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Research Article

Positive Periodic Solution of Second-Order Coupled Systems with Singularities

Tiantian Ma

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

Correspondence should be addressed to Tiantian Ma; matiantian1215@163.com

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This paper establishes the existence of periodic solution for a kind of second-order singular nonautonomous coupled systems. Our approach is based on fixed point theorem in cones. Examples are given to illustrate the main result.

1. Introduction

We are concerned with the existence of positive T-periodic solution for the second-order nonautonomous singular coupled systems:

$$\begin{aligned} &u''(t) = f_1(t, v(t)) + c_1(t), \quad t \in (0,T), \\ &v''(t) = f_2(t, u(t)) + c_2(t), \quad t \in (0,T), \\ &u(0) = u(T), \qquad u'(0) = u'(T), \\ &v(0) = v(T), \qquad v'(0) = v'(T), \end{aligned} \tag{1}$$

where $f_i \in C((0,T) \times [0,+\infty), [0,+\infty))$, $c_i : (0,T) \to [0,+\infty)$, $i = 1, 2$, are Lebesgue integrable, f_i may be singular at $t = 0, T$ and c_i can have finitely many singularities.

Singular differential equations or systems arise from many branches of applied mathematics and physics such as gas dynamics, Newtonian fluid mechanics, and nuclear physics, which have been widely studied by many authors (see [1–7] and references therein). Some classical-tools have been used to study the positive solutions for two point nonperiodic boundary value problems of coupled systems [8, 9]. However, there are few works on periodic solutions of second order nonautonomous singular coupled systems of type (1).

In the recent papers [10, 11], the periodic solutions of singular coupled systems

$$\begin{aligned} x'' + a_1(t)x &= f_1(t, y) + c_1(t), \\ y'' + a_2(t)y &= f_2(t, x) + c_2(t), \end{aligned} \tag{2}$$

were proved by using some fixed point theorems in cones for completely continuous operators, where $a_1, a_2, c_1, c_2 \in L^1(0,T)$, $f_1, f_2 \in \mathrm{Car}([0,T] \times (0,+\infty), (0,+\infty))$. When the Green's function $G_i(t,s)$ $(i = 1, 2)$, associated with the periodic boundary problem

$$\begin{gathered} x'' + a_i(t)x = c_i(t), \qquad x(0) = x(T), \\ x'(0) = x'(T), \end{gathered} \tag{3}$$

is nonnegative for all $(t,s) \in [0,T] \times [0,T]$ and f_i $(i = 1, 2)$ satisfies weak singularities

$$\begin{aligned} 0 \le \frac{\widehat{b}_i(t)}{x^{\alpha_i}} &\le f_i(t,x) \\ &\le \frac{b_i(t)}{x^{\alpha_i}}, \quad \forall x > 0, \ \text{a.e. } t \in (0,T), \end{aligned} \tag{4}$$

where $0 < \alpha_i < 1$, $\widehat{b}_i, b_i \in L^1(0,T)$, $\widehat{b}_i \ge 0, b_i \ge 0$, and $\widehat{b}_i$, b_i are strictly positive on some positive measure subsets of $(0,T)$, some sufficient conditions for the existence of periodic solutions of (2) were obtained in [10, 11].

Motivated by the papers [9–11], we consider the existence of positive T-periodic solution of (1). Owing to the disappearing of the terms $a_i(t)$ $(i = 1, 2)$ in (2), the methods in [9, 10] are no longer valid. In present paper, we will deal with the periodic solutions of (1) under new conditions. Let k be a constant satisfying $0 < k < \pi/T$. Denote by $G(t,s)$ the Green function of

$$x''(t) + k^2 x(t) = 0, \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{5}$$

which can be expressed by

$$G(t,s) = \frac{1}{2k(1-\cos kT)} \begin{cases} \sin k(t-s) + \sin k(T-t+s), & 0 \le s \le t \le T, \\ \sin k(s-t) + \sin k(T-s+t), & 0 \le t \le s \le T. \end{cases} \tag{6}$$

By a direct computation, we can get

$$\begin{aligned} m &:= \min_{0\le t,s\le T} G(t,s) = \frac{1}{2k}\cot\frac{kT}{2}, \\ M &:= \max_{0\le t,s\le T} G(t,s) = \frac{1}{2k\sin(kT/2)}, \\ \sigma &:= \frac{m}{M} = \cos\frac{kT}{2} \in (0,1). \end{aligned} \tag{7}$$

Assume f_i $(i = 1, 2)$ satisfies the following conditions.

(H_1) For $t \in (0,T)$, $f_i(t,1) > 0$ and there exist constants $\mu_1 \ge \mu_2 > 1$, $\lambda_1 \ge \lambda_2 > 1$ such that, for any constants $0 \le \rho_i \le 1$, $i = 1, 2$,

$$\rho_1^{\mu_1} f_1(t,u) \le f_1(t,\rho_1 u) \le \rho_1^{\mu_2} f_1(t,u), \quad (t,u) \in (0,T) \times [0,+\infty), \tag{8}$$

$$\rho_2^{\lambda_1} f_2(t,u) \le f_2(t,\rho_2 u) \le \rho_2^{\lambda_2} f_2(t,u), \quad (t,u) \in (0,T) \times [0,+\infty). \tag{9}$$

(H_2) $\int_0^T c_1^-(t)dt = r_1 > 0$, $\int_0^T c_2^-(t)dt = r_2 > 0$ and

$$\begin{aligned} \int_0^T \left(f_1(t,1) + c_1^+(t)\right) dt &< \frac{k^2 T m r_1}{(2M^2 r_2/m + 1)^{\mu_1}}, \\ \int_0^T \left(f_2(t,1) + c_2^+(t)\right) dt &< \frac{k^2 T m r_2}{(2M^2 r_1/m + 1)^{\lambda_1}}, \end{aligned} \tag{10}$$

where $c_i^+(t) = \max\{c_i(t), 0\}$, $c_i^-(t) = \max\{-c_i(t), 0\}$, $i = 1, 2$.

Remark 1. For any $\rho_1 \ge 1$, we get from (8) that

$$\rho_1^{\mu_2} f_1(t,u) \le f_1(t,\rho_1 u) \le \rho_1^{\mu_1} f_1(t,u), \quad (t,u) \in (0,T) \times [0,+\infty). \tag{11}$$

For any $\rho_2 \ge 1$, we get from (9) that

$$\rho_2^{\lambda_2} f_2(t,u) \le f_2(t,\rho_2 u) \le \rho_2^{\lambda_1} f_2(t,u), \quad (t,u) \in (0,T) \times [0,+\infty). \tag{12}$$

Typical functions that satisfy (8) or (9) are those taking the form

$$f_i(t,u) = \sum_{k=1}^{n} a_{ik}(t) u^{l_{ik}}, \quad t \in (0,T), \tag{13}$$

where $a_{ik} \in C(0,T)$, $a_{ik}(t) > 0$, $l_{ik} > 1$, $i = 1, 2$; $k = 1, 2, \ldots, n$.

Definition 2. Supposing that $(u,v) \in C^1[0,T] \cap C^2(0,T) \times C^1[0,T] \cap C^2(0,T)$ satisfies (1) and $u(t) > 0$, $v(t) > 0$ for any $t \in [0,T]$, then one says that (u,v) is a $C^1[0,T] \times C^1[0,T]$ positive solution of system (1).

By using fixed point theorem in cones, we are able to prove the following result.

Theorem 3. *Assume that (H_1), (H_2) hold. Then* (1) *has at least one positive T-periodic solution.*

The proof of Theorem 3 will be given in Section 3 of this paper.

2. Preliminaries

Lemma 4 (see [12]). *Let X be a Banach space, K a cone in X, Ω_1, Ω_2 two nonempty bounded open sets in K, $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. $T : \overline{\Omega_2}/\Omega_1 \to K$ is a completely continuous operator. If*

(i) $T(x) \ne \lambda x$, $x \in \partial\Omega_1$, $\lambda > 1$,

(ii) $T(x) \ne \lambda x$, $x \in \partial\Omega_2$, $0 < \lambda < 1$, $\inf_{x\in\partial\Omega_2} \|Tx\| > 0$,

then T has a fixed point in $\overline{\Omega_2}/\Omega_1$.

Lemma 5. *If $f_i(t,u)$ satisfy (H_1), then, for $t \in (0,T)$, $f_i(t,u)$ are increasing on u and, for any $[\alpha,\beta] \subset (0,T)$,*

$$\lim_{u\to+\infty} \frac{f_i(t,u)}{u} = +\infty, \quad (i = 1, 2) \tag{14}$$

uniformly with respect to $t \in [\alpha,\beta]$.

Proof. We only deal with f_1. Without loss of generality, let $0 \le x \le y$. If $y = 0$, we get $f_1(t,x) \le f_1(t,y)$. If $y \ne 0$, let $c_0 = x/y$, then $0 \le c_0 \le 1$. From (8), we get

$$f_1(t,x) = f_1(t,c_0 y) \le c_0^{\mu_2} f_1(t,y) \le f_1(t,y), \tag{15}$$

which means $f_1(t,u)$ is increasing on u.

Assume $u > 1$. It follows from (11) that $f_1(t,u) \ge u^{\mu_2} f_1(t,1)$. Thus

$$\frac{f_1(t,u)}{u} \ge u^{\mu_2 - 1} f_1(t,1), \quad t \in (0,T). \tag{16}$$

From (H_1), for any $[\alpha, \beta] \subset (0, T)$, we obtain

$$\frac{f_1(t,u)}{u} \geq u^{\mu_2-1} f_1(t,1) \geq \max_{t\in[\alpha,\beta]} f_1(t,1)\, u^{\mu_2-1} > 0. \quad (17)$$

Therefore

$$\lim_{u\to+\infty} \frac{f_1(t,u)}{u} = +\infty \quad (18)$$

uniformly with respect to $t \in [\alpha, \beta]$. □

Let $X = C[0,T]$. We know X is a Banach space with the norm $\|u\| = \max_{t\in[0,T]}|u(t)|$. Define the sets

$$P = \{u \in X : u(t) \geq 0\},$$
$$Q = \left\{u \in P : \min_{0\leq t\leq T} u(t) \geq \sigma \|u\|\right\}. \quad (19)$$

It is easy to check that P, Q are cones in X and $Q \subset P$. Throughout this paper, we consider the space $X\times X$. It is easy to see $X \times X$ is a Banach space with the norm,

$$\|(u,v)\|_0 = \|u\| + \|v\|, \quad (u,v) \in X \times X. \quad (20)$$

We can get the conclusion that $P \times P$, $Q \times Q$ are cones in $X \times X$ and $Q \times Q \subset P \times P$.

For any $u \in X$, define the function

$$[u(t)]^* = \begin{cases} u(t), & u(t) \geq 0, \\ 0, & u(t) < 0. \end{cases} \quad (21)$$

Then the solution of periodic boundary value problem

$$x''(t) + k^2 x(t) = -c_1^-(t),$$
$$x(0) = x(T), \qquad x'(0) = x'(T) \quad (22)$$

can be expressed by $\tilde{x}_1(t) = -\int_0^T G(t,s)c_1^-(s)ds$, and the solution of periodic boundary value problem

$$x''(t) + k^2 x(t) = -c_2^-(t),$$
$$x(0) = x(T), \qquad x'(0) = x'(T) \quad (23)$$

can be expressed by $\tilde{x}_2(t) = -\int_0^T G(t,s)c_2^-(s)ds$. Obviously, $\tilde{x}_i(t) < 0$, $i = 1, 2$. Then

$$[u(t) + \tilde{x}_1(t)]^* \leq u(t) \leq \|u\|,$$
$$[v(t) + \tilde{x}_2(t)]^* \leq v(t) \leq \|v\|. \quad (24)$$

From (11), (12), and Lemma 5, we have

$$\begin{aligned} f_1\left(t, [v(t) + \tilde{x}_2(t)]^*\right) &\leq f_1(t, \|v\|) \leq f_1(t, \|v\| + 1) \\ &\leq (\|v\| + 1)^{\mu_1} f_1(t,1), \end{aligned} \quad (25)$$

$$\begin{aligned} f_2\left(t, [u(t) + \tilde{x}_1(t)]^*\right) &\leq f_2(t, \|u\|) \leq f_2(t, \|u\| + 1) \\ &\leq (\|u\| + 1)^{\lambda_1} f_2(t,1). \end{aligned} \quad (26)$$

Then, for any fixed $(u, v) \in P \times P$, it follows from (H_2) that

$$\begin{aligned} 0 &\leq \int_0^T G(t,s) \\ &\quad \times \left\{k^2[u(s) + \tilde{x}_1(s)]^* \right. \\ &\qquad \left. + f_1\left(s, [v(s) + \tilde{x}_2(s)]^*\right) + c_1^+(s)\right\} ds \\ &\leq M \int_0^T \left\{k^2 \|u\| + (\|v\| + 1)^{\mu_1} f_1(s,1) + c_1^+(s)\right\} ds \\ &\leq k^2 MT \|u\| + M(\|v\| + 1)^{\mu_1} \int_0^T \{f_1(s,1) + c_1^+(s)\} ds \\ &< +\infty, \\ 0 &\leq \int_0^T G(t,s) \\ &\quad \times \left\{k^2[v(s) + \tilde{x}_2(s)]^* \right. \\ &\qquad \left. + f_2\left(s, [u(s) + \tilde{x}_1(s)]^*\right) + c_2^+(s)\right\} ds \\ &\leq M \int_0^T \left\{k^2 \|v\| + (\|u\| + 1)^{\lambda_1} f_2(s,1) + c_2^+(s)\right\} ds \\ &\leq k^2 MT \|v\| + M(\|u\| + 1)^{\lambda_1} \int_0^T \{f_2(s,1) + c_2^+(s)\} ds \\ &< +\infty. \end{aligned} \quad (27)$$

Thus, we can define the operator $T : P \times P \to P \times P$, $T(u,v) = (T_1 u, T_2 v)$ by

$$\begin{aligned} T_1 u(t) &= \int_0^T G(t,s) \\ &\quad \times \left\{k^2[u(s) + \tilde{x}_1(s)]^* \right. \\ &\qquad \left. + f_1\left(s, [v(s) + \tilde{x}_2(s)]^*\right) + c_1^+(s)\right\} ds, \\ T_2 v(t) &= \int_0^T G(t,s) \\ &\quad \times \left\{k^2[v(s) + \tilde{x}_2(s)]^* \right. \\ &\qquad \left. + f_2\left(s, [u(s) + \tilde{x}_1(s)]^*\right) + c_2^+(s)\right\} ds, \end{aligned} \quad (28)$$

for $(u, v) \in P \times P$. Then, we have the following lemma.

Lemma 6. *Assuming that (H_1), (H_2) hold, then T has a fixed point if and only if*

$$\begin{aligned} &u''(t) + k^2 u(t) = k^2[u(t) + \tilde{x}_1(t)]^* \\ &\quad + f_1\left(t, [v(t) + \tilde{x}_2(t)]^*\right) + c_1^+(t), \quad t \in (0,T), \\ &v''(t) + k^2 v(t) = k^2[v(t) + \tilde{x}_2(t)]^* \\ &\quad + f_2\left(t, [u(t) + \tilde{x}_1(t)]^*\right) + c_2^+(t), \quad t \in (0,T), \\ &u(0) = u(T), \qquad u'(0) = u'(T), \\ &v(0) = v(T), \qquad v'(0) = v'(T) \end{aligned} \quad (29)$$

has one positive T-periodic solution.

Lemma 7. *Assuming that* (H_1), (H_2) *hold, then* $T(Q\times Q)\subset Q\times Q$ *and* $T: Q\times Q \to Q\times Q$ *is completely continuous.*

Proof. For $(u,v)\in Q\times Q$, we have

$$\begin{aligned}
(T_1u)(t) &\geq m\int_0^T \left\{k^2[u(s)+\tilde{x}_1(s)]^* + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\right\}ds\\
&= \sigma\int_0^T \max_{0\leq t,s\leq T} G(t,s)\left\{k^2[u(s)+\tilde{x}_1(s)]^* + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\right\}ds = \sigma\|T_1u\|,\\
(T_2v)(t) &\geq m\int_0^T \left\{k^2[v(s)+\tilde{x}_2(s)]^* + f_2\left(s,[u(s)+\tilde{x}_1(s)]^*\right) + c_2^+(s)\right\}ds\\
&= \sigma\int_0^T \max_{0\leq t,s\leq T} G(t,s) \times\left\{k^2[v(s)+\tilde{x}_2(s)]^* + f_2\left(s,[u(s)+\tilde{x}_1(s)]^*\right) + c_2^+(s)\right\}ds\\
&= \sigma\|T_2v\|.
\end{aligned}\tag{30}$$

Then, we can get

$$\begin{aligned}
\|T(u,v)\|_0 &= \|(T_1u,T_2v)\|_0 = \|T_1u\| + \|T_2v\|\\
&\geq \sigma\left(\|T_1u\| + \|T_2v\|\right) = \sigma\|T(u,v)\|_0,
\end{aligned}\tag{31}$$

which means $T(Q\times Q)\subset Q\times Q$.

Let $B\subset Q\times Q$ be any bounded set. Then there exists a constant N such that, for any $(u,v)\in B$,

$$\|(u,v)\|_0 = \|u\| + \|v\| \leq N. \tag{32}$$

From (27), we have

$$\begin{aligned}
(T_1u)(t) &\leq k^2MT\|u\| + M(\|v\|+1)^{\mu_1}\times\int_0^T \{f_1(s,1)+c_1^+(s)\}ds\\
&\leq k^2MTN + M(N+1)^{\mu_1}\times\int_0^T \{f_1(s,1)+c_1^+(s)\}ds,\\
(T_2v)(t) &\leq k^2MT\|v\| + M(\|u\|+1)^{\lambda_1}\times\int_0^T \{f_2(s,1)+c_2^+(s)\}ds\\
&\leq k^2MTN + M(N+1)^{\lambda_1}\times\int_0^T \{f_2(s,1)+c_2^+(s)\}ds.
\end{aligned}\tag{33}$$

Let

$$\begin{aligned}
\widetilde{N} &= k^2MTN + M(N+1)^{\lambda_1+\mu_1}\times\left(\int_0^T \{f_1(s,1)+c_1^+(s)\}ds + \int_0^T \{f_2(s,1)+c_2^+(s)\}ds\right)\\
&< +\infty.
\end{aligned}\tag{34}$$

Thus

$$\|T(u,v)\|_0 = \|T_1u\| + \|T_2v\| \leq \widetilde{N}, \tag{35}$$

which implies that $T(B)$ is bounded.

Next we prove that $T(B)$ is equicontinuous. For any $(u,v)\in B$, $t\in[0,T]$, we know

$$\begin{aligned}
(T_1u)(t) &= \int_0^T G(t,s)\times\left\{k^2[u(s)+\tilde{x}_1(s)]^* + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\right\}ds\\
&= \frac{1}{2k(1-\cos kT)}\times\int_0^t (\sin k(t-s) + \sin k(T-t+s))\times\left\{k^2[u(s)+\tilde{x}_1(s)]^* + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\right\}ds\\
&\quad + \frac{1}{2k(1-\cos kT)}\times\int_t^T (\sin k(s-t) + \sin k(T-s+t))\times\left\{k^2[u(s)+\tilde{x}_1(s)]^* + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\right\}ds,
\end{aligned}$$

$$
\begin{aligned}
(T_1u)'(t) = &-\frac{1}{2\sin(kT/2)} \\
&\times \int_0^t \sin\frac{2k(t-s)-kT}{2} \\
&\qquad \times \Big\{k^2[u(s)+\tilde{x}_1(s)]^* \\
&\qquad\qquad + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) \\
&\qquad\qquad + c_1^+(s)\Big\}\,ds \\
&+\frac{1}{2\sin(kT/2)} \\
&\times \int_t^T \sin\frac{2k(s-t)-kT}{2} \\
&\qquad \times \Big\{k^2[u(s)+\tilde{x}_1(s)]^* \\
&\qquad\qquad + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) \\
&\qquad\qquad + c_1^+(s)\Big\}\,ds.
\end{aligned}
\tag{36}
$$

From (25) and (H_2), we have

$$
\begin{aligned}
\left|(T_1u)'(t)\right| \le &\frac{1}{2\sin(kT/2)} \\
&\times \int_0^t \Big\{k^2[u(s)+\tilde{x}_1(s)]^* \\
&\qquad + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\Big\}\,ds \\
&+\frac{1}{2\sin(kT/2)} \\
&\times \int_t^T \Big\{k^2[u(s)+\tilde{x}_1(s)]^* \\
&\qquad + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\Big\}\,ds \\
= &\frac{1}{2\sin(kT/2)} \\
&\times \int_0^T \Big\{k^2[u(s)+\tilde{x}_1(s)]^* \\
&\qquad + f_1\left(s,[v(s)+\tilde{x}_2(s)]^*\right) + c_1^+(s)\Big\}\,ds \\
\le &\frac{1}{2\sin(kT/2)} \\
&\times \int_0^T \Big\{k^2\|u\| + (\|v\|+1)^{\mu_1} \\
&\qquad \times f_1(s,1) + c_1^+(s)\Big\}\,ds \\
\le &\frac{k^2T}{2\sin(kT/2)}\|u\| + \frac{1}{2\sin(kT/2)}(\|v\|+1)^{\mu_1} \\
&\times \int_0^T \{f_1(s,1)+c_1^+(s)\}\,ds \\
\le &\frac{k^2T}{2\sin(kT/2)}N + \frac{1}{2\sin(kT/2)}(N+1)^{\mu_1} \\
&\times \int_0^T \{f_1(s,1)+c_1^+(s)\}\,ds < +\infty.
\end{aligned}
\tag{37}
$$

Using the same method, we can obtain $|(T_2v)'(t)| < +\infty$. Therefore, $T(B)$ is equicontinuous. According to Ascoli-Arzela theorem, $T(B)$ is a relatively compact set.

Next, we prove that $T : Q\times Q \to Q\times Q$ is continuous. Suppose $(u_n, v_n), (u_0, v_0) \in Q\times Q$, $(u_n, v_n) \to (u_0, v_0)$, $n \to +\infty$, that is, $u_n \to u_0$, $v_n \to v_0$, $n \to +\infty$. We know that there exists a constant $L > 0$ such that

$$
\begin{aligned}
&\|u_0\| \le L, \qquad \|u_n\| \le L, \qquad \|v_0\| \le L, \\
&\|v_n\| \le L, \quad n = 1, 2 \ldots.
\end{aligned}
\tag{38}
$$

We shall prove $T(u_n, v_n) \to T(u_0, v_0)$, $n \to +\infty$, that is,

$$
T_1u_n \longrightarrow T_1u_0, \qquad T_2v_n \longrightarrow T_2v_0, \quad n \longrightarrow +\infty.
\tag{39}
$$

We first deal with $T_1u_n \to T_1u_0, n \to +\infty$. Otherwise, there exist $\varepsilon_0 > 0$, $\{t_n\} \in [0,T]$ such that $|T_1u_n(t_n) - T_1u_0(t_n)| \ge \varepsilon_0$. Without loss of generality, we can assume $t_n \to t_0 \in [0,T]$. We know

$$
\begin{aligned}
&T_1u_n(t_n) - T_1u_n(t_0) \longrightarrow 0, \\
&T_1u_0(t_n) - T_1u_0(t_0) \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned}
\tag{40}
$$

Next, we show $T_1u_n(t_0) - T_1u_0(t_0) \to 0, n \to \infty$. In fact,

$$
\begin{aligned}
T_1u_n(t_0) - T_1u_0(t_0) = &\int_0^T G(t_0,s) \\
&\times \Big\{k^2[u_n(s)+\tilde{x}_1(s)]^* \\
&\quad - k^2[u_0(s)+\tilde{x}_1(s)]^* \\
&\quad + f_1\left(s,[v_n(s)+\tilde{x}_2(s)]^*\right) \\
&\quad - f_1\left(s,[v_0(s)+\tilde{x}_2(s)]^*\right)\Big\}\,ds.
\end{aligned}
\tag{41}
$$

Let

$$
\begin{aligned}
r_n(s) = &G(t_0,s) \\
&\times \Big\{k^2[u_n(s)+\tilde{x}_1(s)]^* - k^2[u_0(s)+\tilde{x}_1(s)]^* \\
&\quad + f_1\left(s,[v_n(s)+\tilde{x}_2(s)]^*\right) \\
&\quad - f_1\left(s,[v_0(s)+\tilde{x}_2(s)]^*\right)\Big\}.
\end{aligned}
\tag{42}
$$

Since

$$\begin{aligned} &\left|[v_n(s)+\tilde{x}_2(s)]^* - [v_0(s)+\tilde{x}_2(s)]^*\right| \\ &= \left|\frac{|v_n(s)+\tilde{x}_2(s)| + v_n(s)+\tilde{x}_2(s)}{2} - \frac{|v_0(s)+\tilde{x}_2(s)| + v_0(s)+\tilde{x}_2(s)}{2}\right| \\ &= \left|\frac{|v_n(s)+\tilde{x}_2(s)| - |v_0(s)+\tilde{x}_2(s)|}{2} + \frac{v_n(s)-v_0(s)}{2}\right| \\ &\le |v_n(s)-v_0(s)| \longrightarrow 0, \quad n\longrightarrow +\infty, \end{aligned} \tag{43}$$

and f_1 is continuous, we know $r_n(s) \to 0$, $n \to \infty$. From (24), we know

$$\begin{gathered} [u_n(s)+\tilde{x}_1(s)]^* \le L, \qquad [u_0(s)+\tilde{x}_1(s)]^* \le L, \\ [v_n(s)+\tilde{x}_2(s)]^* \le L < L+1, \\ [v_0(s)+\tilde{x}_2(s)]^* \le L < L+1. \end{gathered} \tag{44}$$

It can be inferred from (8) that

$$\begin{aligned} |r_n(s)| &\le M\Big\{\left|k^2[u_n(s)+\tilde{x}_1(s)]^* - k^2[u_0(s)+\tilde{x}_1(s)]^*\right| \\ &\quad + \Big| f_1\left(s,[v_n(s)+\tilde{x}_2(s)]^*\right) - f_1\left(s,[v_0(s)+\tilde{x}_2(s)]^*\right)\Big|\Big\} \\ &\le 2Mk^2L + 2M(L+1)^{\mu_1} f_1(s,1). \end{aligned} \tag{45}$$

Set $F(s) = 2Mk^2L + 2M(L+1)^{\mu_1}(f_1(s,1)+c_1^+(s))$. Thus, we get

$$|r_n(s)| \le F(s), \quad s\in(0,T), \ n=1,2,\ldots. \tag{46}$$

From (H_2), we know $\int_0^T F(s)ds < +\infty$. Using Lebesgue-dominated convergence theorem, we get

$$T_1u_n(t_0) - T_1u_0(t_0) = \int_0^T r_n(s)\,ds \longrightarrow 0, \quad n\longrightarrow +\infty. \tag{47}$$

From (40) and (47), we obtain

$$\begin{aligned} \varepsilon_0 &\le |T_1u_n(t_n) - T_1u_0(t_n)| \\ &= |T_1u_n(t_n) - T_1u_n(t_0) + T_1u_n(t_0) - T_1u_0(t_0) + T_1u_0(t_0) - T_1u_0(t_n)| \\ &\le |T_1u_n(t_n) - T_1u_n(t_0)| + |T_1u_n(t_0) - T_1u_0(t_0)| \\ &\quad + |T_1u_0(t_0) - T_1u_0(t_n)| \longrightarrow 0, \quad n\longrightarrow\infty, \end{aligned} \tag{48}$$

which is a contradiction. Thus, we know $T_1u_n \to T_1u_0$, $n\to +\infty$. Using the same method, we can obtain $T_2v_n \to T_2v_0$, $n\to+\infty$. Then

$$\begin{aligned} T(u_n,v_n) &= (T_1u_n, T_2v_n) \longrightarrow (T_1u_0, T_2v_0) \\ &= T(u_0,v_0), \quad n\longrightarrow +\infty, \end{aligned} \tag{49}$$

which means $T: Q\times Q \to Q\times Q$ is continuous. Therefore $T: Q\times Q \to Q\times Q$ is a completely continuous operator. □

3. Proof of Theorem 3

We proceed to prove Theorem 3 in two steps.

(1) Let $\Omega_1 = \{(u,v)\in Q\times Q : \|u\| < 2M^2r_1/m, \|v\| < 2M^2r_2/m\}$. We can get

$$T(u,v) \ne \lambda(u,v), \quad (u,v)\in\partial\Omega_1, \ \lambda>1. \tag{50}$$

For $(u,v)\in\partial\Omega_1$, we have the following two cases.

Case I. One has $\{(u,v)\in Q\times Q : \|u\| = 2M^2r_1/m, \|v\| \le 2M^2r_2/m\}$. Under this condition, we can get

$$T_1u \ne \lambda u, \quad (u,v)\in\partial\Omega_1, \ \lambda>1. \tag{51}$$

Otherwise, there exist $(u_0,v_0)\in\partial\Omega_1$, $\lambda_0>1$ such that $T_1u_0 = \lambda_0u_0$. As

$$\begin{aligned} u_0(t) + \tilde{x}_1(t) &= u_0(t) - \int_0^T G(t,s)c_1^-(s)\,ds \\ &\ge \sigma\|u_0\| - Mr_1 > 0, \end{aligned} \tag{52}$$

then $[u_0(t)+\tilde{x}_1(t)]^* = u_0(t)+\tilde{x}_1(t)$. By a direct computation, we know $u_0 = (1/\lambda_0)T_1u_0$ satisfies

$$\begin{gathered} u_0''(t) + k^2u_0(t) = \frac{1}{\lambda_0}\left\{k^2(u_0(t)+\tilde{x}_1(t)) + f_1\left(t,[v_0(t)+\tilde{x}_2(t)]^*\right) + c_1^+(t)\right\}, \\ u_0(0) = u_0(T), \qquad u_0'(0) = u_0'(T). \end{gathered} \tag{53}$$

Then, we get

$$\begin{aligned} u_0''(t) &= \frac{1}{\lambda_0}\left\{k^2(u_0(t)+\tilde{x}_1(t)) + f_1\left(t,[v_0(t)+\tilde{x}_2(t)]^*\right) + c_1^+(t)\right\} - k^2u_0(t) \\ &\le k^2u_0(t) + k^2\tilde{x}_1(t) + f_1\left(t,[v_0(t)+\tilde{x}_2(t)]^*\right) + c_1^+(t) - k^2u_0(t) \\ &= k^2\tilde{x}_1(t) + f_1\left(t,[v_0(t)+\tilde{x}_2(t)]^*\right) + c_1^+(t). \end{aligned} \tag{54}$$

Since $[v_0(t)+\widetilde{x}_2(t)]^* \le v_0(t) < 2M^2 r_2/m+1$, integrating both sides of (54) on $[0,T]$, we get

$$\begin{aligned}
0 &\le k^2 \int_0^T \widetilde{x}_1(t)\,dt \\
&\quad + \int_0^T \left\{ f_1\left(t,[v_0(t)+\widetilde{x}_2(t)]^*\right) + c_1^+(t) \right\} dt \\
&\le k^2 \int_0^T \widetilde{x}_1(t)\,dt + \left(\frac{2M^2 r_2}{m}+1\right)^{\mu_1} \\
&\quad \times \int_0^T \left\{ f_1(t,1) + c_1^+(t) \right\} dt.
\end{aligned} \tag{55}$$

Then

$$\begin{aligned}
&\left(\frac{2M^2 r_2}{m}+1\right)^{\mu_1} \int_0^T \left[f_1(t,1) + c_1^+(t) \right] dt \\
&\ge -k^2 \int_0^T \widetilde{x}_1(t)\,dt = k^2 \int_0^T \int_0^T G(t,s)\,c_1^-(s)\,ds\,dt \\
&\ge k^2 T m r_1.
\end{aligned} \tag{56}$$

That is,

$$\int_0^T \left[f_1(t,1) + c_1^+(t) \right] dt \ge \frac{k^2 T m r_1}{(2M^2 r_2/m+1)^{\mu_1}}, \tag{57}$$

which contradicts with (H_2).

Case II. One has $\{(u,v) \in Q\times Q : \|u\| \le 2M^2 r_1/m, \|v\| = 2M^2 r_2/m\}$. Under this condition, we can get

$$T_2 v \ne \lambda v, \quad (u,v) \in \partial\Omega_1, \ \lambda > 1. \tag{58}$$

Otherwise, there exist $(u_0,v_0) \in \partial\Omega_1$, $\lambda_0 > 1$ such that $T_2 v_0 = \lambda_0 v_0$. As

$$\begin{aligned}
v_0(t) + \widetilde{x}_2(t) &= v_0(t) - \int_0^T G(t,s)\,c_2^-(s)\,ds \\
&\ge \sigma \|v_0\| - M r_2 > 0,
\end{aligned} \tag{59}$$

then $[v_0(t)+\widetilde{x}_2(t)]^* = v_0(t)+\widetilde{x}_2(t)$. By a direct computation, we know $v_0 = (1/\lambda_0) T_2 v_0$ satisfies

$$\begin{gathered}
v_0''(t) + k^2 v_0(t) = \frac{1}{\lambda_0} \left\{ k^2 \left(v_0(t)+\widetilde{x}_2(t)\right) \right. \\
+ f_2\left(t,[u_0(t)+\widetilde{x}_1(t)]^*\right) \\
\left. + c_2^+(t) \right\}, \\
v_0(0) = v_0(T), \qquad v_0'(0) = v_0'(T).
\end{gathered} \tag{60}$$

For $[u_0(t)+\widetilde{x}_1(t)]^* \le u_0(t) < 2M^2 r_1/m+1$, using the same method as condition I, we obtain

$$\int_0^T \left[f_2(t,1) + c_2^+(t) \right] dt \ge \frac{k^2 T m r_2}{(2M^2 r_1/m+1)^{\lambda_1}}, \tag{61}$$

which is also a contradiction.

(2) Choose an interval $[\alpha,\beta] \subset (0,T)$ satisfying $\beta-\alpha = T/2$. Set $\widetilde{M} > 4M/Tm^2$. From Lemma 5, there exists $R_1 > 2M^2 r/m$, $r = \max\{r_1,r_2\}$ such that

$$f_i(t,u) \ge \widetilde{M} u, \quad u \ge R_1, \ t \in [\alpha,\beta], \ i = 1,2. \tag{62}$$

Let $R \ge (2M/m)R_1 > R_1 > 2M^2 r/m$. Define $\Omega_2 = \{(u,v) \in Q\times Q : \|u\| < R, \|v\| < R\}$. We can get

$$T(u,v) \ne \lambda(u,v), \quad (u,v) \in \partial\Omega_2, \ 0 < \lambda < 1. \tag{63}$$

For $(u,v) \in \partial\Omega_2$, we have the following two cases.

Case I. One has $\{(u,v) \in Q\times Q : \|u\| = R, \|v\| \le R\}$. Under this condition, we know

$$u(t) + \widetilde{x}_1(t) \ge \sigma\|u\| - M r_1 \ge \sigma R - M r > \frac{m}{2M} R \,(> R_1 > 0). \tag{64}$$

Thus $[u(t)+\widetilde{x}_1(t)]^* = u(t)+\widetilde{x}_1(t)$. Furthermore, we obtain from (62) that

$$\begin{aligned}
T_2 v(t) &= \int_0^T G(t,s) \left\{ k^2 [v(s)+\widetilde{x}_2(s)]^* \right. \\
&\qquad + f_2\left(s, u(s)+\widetilde{x}_1(s)\right) \\
&\qquad \left. + c_2^+(s) \right\} ds \\
&\ge \int_\alpha^\beta G(t,s)\, f_2\left(s, u(s)+\widetilde{x}_1(s)\right) ds \\
&\ge \frac{\widetilde{M} T R m^2}{4M}.
\end{aligned} \tag{65}$$

On the other hand, for $0 < \lambda < 1$, we know $\lambda v(t) < v(t) \le R$. From the choice of $\widetilde{M}$, we get $T_2 v \ne \lambda v$.

Case II. One has $\{(u,v) \in Q\times Q : \|u\| \le R, \|v\| = R\}$. Under this condition, we get

$$\begin{aligned}
v(t) + \widetilde{x}_2(t) &\ge \sigma\|v\| - M r_2 \\
&\ge \sigma R - M r > \frac{m}{2M} R \,(> R_1 > 0).
\end{aligned} \tag{66}$$

Thus $[v(t)+\widetilde{x}_2(t)]^* = v(t)+\widetilde{x}_2(t)$. From (62), we get

$$\begin{aligned}
T_1 u(t) &= \int_0^T G(t,s) \left\{ k^2 [u(s)+\widetilde{x}_1(s)]^* \right. \\
&\qquad + f_1\left(s, v(s)+\widetilde{x}_2(s)\right) \\
&\qquad \left. + c_1^+(s) \right\} ds \\
&\ge \int_\alpha^\beta G(t,s)\, f_1\left(s, u(s)+\widetilde{x}_1(s)\right) ds \\
&\ge \frac{\widetilde{M} T R m^2}{4M}.
\end{aligned} \tag{67}$$

For $0 < \lambda < 1$, $\lambda u(t) < u(t) \le R$, from the choice of $\widetilde{M}$, we know $T_1 u \ne \lambda u$.

Furthermore, we can obtain

$$\begin{aligned}\|T(u,v)\|_0 &= \|T_1 u\| + \|T_2 v\| \\ &\geq \frac{\widetilde{M}TRm^2}{4M}, \quad (u,v) \in \partial\Omega_2.\end{aligned} \tag{68}$$

It implies $\inf_{(u,v)\in\partial\Omega_2} \| T(u,v)\|_0 > 0$.

From Lemma 4, we know T has a fixed point $(\tilde{u},\tilde{v})$ in $\overline{\Omega_2}/\Omega_1$. For $(\tilde{u},\tilde{v}) \in \overline{\Omega_2}/\Omega_1$, we have the following three cases.

Case 1. One has

$$\begin{aligned}(\tilde{u},\tilde{v}) &\in \left\{Q\times Q : \frac{2M^2 r}{m} \leq \|\tilde{u}\| \leq R, \|\tilde{v}\| < \frac{2M^2 r}{m}\right\} \\ &:= S_1.\end{aligned} \tag{69}$$

Case 2. One has

$$\begin{aligned}(\tilde{u},\tilde{v}) &\in \left\{Q\times Q : \|\tilde{u}\| < \frac{2M^2 r}{m}, \frac{2M^2 r}{m} \leq \|\tilde{v}\| \leq R\right\} \\ &:= S_2.\end{aligned} \tag{70}$$

Case 3. One has

$$\begin{aligned}(\tilde{u},\tilde{v}) &\in \left\{Q\times Q : \frac{2M^2 r}{m} \leq \|\tilde{u}\| \leq R, \frac{2M^2 r}{m} \leq \|\tilde{v}\| \leq R\right\} \\ &:= S_3.\end{aligned} \tag{71}$$

Next, we show Cases 1 and 2 are impossible. In Case 1, we have

$$\begin{aligned}\tilde{u}(t) + \tilde{x}_1(t) &= \tilde{u}(t) - \int_0^T G(t,s)\, c_1^-(s)\, ds \\ &\geq \sigma\|\tilde{u}\| - Mr_1 > 0.\end{aligned} \tag{72}$$

It follows that $[\tilde{u}(t) + \tilde{x}_1(t)]^* = \tilde{u}(t) + \tilde{x}_1(t)$. By a direct computation, we know $\tilde{u} = T_1\tilde{u}$ satisfies

$$\begin{aligned}\tilde{u}''(t) + k^2\tilde{u}(t) &= k^2\left(\tilde{u}(t) + \tilde{x}_1(t)\right) \\ &\quad + f_1\left(t, [\tilde{v}(t) + \tilde{x}_2(t)]^*\right) + c_1^+(t), \\ \tilde{u}(0) = \tilde{u}(T), &\qquad \tilde{u}'(0) = \tilde{u}'(T).\end{aligned} \tag{73}$$

Then, we get

$$\tilde{u}''(t) = k^2\tilde{x}_1(t) + f_1\left(t, [\tilde{v}(t) + \tilde{x}_2(t)]^*\right) + c_1^+(t). \tag{74}$$

Since $[\tilde{v}(t) + \tilde{x}_2(t)]^* \leq \tilde{v}(t) < 2M^2 r_2/m + 1$, integrating both sides of (74) on $[0,T]$, we get

$$\begin{aligned}0 &= k^2\int_0^T \tilde{x}_1(t)\, dt \\ &\quad + \int_0^T \left\{f_1\left(t, [\tilde{v}(t) + \tilde{x}_2(t)]^*\right) + c_1^+(t)\right\} dt \\ &\leq k^2\int_0^T \tilde{x}_1(t)\, dt + \left(\frac{2M^2 r_2}{m} + 1\right)^{\mu_1} \\ &\quad \times \int_0^T \left\{f_1(t,1) + c_1^+(t)\right\} dt.\end{aligned} \tag{75}$$

Then

$$\begin{aligned}&\left(\frac{2M^2 r_2}{m} + 1\right)^{\mu_1} \int_0^T \left[f_1(t,1) + c_1^+(t)\right] dt \\ &\geq -k^2\int_0^T \tilde{x}_1(t)\, dt = k^2\int_0^T\int_0^T G(t,s)\, c_1^-(s)\, ds\, dt \\ &\geq k^2 Tmr_1.\end{aligned} \tag{76}$$

That is,

$$\int_0^T \left[f_1(t,1) + c_1^+(t)\right] dt \geq \frac{k^2 Tmr_1}{(2M^2 r_2/m + 1)^{\mu_1}}, \tag{77}$$

which contradicts with (H_2). Using the same method, we can prove that Case 2 is also impossible. Therefore, Case 3 is satisfied and T has a fixed point $(\tilde{u},\tilde{v})$ in $\overline{\Omega_2}/\Omega_1$ satisfying

$$\frac{2M^2 r}{m} \leq \|\tilde{u}\| \leq R, \qquad \frac{2M^2 r}{m} \leq \|\tilde{v}\| \leq R. \tag{78}$$

Since

$$\begin{aligned}\tilde{u}(t) + \tilde{x}_1(t) &\geq \sigma\|\tilde{u}\| - Mr_1 > 0, \\ \tilde{v}(t) + \tilde{x}_2(t) &\geq \sigma\|\tilde{v}\| - Mr_2 > 0,\end{aligned} \tag{79}$$

from Lemma 6, we know $\tilde{u}(t), \tilde{v}(t)$ satisfy

$$\begin{aligned}\tilde{u}''(t) + k^2\tilde{u}(t) &= k^2\left(\tilde{u}(t) + \tilde{x}_1(t)\right) \\ &\quad + f_1\left(t, \tilde{v}(t) + \tilde{x}_2(t)\right) + c_1^+(t), \\ \tilde{v}''(t) + k^2\tilde{v}(t) &= k^2\left(\tilde{v}(t) + \tilde{x}_2(t)\right) \\ &\quad + f_2\left(t, \tilde{u}(t) + \tilde{x}_1(t)\right) + c_2^+(t), \\ \tilde{u}(0) = \tilde{u}(T), &\qquad \tilde{u}'(0) = \tilde{u}'(T), \\ \tilde{v}(0) = \tilde{v}(T), &\qquad \tilde{v}'(0) = \tilde{v}'(T).\end{aligned} \tag{80}$$

Let $(u^*(t), v^*(t)) = (\tilde{u}(t) + \tilde{x}_1(t), \tilde{v}(t) + \tilde{x}_2(t))$. For

$$c_1(t) = c_1^+(t) - c_1^-(t), \qquad c_2(t) = c_2^+(t) - c_2^-(t), \tag{81}$$

we obtain

$$\begin{aligned} u^{*\prime\prime}(t) &= f(t, v^*(t)) + c_1(t), \quad t \in (0,T), \\ v^{*\prime\prime}(t) &= f(t, u^*(t)) + c_2(t), \quad t \in (0,T), \\ u^*(0) &= u^*(T), \qquad u^{*\prime}(0) = u^{*\prime}(T), \\ v^*(0) &= v^*(T), \qquad v^{*\prime}(0) = v^{*\prime}(T). \end{aligned} \tag{82}$$

This means (u^*, v^*) is one positive T-periodic solution of (1).

4. Applications of Theorem 3

Finally, we give some examples as the applications of Theorem 3:

$$(e1)\begin{cases} u''(t) = \dfrac{k^2 T m v^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{t}} - \dfrac{2}{\sqrt{t}}, \\ v''(t) = \dfrac{k^2 T m u^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{t}} - \dfrac{2}{\sqrt{t}}, \\ u(0) = u(T), \qquad u'(0) = u'(T), \\ v(0) = v(T), \qquad v'(0) = v'(T). \end{cases} \tag{83}$$

$$(e2)\begin{cases} u''(t) = \dfrac{k^2 T m v^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{T-t}} - \dfrac{2}{\sqrt{t}}, \\ v''(t) = \dfrac{k^2 T m u^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{T-t}} - \dfrac{2}{\sqrt{t}}, \\ u(0) = u(T), \qquad u'(0) = u'(T), \\ v(0) = v(T), \qquad v'(0) = v'(T). \end{cases}$$

Choosing $f_1(t,v) = k^2 T m v^{3/2}/(2M^2\sqrt{T}/m + 1)^2\sqrt{t}$, $f_2(t,u) = k^2 T m u^{3/2}/(2M^2\sqrt{T}/m+1)^2\sqrt{T-t}$, $c_i^-(t) = 2/\sqrt{t}$, $c_i^+(t) \equiv 0$, $r_i = \int_0^T c_i^-(t)dt 4\sqrt{T}$, $i = 1, 2$, and $\mu_1 = 2 > \mu_2 = 5/4 > 1$, $\lambda_1 = 2 > \lambda_2 = 5/4 > 1$, then (H_1) is satisfied. Notice (H_2) also holds, since

$$\int_0^T \left(f_i(t,1) + c_i^+(t)\right) dt = \frac{2k^2 T\sqrt{T}m}{\left(2M^2\sqrt{T}/m + 1\right)^2}, \quad i = 1, 2. \tag{84}$$

Existence of the positive T-periodic solutions is guaranteed from Theorem 3. We can also consider the following examples and the same result can be obtained:

$$(e3)\begin{cases} u''(t) = \dfrac{k^2 T m v^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{T-t}} - \dfrac{2}{\sqrt{t}}, \\ v''(t) = \dfrac{k^2 T m u^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{t}} - \dfrac{2}{\sqrt{T-t}}, \\ u(0) = u(T), \qquad u'(0) = u'(T), \\ v(0) = v(T), \qquad v'(0) = v'(T), \end{cases} \tag{85}$$

$$(e4)\begin{cases} u''(t) = \dfrac{k^2 T m v^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{T-t}} - \dfrac{2}{\sqrt{T-t}}, \\ v''(t) = \dfrac{k^2 T m u^{3/2}}{\left(2M^2\sqrt{T}/m + 1\right)^2\sqrt{T-t}} - \dfrac{2}{\sqrt{T-t}}, \\ u(0) = u(T), \qquad u'(0) = u'(T), \\ v(0) = v(T), \qquad v'(0) = v'(T). \end{cases} \tag{86}$$

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