k+\gamma_k\right\}\left(\mathscr{R}_0-1\right)\mathbf{1}_{\mathscr{R}_0\le 1}\\
&\quad + \max_{1\le k\le n}\left\{\epsilon_k+\gamma_k\right\}\left(\mathscr{R}_0-1\right)\mathbf{1}_{\mathscr{R}_0\ge 1} := m_1,\\
H_3 &= -\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2V^2} = -\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2\left(\sum_{k=1}^{n}\left(a_k/\beta_k\right)\beta_kI_k\right)^2}\\
&\le -\frac{\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2}{2\left(\sum_{k=1}^{n}a_k^2\beta_k^2I_k^2\right)\left(\sum_{k=1}^{n}\left(1/\beta_k^2\right)\right)}\\
&= -\frac{1}{2\sum_{k=1}^{n}\left(1/\beta_k^2\right)} := m_2.
\end{aligned} \tag{21}$$

Hence,

$$\begin{aligned}
L\left(\log V\right) &\le \max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\\
&\quad \times\sum_{k=1}^{n}\left(a_k\left|X_k-S_k^0\right|\right)+m_1+m_2,
\end{aligned} \tag{22}$$

which combined with (17) yields

$$\begin{aligned}
\frac{\log V(t)}{t} &\le \frac{\log V(0)}{t}\\
&\quad + \max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\sum_{k=1}^{n}\left(\frac{a_k\int_0^t\left|X_k-S_k^0\right|dt}{t}\right)\\
&\quad + m_1+m_2-\sum_{k=1}^{n}a_k\beta_k\frac{\int_0^t I_k/V dB_{2k}}{t}.
\end{aligned} \tag{23}$$

By [22, Lemma 2.6], we get

$$\lim_{t\to+\infty}\frac{\int_0^t I_k/V dB_{2k}}{t} = 0, \quad \text{a.s.} \tag{24}$$

This together with (14) implies

$$\begin{aligned}
\lim_{t\to\infty}\frac{\log V}{t} &\le \max_{0\le k,j\le n}\left\{\frac{\beta_{kj}c_j}{a_j}\right\}\sum_{k=1}^{n}\frac{a_k\alpha_kS_k^0}{\sqrt{2d_k-\alpha_k^2}}\\
&\quad + m_1+m_2, \quad \text{a.s.}
\end{aligned} \tag{25}$$

Thus, the proof of Theorem 3 is completed. □

Remark 4. From Theorem 3 we know that if $\mathscr{R}_0<1$ and α_k, $1\le k\le n$ are small, the disease will extinct exponentially; that is, $I_k\to 0$ as $t\to\infty$. By using the same arguments as in [20, 21], we get $S_k(t)\xrightarrow{w}\nu_k$ and $R_k(t)\to 0$ $(t\to+\infty)$, where $\xrightarrow{w}$ means the weak convergence and ν_k is a distribution in R_+ such that $\int_0^\infty x\nu_k(dx) = S_k^0$ and its density is $(M_kx^2p(x))^{-1}$, where M_k is a normal constant and $p(x) = x^{2d_k/\alpha_k^2}\exp(2d_kS_k^0/\alpha_k^2x)$, $x>0$. That is, the solution of system (3) converges weakly to a singular measure (i.e., a distribution) when $\mathscr{R}_0<1$ and α_k, $1\le k\le n$ are small.

Theorem 5. *Assume* (C_1), (C_2) *hold and* $B=(\beta_{kj})_{n\times n}$ *is irreducible. Then the solution* $Y(t)$ *of system* (3) *with initial value* $Y(0)\in R_+^{3n}$ *has the property* $\lim_{t\to+\infty}(\log(\sum_{k=1}^{n}I_k)/t)\le c$, *a.s. If* $c<0$, *the disease will extinct exponentially almost surely. Here* $c := \max_{1\le k,j\le n}\{\beta_{kj}c_j\}\sum_{k=1}^{n}S_k^0 - \min_{1\le k\le n}\{\epsilon_k+\gamma_k\} - 1/2\sum_{k=1}^{n}(1/\beta_k^2)$.

Proof. Define C^2-function $V: R_+^n\to R_+$ by $V(I_1,I_2,\ldots,I_n)=\sum_{k=1}^{n}I_k$. By Itô's formula, we compute

$$d\left(\log V\right) = L\left(\log V\right)dt - \frac{1}{V}\sum_{k=1}^{n}\beta_kI_kdB_{2k}, \tag{26}$$

where

$$\begin{aligned}
L\left(\log V\right) &= \frac{1}{V}\sum_{k=1}^{n}\left[\sum_{j=1}^{n}\beta_{kj}S_k\phi_j\left(I_j\right)-\left(\epsilon_k+\gamma_k\right)I_k\right]\\
&\quad -\frac{\sum_{k=1}^{n}\beta_k^2I_k^2}{2V^2}\\
&\le \frac{1}{V}\sum_{k=1}^{n}\left[\sum_{j=1}^{n}\beta_{kj}X_kc_jI_j-\left(\epsilon_k+\gamma_k\right)I_k\right]\\
&\quad -\frac{\sum_{k=1}^{n}\beta_k^2I_k^2}{2\left(\sum_{k=1}^{n}\left(1/\beta_k\right)\beta_kI_k\right)^2}\\
&\le \max_{1\le k,j\le n}\left\{\beta_{kj}c_j\right\}\sum_{k=1}^{n}X_k\\
&\quad -\min_{1\le k\le n}\left\{\epsilon_k+\gamma_k\right\}-\frac{1}{2\sum_{k=1}^{n}\left(1/\beta_k^2\right)}.
\end{aligned} \tag{27}$$

Then

$$\frac{\log V(t)}{t} \le \frac{\log V(0)}{t} + \max_{1\le k,j\le n}\left\{\beta_{kj}c_j\right\}\sum_{k=1}^{n}\frac{\int_0^t X_k dt}{t} - \min_{1\le k\le n}\left\{\epsilon_k+\gamma_k\right\} - \frac{1}{2\sum_{k=1}^{n}\left(1/\beta_k^2\right)} - \sum_{k=1}^{n}\frac{\beta_k\int_0^t I_k/V dB_{2k}}{t}. \tag{28}$$

Taking $t \to \infty$, in view of (24) and (13), we get

$$\lim_{t\to+\infty}\frac{\log V(t)}{t} \le \max_{1\le k,j\le n}\left\{\beta_{kj}c_j\right\}\sum_{k=1}^{n}S_k^0 - \min_{1\le k\le n}\left\{\epsilon_k+\gamma_k\right\} - \frac{1}{2\sum_{k=1}^{n}\left(1/\beta_k^2\right)}, \quad \text{a.s.} \tag{29}$$

□

Remark 6. Theorem 5 tells us that the large perturbation forces the infective to expire regardless of the magnitude of $\mathcal{R}_0$.

4. Ergodicity of System (3)

When studying epidemic dynamical systems, we are also interested in when the disease will prevail and persist in a population. For a deterministic model, this is usually solved by showing that the endemic equilibrium is a global attractor or is globally asymptotically stable. But, for the stochastic system (3), there is no endemic equilibrium. In this section, we explore a weak stability. We show that there is a stationary distribution based on the theory of [23] (see Appendix), which reveals the disease will prevail.

From the proof of Theorem 2, we obtain $LV \le \widetilde{M}$. Let $\widetilde{V} = V + \widetilde{M}$. Then $L\widetilde{V} \le \widetilde{V}$ and it is clear that $\inf_{Y\in R_+^{3n}\setminus D_k}\widetilde{V}(Y) \to \infty$ as $k \to \infty$, where $D_k = (1/k,k)\times(1/k,k)\times\cdots\times(1/k,k)$. Hence, by Remark 2 of Theorem 4.1 of Hasminskii ([23], p. 86), we obtain that the solution $Y(t)$ is a homogeneous Markov process in R_+^{3n}.

Theorem 7. *Assume (C_1)–(C_4) hold, $B = (\beta_{kj})_{n\times n}$ is irreducible, and $\mathcal{R}_0 = \rho(M_0) > 1$. If $0 < \alpha_k^2 < d_k$, $0 < \beta_k^2 < \gamma_k + \epsilon_k$, $0 < \sigma_k^2 < \delta_k$, $k = 1,2,\ldots,n$ such that*

$$\delta < \min\left\{\frac{\bar{c}_k}{2}\left(d_k-\alpha_k^2\right)S_k^*,\ \frac{b_k}{4}\left(\epsilon_k+\gamma_k-\beta_k^2\right)\left(I_k^*\right)^2,\ \frac{m_k}{2}\left(\delta_k-\sigma_k^2\right)\left(R_k^*\right)^2,\ k=1,2,\ldots,n\right\}. \tag{30}$$

Then, for any initial value $Y(0) \in R_+^{3n}$, there is a stationary distribution $\mu(\cdot)$ for system (3) and it has ergodic property, where $M_0 = M(S^0) = \left(\beta_{kj}S_k^0c_j/(\epsilon_k+\gamma_k)\right)_{n\times n}$, $S_k^0 = \Lambda_k/d_k$,

$$\delta = \sum_{k=1}^{n}\left[\left(\frac{a+2}{2}\bar{c}_k + b_kS_k^*\right)S_k^*\alpha_k^2 + \left(\frac{a+1}{2}\bar{c}_k + b_kI_k^*\right)I_k^*\beta_k^2 + m_k\left(R_k^*\right)^2\sigma_k^2\right], \tag{31}$$

a, b_k, m_k, $k = 1,2,\ldots,n$ are defined as in the proof, $P^* = (S_1^*, I_1^*, R_1^*, \ldots, S_n^*, I_n^*, R_n^*)$ is the endemic equilibrium of system (1), and $\bar{c}_k$, $k = 1,2,\ldots,n$ denote the cofactor of the kth diagonal element of $L_{\overline{B}}$, $\overline{B} = (\overline{\beta}_{kj})_{n\times n} = (\beta_{kj}S_k^*\phi_j(I_j^*))_{n\times n}$.

Proof. Since $\mathcal{R}_0 > 1$ and (C_1)–(C_4) hold, from Proposition 1 there is an endemic equilibrium P^* of system (1). Then

$$\sum_{j=1}^{n}\beta_{kj}S_k^*\phi_j\left(I_j^*\right) + d_kS_k^* = \Lambda_k, \qquad \sum_{j=1}^{n}\beta_{kj}S_k^*\phi_j\left(I_j^*\right) = \left(\epsilon_k+\gamma_k\right)I_k^*. \tag{32}$$

Define

$$\begin{aligned} V(Y) &= a\sum_{k=1}^{n}\bar{c}_k\left(S_k - S_k^* - S_k^*\log\frac{S_k}{S_k^*} + I_k - I_k^* - I_k^*\log\frac{I_k}{I_k^*}\right) + \sum_{k=1}^{n}\bar{c}_k\left(I_k - I_k^* - \log\frac{I_k}{I_k^*}\right) \\ &\quad + \frac{1}{2}\sum_{k=1}^{n}b_k\left(S_k - S_k^* + I_k - I_k^*\right)^2 + \frac{1}{2}\sum_{k=1}^{n}l_k\left(S_k - S_k^*\right)^2 + \frac{1}{2}\sum_{k=1}^{n}m_k\left(R_k - R_k^*\right)^2 \\ &:= aV_1 + V_2 + V_3 + V_4 + V_5, \end{aligned} \tag{33}$$

where a, b_k, m_k, l_k, $k = 1,2,\ldots,n$ are positive constants to be determined later, and $\bar{c}_k > 0$, $k = 1,2,\ldots,n$ according to Lemma A.1. Then V is positive definite. In view of (32) and the inequality $(x+y)^2 \le 2(x^2+y^2)$, by direct calculation, we get LV_i $(i = 1,2,\ldots,5)$ as follows:

$$\begin{aligned} LV_1 &= \sum_{k=1}^{n}\bar{c}_k\left(1-\frac{S_k^*}{S_k}\right)\left(\Lambda_k - \sum_{j=1}^{n}\beta_{kj}S_k\phi_j\left(I_j\right) - d_kS_k\right) + \sum_{k=1}^{n}\frac{\bar{c}_kS_k^*\alpha_k^2}{2} \\ &\quad + \sum_{k=1}^{n}\bar{c}_k\left(1-\frac{I_k^*}{I_k}\right)\left(\sum_{j=1}^{n}\beta_{kj}S_k\phi_j\left(I_j\right) - \left(\epsilon_k+\gamma_k\right)I_k\right) + \sum_{k=1}^{n}\frac{\bar{c}_kI_k^*\beta_k^2}{2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \overline{c}_k \left(1 - \frac{S_k^*}{S_k}\right) \\
&\quad \times \left(\sum_{j=1}^{n} \overline{\beta}_{kj} \left(1 - \frac{S_k \phi_j(I_j)}{S_k^* \phi_j(I_j^*)}\right) - d_k (S_k - S_k^*)\right) \\
&\quad + \sum_{k=1}^{n} \frac{\overline{c}_k S_k^* \alpha_k^2}{2} \\
&\quad + \sum_{k=1}^{n} \overline{c}_k \left(1 - \frac{I_k^*}{I_k}\right) \\
&\quad \times \left(\sum_{j=1}^{n} \overline{\beta}_{kj} \frac{S_k \phi_j(I_j)}{S_k^* \phi_j(I_j^*)} - \sum_{j=1}^{n} \overline{\beta}_{kj} \frac{I_k}{I_k^*}\right) \\
&\quad + \sum_{k=1}^{n} \frac{\overline{c}_k I_k^* \beta_k^2}{2} \\
&= \sum_{k=1}^{n} \overline{c}_k \left[\sum_{j=1}^{n} \overline{\beta}_{kj} \left(1 - \frac{S_k^*}{S_k} + \frac{\phi_j(I_j)}{\phi_j(I_j^*)} - \frac{S_k \phi_j(I_j)}{S_k^* \phi_j(I_j^*)}\right) \right. \\
&\qquad \left. - d_k \frac{(S_k - S_k^*)^2}{S_k}\right] + \sum_{k=1}^{n} \frac{\overline{c}_k S_k^* \alpha_k^2}{2} \\
&\quad + \sum_{k=1}^{n} \overline{c}_k \left[\sum_{j=1}^{n} \overline{\beta}_{kj} \left(1 - \frac{I_k}{I_k^*} + \frac{S_k \phi_j(I_j)}{S_k^* \phi_j(I_j^*)} \right.\right. \\
&\qquad \left.\left. - \frac{S_k \phi_j(I_j) I_k^*}{S_k^* \phi_j(I_j^*) I_k}\right)\right] \\
&\quad + \sum_{k=1}^{n} \frac{\overline{c}_k I_k^* \beta_k^2}{2} \\
&= \sum_{k=1}^{n} \overline{c}_k \left[-d_k \frac{(S_k - S_k^*)^2}{S_k} \right. \\
&\qquad + \sum_{j=1}^{n} \overline{\beta}_{kj} \left(2 - \frac{S_k^*}{S_k} - \frac{I_k}{I_k^*} + \frac{\phi_j(I_j)}{\phi_j(I_j^*)} \right. \\
&\qquad \left. \left. - \frac{S_k \phi_j(I_j) I_k^*}{S_k^* \phi_j(I_j^*) I_k}\right) \right. \\
&\qquad \left. + \frac{S_k^* \alpha_k^2}{2} + \frac{I_k^* \beta_k^2}{2}\right].
\end{aligned} \tag{34}
$$

From the calculation of LV_1, we directly get

$$
\begin{aligned}
LV_2 &= \sum_{k=1}^{n} \overline{c}_k \left[\sum_{j=1}^{n} \overline{\beta}_{kj} \left(1 - \frac{I_k}{I_k^*} + \frac{S_k \phi_j(I_j)}{S_k^* \phi_j(I_j^*)} - \frac{S_k \phi_j(I_j) I_k^*}{S_k^* \phi_j(I_j^*) I_k}\right)\right] \\
&\quad + \sum_{k=1}^{n} \frac{\overline{c}_k I_k^* \beta_k^2}{2}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
LV_3 &= \sum_{k=1}^{n} b_k \left\{ (S_k - S_k^* + I_k - I_k^*) \right. \\
&\qquad \times [\Lambda_k - d_k S_k - (\epsilon_k + \gamma_k) I_k] \\
&\qquad \left. + \frac{S_k^2 \alpha_k^2}{2} + \frac{I_k^2 \beta_k^2}{2}\right\} \\
&= \sum_{k=1}^{n} b_k \left\{ (S_k - S_k^* + I_k - I_k^*) \right. \\
&\qquad \times [-d_k (S_k - S_k^*) - (\epsilon_k + \gamma_k)(I_k - I_k^*)] \\
&\qquad \left. + \frac{S_k^2 \alpha_k^2}{2} + \frac{I_k^2 \beta_k^2}{2}\right\} \\
&\le -\sum_{k=1}^{n} b_k \left[d_k (S_k - S_k^*)^2 + (\epsilon_k + \gamma_k)(I_k - I_k^*)^2 \right. \\
&\qquad \left. - (\gamma_k + d_k + \epsilon_k)(S_k - S_k^*)(I_k - I_k^*)\right] \\
&\quad + \sum_{k=1}^{n} b_k \alpha_k^2 \left[(S_k - S_k^*)^2 + (S_k^*)^2\right] \\
&\quad + \sum_{k=1}^{n} b_k \beta_k^2 \left[(I_k - I_k^*)^2 + (I_k^*)^2\right] \\
&= -\sum_{k=1}^{n} b_k \left[\left(d_k - \alpha_k^2\right)(S_k - S_k^*)^2 \right. \\
&\qquad \left. + \left(\epsilon_k + \gamma_k - \beta_k^2\right)(I_k - I_k^*)^2\right] \\
&\quad - \sum_{k=1}^{n} b_k (\gamma_k + d_k + \epsilon_k)(S_k - S_k^*)(I_k - I_k^*) \\
&\quad + \sum_{k=1}^{n} b_k \left[\alpha_k^2 (S_k^*)^2 + \beta_k^2 (I_k^*)^2\right] \\
&\le -\sum_{k=1}^{n} b_k \left[\left(d_k - \alpha_k^2 - \frac{(d_k + \gamma_k + \epsilon_k)^2}{2(\gamma_k + d_k + \epsilon_k - \beta_k^2)}\right) \right. \\
&\qquad \left. \times (S_k - S_k^*)^2 + \frac{1}{2}\left(\epsilon_k + \gamma_k - \beta_k^2\right)(I_k - I_k^*)^2\right] \\
&\quad + \sum_{k=1}^{n} b_k \left[\alpha_k^2 (S_k^*)^2 + \beta_k^2 (I_k^*)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
LV_4 &= \sum_{k=1}^{n} l_k (S_k - S_k^*) \\
&\qquad \times \left(\Lambda_k - \sum_{j=1}^{n} \beta_{kj} S_k \phi_j(I_j) - d_k S_k\right) \\
&\quad + \sum_{k=1}^{n} \frac{l_k S_k^2 \alpha_k^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} l_k \left(S_k - S_k^*\right) \\
&\quad \times \left(\sum_{j=1}^{n} \beta_{kj} \left(S_k^* \phi_j \left(I_j^*\right) - S_k \phi_j \left(I_j\right)\right) \right. \\
&\quad \left. -d_k \left(S_k - S_k^*\right) \right) \\
&\quad + \sum_{k=1}^{n} \frac{l_k S_k^2 \alpha_k^2}{2} \\
&= -\sum_{k=1}^{n} \sum_{j=1}^{n} l_k \beta_{kj} S_k^* \\
&\quad \times \left(S_k - S_k^*\right) \left(\phi_j \left(I_j\right) - \phi_j \left(I_j^*\right)\right) \\
&\quad - \sum_{k=1}^{n} \sum_{j=1}^{n} l_k \beta_{kj} \left(S_k - S_k^*\right)^2 \phi_j \left(I_j\right) \\
&\quad - \sum_{k=1}^{n} l_k d_k \left(S_k - S_k^*\right)^2 + \sum_{k=1}^{n} \frac{l_k S_k^2 \alpha_k^2}{2} \\
&\le -\sum_{k=1}^{n} \sum_{j=1}^{n} l_k \beta_{kj} S_k^* \left(S_k - S_k^*\right) \left(\phi_j \left(I_j\right) - \phi_j \left(I_j^*\right)\right) \\
&\quad - \sum_{k=1}^{n} l_k \left(d_k - \alpha_k^2\right) \left(S_k - S_k^*\right)^2 \\
&\quad + \sum_{k=1}^{n} l_k \left(S_k^*\right)^2 \alpha_k^2,
\end{aligned}
$$

$$
\begin{aligned}
LV_5 &= \sum_{k=1}^{n} m_k \left(R_k - R_k^*\right) \left(\gamma_k I_k - \delta_k R_k\right) \\
&\quad + \sum_{k=1}^{n} \frac{m_k R_k^2 \sigma_k^2}{2} \\
&= \sum_{k=1}^{n} m_k \left(R_k - R_k^*\right) \\
&\quad \times \left[\gamma_k \left(I_k - I_k^*\right) - \delta_k \left(R_k - R_k^*\right)\right] \\
&\quad + \sum_{k=1}^{n} \frac{m_k R_k^2 \sigma_k^2}{2} \\
&\le \sum_{k=1}^{n} m_k \gamma_k \left(I_k - I_k^*\right) \left(R_k - R_k^*\right) \\
&\quad - \sum_{k=1}^{n} m_k \left(\delta_k - \sigma_k^2\right) \left(R_k - R_k^*\right)^2 \\
&\quad + \sum_{k=1}^{n} m_k \left(R_k^*\right)^2 \sigma_k^2 \\
&\le \sum_{k=1}^{n} \frac{m_k \gamma_k^2}{2 \left(\delta_k - \sigma_k^2\right)} \left(I_k - I_k^*\right)^2 \\
&\quad - \frac{1}{2} \sum_{k=1}^{n} m_k \left(\delta_k - \sigma_k^2\right) \left(R_k - R_k^*\right)^2 \\
&\quad + \sum_{k=1}^{n} m_k \left(R_k^*\right)^2 \sigma_k^2.
\end{aligned} \tag{36}
$$

By Lemma A.3, we know

$$
\sum_{k=1}^{n} \bar{c}_k \left(\sum_{j=1}^{n} \overline{\beta}_{kj} \frac{\phi_j \left(I_j\right)}{\phi_j \left(I_j^*\right)} - \sum_{j=1}^{n} \overline{\beta}_{kj} \frac{\phi_k \left(I_k\right)}{\phi_k \left(I_k^*\right)} \right) = 0, \tag{37}
$$

$$
\sum_{k=1}^{n} \bar{c}_k \left(\sum_{j=1}^{n} \overline{\beta}_{kj} \log \frac{\phi_j \left(I_j\right)}{\phi_j \left(I_j^*\right)} - \sum_{j=1}^{n} \overline{\beta}_{kj} \log \frac{\phi_k \left(I_k\right)}{\phi_k \left(I_k^*\right)} \right) = 0. \tag{38}
$$

Besides note that $a - 1 - \log a \ge 0$ for $a > 0$; then

$$
1 - \frac{S_k^*}{S_k} \le -\log \frac{S_k^*}{S_k}, \tag{39}
$$

$$
\begin{aligned}
1 - \frac{S_k \phi_j \left(I_j\right) I_k^*}{S_k^* \phi_j \left(I_j^*\right) I_k} &\le -\log \frac{S_k \phi_j \left(I_j\right) I_k^*}{S_k^* \phi_j \left(I_j^*\right) I_k} \\
&= -\log \frac{S_k}{S_k^*} - \log \frac{\phi_j \left(I_j\right)}{\phi_j \left(I_j^*\right)} - \log \frac{I_k^*}{I_k},
\end{aligned} \tag{40}
$$

$$
-\log \frac{\phi_k \left(I_k\right)}{\phi_k \left(I_k^*\right)} - \log \frac{I_k^*}{I_k} = -\log \frac{\phi_k \left(I_k\right) I_k^*}{\phi_k \left(I_k^*\right) I_k} \le \frac{\phi_k \left(I_k^*\right) I_k}{\phi_k \left(I_k\right) I_k^*} - 1. \tag{41}
$$

Successively substituting (39), (40), (37), (38), and (41) into (34) yields

$$
\begin{aligned}
LV_1 \le & -\sum_{k=1}^{n} \frac{\bar{c}_k d_k \left(S_k - S_k^*\right)^2}{S_k} \\
& + \sum_{k=1}^{n} \bar{c}_k \sum_{j=1}^{n} \overline{\beta}_{kj} \left(-\log \frac{S_k^*}{S_k} - \log \frac{S_k}{S_k^*} \right. \\
& \qquad - \log \frac{\phi_j \left(I_j\right)}{\phi_j \left(I_j^*\right)} - \log \frac{I_k^*}{I_k} \\
& \qquad \left. - \frac{I_k}{I_k^*} + \frac{\phi_j \left(I_j\right)}{\phi_j \left(I_j^*\right)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\overline{c}_k\left[\frac{S_k^*\alpha_k^2}{2}+\frac{I_k^*\beta_k^2}{2}\right]\\
&=-\sum_{k=1}^{n}\frac{\overline{c}_k d_k\left(S_k-S_k^*\right)^2}{S_k}\\
&\quad+\sum_{k=1}^{n}\overline{c}_k\sum_{j=1}^{n}\overline{\beta}_{kj}\left(-\log\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}\right.\\
&\qquad\left.-\log\frac{I_k^*}{I_k}-\frac{I_k}{I_k^*}+\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}\right)\\
&\quad+\sum_{k=1}^{n}\overline{c}_k\left[\frac{S_k^*\alpha_k^2}{2}+\frac{I_k^*\beta_k^2}{2}\right]\\
&\le-\sum_{k=1}^{n}\frac{\overline{c}_k d_k\left(S_k-S_k^*\right)^2}{S_k}\\
&\quad+\sum_{k=1}^{n}\overline{c}_k\sum_{j=1}^{n}\overline{\beta}_{kj}\left(\frac{\phi_k\left(I_k^*\right)I_k}{\phi_k\left(I_k\right)I_k^*}-1-\frac{I_k}{I_k^*}+\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}\right)\\
&\quad+\sum_{k=1}^{n}\overline{c}_k\left[\frac{S_k^*\alpha_k^2}{2}+\frac{I_k^*\beta_k^2}{2}\right].
\end{aligned}
\tag{42}
$$

In view of (C_4), we obtain

$$
\begin{aligned}
&\frac{\phi_k\left(I_k^*\right)I_k}{\phi_k\left(I_k\right)I_k^*}-1-\frac{I_k}{I_k^*}+\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}\\
&\quad=\frac{1}{\phi_k\left(I_k^*\right)}\frac{I_k}{\phi_k\left(I_k\right)}\left(\phi_k\left(I_k^*\right)-\phi_k\left(I_k\right)\right)\\
&\qquad\times\left(\frac{\phi_k\left(I_k^*\right)}{I_k^*}-\frac{\phi_k\left(I_k\right)}{I_k}\right)\le 0.
\end{aligned}
\tag{43}
$$

Then

$$
\begin{aligned}
LV_1\le&-\sum_{k=1}^{n}\frac{\overline{c}_k d_k\left(S_k-S_k^*\right)^2}{S_k}\\
&+\sum_{k=1}^{n}\overline{c}_k\left[\frac{S_k^*\alpha_k^2}{2}+\frac{I_k^*\beta_k^2}{2}\right].
\end{aligned}
\tag{44}
$$

By using inequality $a-1-\log a\ge 0$ for $a>0$ again, it follows that

$$
-\log\frac{S_k}{S_k^*}\le\frac{S_k^*}{S_k}-1.
\tag{45}
$$

Successively substituting (40), (37), (45), (41), and (43) into (35), we obtain

$$
\begin{aligned}
LV_2\le&\sum_{k=1}^{n}\overline{c}_k\left[\sum_{j=1}^{n}\overline{\beta}_{kj}\left(-\frac{I_k}{I_k^*}+\frac{S_k\phi_j\left(I_j\right)}{S_k^*\phi_j\left(I_j^*\right)}-\log\frac{S_k}{S_k^*}\right.\right.\\
&\left.\left.-\log\frac{\phi_j\left(I_j\right)}{\phi_j\left(I_j^*\right)}-\log\frac{I_k^*}{I_k}\right)\right]\\
&+\sum_{k=1}^{n}\frac{\overline{c}_k I_k^*\beta_k^2}{2}\\
=&\sum_{k=1}^{n}\overline{c}_k\left[\sum_{j=1}^{n}\overline{\beta}_{kj}\left(-\frac{I_k}{I_k^*}+\frac{S_k\phi_j\left(I_j\right)}{S_k^*\phi_j\left(I_j^*\right)}-\log\frac{S_k}{S_k^*}\right.\right.\\
&\left.\left.-\log\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}-\log\frac{I_k^*}{I_k}\right)\right]\\
&+\sum_{k=1}^{n}\frac{\overline{c}_k I_k^*\beta_k^2}{2}\\
\le&\sum_{k=1}^{n}\overline{c}_k\left[\sum_{j=1}^{n}\overline{\beta}_{kj}\left(-\frac{I_k}{I_k^*}+\frac{S_k\phi_j\left(I_j\right)}{S_k^*\phi_j\left(I_j^*\right)}+\frac{S_k^*}{S_k}\right.\right.\\
&\left.\left.-1+\frac{I_k\phi_k\left(I_k^*\right)}{I_k^*\phi_k\left(I_k\right)}-1\right)\right]\\
&+\sum_{k=1}^{n}\frac{\overline{c}_k I_k^*\beta_k^2}{2}\\
=&\sum_{k=1}^{n}\overline{c}_k\left(\sum_{j=1}^{n}\overline{\beta}_{kj}\left[\left(\frac{S_k}{S_k^*}-1\right)\left(\frac{\phi_j\left(I_j\right)}{\phi_j\left(I_j^*\right)}-1\right)\right.\right.\\
&+\left(\frac{S_k}{S_k^*}+\frac{S_k^*}{S_k}-2\right)\\
&+\left(\frac{\phi_k\left(I_k^*\right)I_k}{\phi_k\left(I_k\right)I_k^*}-1-\frac{I_k}{I_k^*}\right.\\
&\left.\left.\left.+\frac{\phi_k\left(I_k\right)}{\phi_k\left(I_k^*\right)}\right)\right]\right)\\
&+\sum_{k=1}^{n}\frac{\overline{c}_k I_k^*\beta_k^2}{2}\\
\le&\sum_{k=1}^{n}\overline{c}_k\left(\sum_{j=1}^{n}\overline{\beta}_{kj}\left[\left(\frac{S_k}{S_k^*}-1\right)\left(\frac{\phi_j\left(I_j\right)}{\phi_j\left(I_j^*\right)}-1\right)\right.\right.\\
&\left.\left.+\left(\frac{S_k}{S_k^*}+\frac{S_k^*}{S_k}-2\right)\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\frac{\bar{c}_k I_k^* \beta_k^2}{2}\\
&=\sum_{k=1}^{n}\bar{c}_k\left(\sum_{j=1}^{n}\beta_{kj}\left(S_k-S_k^*\right)\left(\phi_j\left(I_j\right)-\phi_j\left(I_j^*\right)\right)\right.\\
&\qquad\left.+\sum_{j=1}^{n}\beta_{kj}\phi_j\left(I_j^*\right)\frac{\left(S_k-S_k^*\right)^2}{S_k}\right)\\
&+\sum_{k=1}^{n}\frac{\bar{c}_k I_k^* \beta_k^2}{2}.
\end{aligned}
\tag{46}
$$

Hence

$$
\begin{aligned}
&aLV_1+LV_2+LV_4\\
&\le-\sum_{k=1}^{n}\left[\sum_{j=1}^{n}\beta_{kj}\left(l_k S_k^*-\bar{c}_k\right)\left(S_k-S_k^*\right)\left(\phi_j\left(I_j\right)-\phi_j\left(I_j^*\right)\right)\right.\\
&\qquad\left.+\bar{c}_k\left(ad_k-\sum_{j=1}^{n}\beta_{kj}\phi_j\left(I_j^*\right)\right)\frac{\left(S_k-S_k^*\right)^2}{S_k}\right]\\
&-\sum_{k=1}^{n}l_k\left(d_k-\alpha_k^2\right)\left(S_k-S_k^*\right)^2\\
&+\sum_{k=1}^{n}\bar{c}_k\left[\frac{\beta_k^2}{2}(a+1)I_k^*+\frac{\alpha_k^2}{2}(a+2)S_k^*\right].
\end{aligned}
\tag{47}
$$

Choose $l_k=\bar{c}_k/S_k^*$ and $a=\max\{\sum_{j=1}^{n}\beta_{kj}I_j^*/d_k,\ k=1,2,\ldots,n\}$ such that $l_kS_k^*-\bar{c}_k=0$ and $ad_k-\sum_{j=1}^{n}\beta_{kj}I_j^*\ge 0$, $k=1,2,\ldots,n$. Then

$$
\begin{aligned}
&aLV_1+LV_2+LV_4\\
&\le-\sum_{k=1}^{n}\frac{\bar{c}_k}{S_k^*}\left(d_k-\alpha_k^2\right)\left(S_k-S_k^*\right)^2\\
&\quad+\sum_{k=1}^{n}\bar{c}_k\left[\frac{\beta_k^2}{2}(a+1)I_k^*+\frac{\alpha_k^2}{2}(a+2)S_k^*\right].
\end{aligned}
\tag{48}
$$

Furthermore,

$$
\begin{aligned}
&aLV_1+LV_2+LV_3+LV_4+LV_5\\
&\le-\sum_{k=1}^{n}\left[b_k\left(d_k-\alpha_k^2-\frac{\left(d_k+\gamma_k+\epsilon_k\right)^2}{2\left(\gamma_k+d_k+\epsilon_k-\beta_k^2\right)}\right)\right.\\
&\qquad\left.+\frac{\bar{c}_k}{S_k^*}\left(d_k-\alpha_k^2\right)\right]\left(S_k-S_k^*\right)^2\\
&-\frac{1}{2}\sum_{k=1}^{n}\left(b_k\left(\epsilon_k+\gamma_k-\beta_k^2\right)-\frac{m_k\gamma_k^2}{\delta_k-\sigma_k^2}\right)\left(I_k-I_k^*\right)^2\\
&-\frac{1}{2}\sum_{k=1}^{n}m_k\left(\delta_k-\sigma_k^2\right)\left(R_k-R_k^*\right)^2\\
&+\sum_{k=1}^{n}\left[\left(\frac{a+2}{2}\bar{c}_k+b_kS_k^*\right)S_k^*\alpha_k^2+\left(\frac{a+1}{2}\bar{c}_k+b_kI_k^*\right)\right.\\
&\qquad\left.\times I_k^*\beta_k^2+m_k\left(R_k^*\right)^2\sigma_k^2\right].
\end{aligned}
\tag{49}
$$

Choose $b_k>0$ such that $b_k(d_k-\alpha_k^2-(d_k+\gamma_k+\epsilon_k)^2/2(\gamma_k+d_k+\epsilon_k-\beta_k^2))+(\bar{c}_k/S_k^*)(d_k-\alpha_k^2)\ge(\bar{c}_k/2S_k^*)(d_k-\alpha_k^2)$ and choose $m_k=(b_k/2)(\epsilon_k+\gamma_k-\beta_k^2)((\delta_k-\sigma_k^2)/\gamma_k^2)$ such that $b_k(\epsilon_k+\gamma_k-\beta_k^2)-m_k\gamma_k^2/(\delta_k-\sigma_k^2)=(b_k/2)(\epsilon_k+\gamma_k-\beta_k^2)$; then

$$
\begin{aligned}
&aLV_1+LV_2+LV_3+LV_4+LV_5\\
&\le-\sum_{k=1}^{n}\frac{\bar{c}_k}{2S_k^*}\left(d_k-\alpha_k^2\right)\left(S_k-S_k^*\right)^2\\
&\quad-\frac{1}{4}\sum_{k=1}^{n}b_k\left(\epsilon_k+\gamma_k-\beta_k^2\right)\left(I_k-I_k^*\right)^2\\
&\quad-\frac{1}{2}\sum_{k=1}^{n}m_k\left(\delta_k-\sigma_k^2\right)\left(R_k-R_k^*\right)^2\\
&\quad+\sum_{k=1}^{n}\left[\left(\frac{a+2}{2}\bar{c}_k+b_kS_k^*\right)S_k^*\alpha_k^2\right.\\
&\qquad\left.+\left(\frac{a+1}{2}\bar{c}_k+b_kI_k^*\right)I_k^*\beta_k^2\right.\\
&\qquad\left.+m_k\left(R_k^*\right)^2\sigma_k^2\right]\\
&:=-\sum_{k=1}^{n}\frac{\bar{c}_k}{2S_k^*}\left(d_k-\alpha_k^2\right)\left(S_k-S_k^*\right)^2\\
&\quad-\sum_{k=1}^{n}\frac{b_k}{4}\left(\epsilon_k+\gamma_k-\beta_k^2\right)\left(I_k-I_k^*\right)^2\\
&\quad-\sum_{k=1}^{n}\frac{m_k}{2}\left(\delta_k-\sigma_k^2\right)\left(R_k-R_k^*\right)^2+\delta.
\end{aligned}
\tag{50}
$$

Note that

$$
\begin{aligned}
\delta<\min\Bigg\{&\frac{\bar{c}_k}{2}\left(d_k-\alpha_k^2\right)S_k^*,\ \frac{b_k}{4}\left(\epsilon_k+\gamma_k-\beta_k^2\right)\left(I_k^*\right)^2,\\
&\frac{m_k}{2}\left(\delta_k-\sigma_k^2\right)\left(R_k^*\right)^2,\ k=1,2,\ldots,n\Bigg\}.
\end{aligned}
\tag{51}
$$

Then the ellipsoid

$$-\sum_{k=1}^{n}\frac{\bar{c}_k}{2S_k^*}\left(d_k-\alpha_k^2\right)\left(S_k-S_k^*\right)^2 -\sum_{k=1}^{n}\frac{b_k}{4}\left(\epsilon_k+\gamma_k-\beta_k^2\right)\left(I_k-I_k^*\right)^2 -\sum_{k=1}^{n}\frac{m_k}{2}\left(\delta_k-\sigma_k^2\right)\left(R_k-R_k^*\right)^2+\delta=0 \tag{52}$$

lies entirely in R_+^{3n}. We can take U to be a neighborhood of the ellipsoid with $\overline{U}\subseteq E_{3n}=R_+^{3n}$, so for $x\in E_{3n}\setminus U$, $LV\le -K$ (K is a positive constant), which implies condition (B.2) in Lemma A.5 is satisfied.

Note that the diffusion matrix of system (3) is

$$A=\left(a_{ij}\right)_{n\times n}=\operatorname{diag}\left(\alpha_1^2S_1^2,\beta_1^2I_1^2,\sigma_1^2R_1^2,\ldots,\alpha_n^2S_n^2,\beta_n^2I_n^2,\sigma_n^2R_n^2\right). \tag{53}$$

For any bounded domain $D\subset R_+^{3n}$ there is

$$M=\min\left\{\alpha_i^2S_i^2,\beta_i^2I_i^2,\sigma_i^2R_i^2,i=1,2,\ldots,n,Y\in\overline{D}\right\}>0 \tag{54}$$

such that

$$\sum_{i,j=1}^{3n}a_{ij}\xi_i\xi_j=\sum_{i=1}^{n}\alpha_i^2S_i^2\xi_{3i-2}^2+\sum_{i=1}^{n}\beta_i^2I_i^2\xi_{3i-1}^2+\sum_{i=1}^{n}\sigma_i^2R_i^2\xi_{3i}^2\ge M\left\|\xi^2\right\| \tag{55}$$

for all $Y\in\overline{D}$, $\xi\in R^{3n}$, which implies condition (B.1) is also satisfied. Therefore, by Lemma A.5, the stochastic system (3) has a stationary distribution $\mu(\cdot)$ and it is ergodic. □

Appendix

For the completeness of the paper, in this section, we list some theory used in the upper sections.

(1) Some graph theory (see [9, 24]): a directed graph or a digraph $\mathcal{G}=(V,E)$ contains a set $V=\{1,2,\ldots,n\}$ of vertices and a set E of arcs (k,j) leading from initial vertex k to terminal vertex j. A digraph $\mathcal{G}$ is weighted if each arc (j,k) is assigned a positive weight a_{kj}. Given a weighted digraph $\mathcal{G}$ with n vertices, define the weight matrix $A=(a_{kj})_{n\times n}$ whose entry a_{kj} equals the weight of arc (j,k) if it exists, and 0 otherwise. A weighted digraph is denoted as $(\mathcal{G},A)$. A digraph $\mathcal{G}$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph $(\mathcal{G},A)$ is strongly connected if and only if the weight matrix A is irreducible [25].

Lemma A.1. *If A is nonnegative and irreducible, then the spectral radius $\rho(A)$ of A is a simple eigenvalue, and A has a positive eigenvector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ corresponding to $\rho(A)$.*

Remark A.2. The result of this lemma is given in P. 27, [25, Theorem 1.4].

The Laplacian matrix of $(\mathcal{G},A)$ is defined as

$$L_A=\begin{bmatrix}\sum_{k\ne1}a_{1k} & -a_{12} & \cdots & -a_{1n}\\ -a_{21} & \sum_{k\ne2}a_{2k} & \cdots & -a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1} & -a_{n2} & \cdots & \sum_{k\ne n}a_{nk}\end{bmatrix}. \tag{A.1}$$

Let c_k denote the cofactor of the kth diagonal element of L_A, which has following property.

Lemma A.3 (see [9]). *Assume $n\ge2$.*

(1) *If $(\mathcal{G},A)$ is strongly connected, then $c_k>0$ for $1\le k\le n$.*

(2) *The following identity holds:*

$$\sum_{k,j=1}^{n}c_ka_{kj}G_k\left(x_k\right)=\sum_{k,j=1}^{n}c_ka_{kj}G_j\left(x_j\right), \tag{A.2}$$

where $G_k(x_k)$, $1\le k\le n$ are arbitrary functions.

Remark A.4. Lemma 2.1 of [8] gives the result (1) of Lemma A.3, and the result (2) is referred to [9, Theorem 2.3].

(2) Some theory about stationary distribution (see [23]): let $X(t)$ be a homogeneous Markov process in E_l (E_l denotes Euclidean l-space) described by the stochastic equation

$$dX(t)=b(X)\,dt+\sum_{r=1}^{k}g_r(X)\,dB_r(t). \tag{A.3}$$

The diffusion matrix is

$$A(x)=\left(a_{ij}(x)\right),\quad a_{ij}(x)=\sum_{r=1}^{k}g_r^i(x)\,g_r^j(x). \tag{A.4}$$

Assumption B. There exists a bounded domain $U\subset E_l$ with regular boundary Γ, having the following properties.

(B.1) In the domain U and some neighbourhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x\in E_l\setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K}E_x\tau<\infty$ for every compact subset $K\subset E_l$.

Lemma A.5 (see [23]). *If (B) holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))\,dt=\int_{E_l}f(x)\,\mu(dx)\right\}=1 \tag{A.5}$$

for all $x\in E_l$.

Remark A.6. The proof is given in [23]. Exactly, the existence of stationary distribution with density is referred to Theorem 4.1, P. 119 and Lemma 9.4, P. 138. The weak convergence and the ergodicity are obtained in Theorem 5.1, P. 121 and Theorem 7.1, P. 130.

To validate (B.1), it suffices to prove F is uniformly elliptical in U, where $Fu = b(x) \cdot u_x + (1/2)\,\mathrm{tr}(A(x)u_{xx})$; that is, there is a positive number M such that

$$\sum_{i,j=1}^{k} a_{ij}(x)\,\xi_i\xi_j \geq M\left|\xi\right|^2, \quad x \in U,\ \xi \in R^k \tag{A.6}$$

(see Chapter 3, P. 103 of [26] and Rayleigh's principle in [27] Chapter 6, P. 349). To verify (B.2), it is sufficient to show that there exists some neighborhood U and a nonnegative C^2-function such that and for any $E_l \setminus U$, LV is negative (for details refer to [28], P. 1163).

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