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Research Article

Successive Iteration of Positive Solutions for Fourth-Order Two-Point Boundary Value Problems

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We are concerned with a fourth-order two-point boundary value problem. We prove the existence of positive solutions and establish iterative schemes for approximating the solutions. The interesting point of our method is that the nonlinear term is involved with all lower-order derivatives of unknown function, and the iterative scheme starts off with a known cubic function or the zero function. Finally we give two examples to verify the effectiveness of the main results.

1. Introduction

The bending of an elastic beam can be described with some fourth-order boundary value problems. Owing to their importance in engineering, physics, and material mechanics, boundary value problems for elastic beam equations have attracted much attention of many authors. Many methods, such as upper and lower solutions method, shooting method, coincidence degree theory, or fixed point theorem and so on, have been developed to derive existence of solutions for fourth-order differential equations with two-point boundary conditions; see [1–20] and many references cited therein. However, almost all of the papers we mentioned focused their attention on the existence of solutions or positive solutions. In the existing literature, there are few papers that concern with the computational methods of solutions or positive solutions; see [21–24]. In this paper, by the successively iterative technique, we study the existence and iteration of monotone positive solutions for the following fourth-order differential equation:

$$u''''(t) = q(t) f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad \text{a.e. } t \in [0,1], \tag{1}$$

together with two-point boundary conditions

$$u(0) = u'(1) = u''(0) = u'''(1) = 0. \tag{2}$$

Throughout this paper, the following conditions are satisfied:

(A1) $f \in C([0,1] \times [0,\infty) \times [0,\infty) \times (-\infty,0] \times (-\infty,0], [0,\infty))$;

(A2) $q \in L^1[0,1]$ is nonnegative and $0 < \int_0^1 q(s)ds < \infty$.

In material mechanics, (1) describes the deflection of a beam under a certain force. The boundary conditions of the forms (2) mean that the elastic beam is simply fixed at the end $t = 0$ and fastened with a sliding clamp at the end $t = 1$.

Fourth-order differential equations with boundary conditions (2) have been studied. For example, Graef and Yang in [7] studied the nonlinear fourth-order differential equation:

$$u''''(t) = \lambda a(t) f(u(t)), \quad 0 < t < 1, \tag{3}$$

together with boundary conditions (2). The authors established some sufficient conditions for the existence and nonexistence of positive solutions. The main tool is the Kranosel'skii fixed point theorem. Using the Leggett-Williams fixed point theorem, Yang [18] established an existence criterion for triple positive solutions of the nonlinear fourth-order differential equation:

$$u''''(t) = g(t) f\left(t, u(t), u'(t)\right), \quad 0 < t < 1, \tag{4}$$

with two-point boundary conditions (2). By applying iterative techniques, Sun and Wang [21] investigated the existence

and iteration of monotone positive solution for the following elastic beam equation

$$u''''(t) = f\left(t, u(t), u'(t)\right), \quad 0 < t < 1, \tag{5}$$

with boundary conditions (2). By using Leray-Schauder nonlinear alternate, Leray-Schauder fixed point theorem and a fixed point theorem due to Avery and Peterson, Sun [15] presented some results on the existence and multiplicity of positive solutions to problem (1)-(2). Bai [3] proposed a constructive method which yields two monotone sequences that converge uniformly to extremal solutions to the following differential equations

$$\begin{aligned} &u''''(t) - Mu''(t) \\ &\quad = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad 0 < t < 1, \end{aligned} \tag{6}$$

or

$$\begin{aligned} &u''''(t) - Mu''(t) \\ &\quad = f\left(t, u(t), u'(t), u''(t)\right), \quad 0 < t < 1, \end{aligned} \tag{7}$$

with boundary conditions (2), if there exists an upper solution β and a lower solution α with $\beta \leqslant \alpha$. By placing some restrictions on the nonlinear term f, Bai [4] obtained the existence results for the fourth-order two-point boundary value problem (1)-(2) via the lower and upper solution method. In particular, a new truncating technique and an appropriate Nagumo-type condition are introduced and employed. In addition, Yao studied (1) with nonhomogeneous boundary condition

$$u(0) = a, \qquad u'(1) = b, \qquad u''(0) = c, \qquad u'''(1) = d. \tag{8}$$

In [20] the author obtained local existence theorem via Leray-Schauder fixed point theorem. In [22] Yao constructed an iterative sequence by the monotonic technique and proved that the sequence approximates successively to the solution of the equation under suitable assumptions.

Motivated by the above-mentioned works, this paper investigates the existence and iteration of positive solutions for problem (1)-(2). It is noted that the successively iterative scheme starts off with the zero function or a simple cubic function, which is feasible for the computational purpose. At the end of the paper, two examples are included to illustrate the main results.

2. Preliminaries

In this section, we introduce some necessary definitions and properties of the Green function.

Definition 1 (see [25, 26]). Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a cone of E if it satisfies the following two conditions:

(1) $x \in K$, $\lambda \geqslant 0$ imply $\lambda x \in K$;

(2) $x \in K$, $-x \in K$ imply $x = 0$.

Definition 2 (see [25, 26]). An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

The Green's function $G : [0,1] \times [0,1] \to [0, +\infty)$ for problem (1)-(2) is given by

$$G(t,s) = \frac{1}{6}\begin{cases} \left(6t - 3t^2 - s^2\right)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ \left(6s - 3s^2 - t^2\right)t, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{9}$$

so solving the boundary value problem (1)-(2) is equivalent to finding a solution to the integral equation

$$u(t) = \int_0^1 G(t,s)\, q(s)\, f\left(s, u(s), u'(s), u''(s), u'''(s)\right) ds. \tag{10}$$

Some properties of functions $G(t,s)$ are summarized as follows.

Lemma 3. *For all $t, s \in [0,1]$, the following inequalities hold true:*

$$\begin{aligned} &0 \leqslant -\frac{\partial^3 G(t,s)}{\partial t^3} \leqslant 1, \qquad 0 \leqslant -\frac{\partial^2 G(t,s)}{\partial t^2} \leqslant t, \\ &0 \leqslant \frac{\partial G(t,s)}{\partial t} \leqslant \frac{1}{2}\left(1 - t^2\right), \qquad 0 \leqslant G(t,s) \leqslant \frac{1}{6}\left(3t - t^3\right). \end{aligned} \tag{11}$$

Proof. For any fixed $s \in [0,1]$, from (9) we obtain

$$\begin{aligned} \frac{\partial^3 G(t,s)}{\partial t^3} &= \begin{cases} 0, & 0 \leqslant s < t \leqslant 1, \\ -1, & 0 \leqslant t < s \leqslant 1, \end{cases} \\ \frac{\partial^2 G(t,s)}{\partial t^2} &= \begin{cases} -s, & 0 \leqslant s \leqslant t \leqslant 1, \\ -t, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \end{aligned} \tag{12}$$

then,

$$0 \leqslant -\frac{\partial^3 G(t,s)}{\partial t^3} \leqslant 1, \quad 0 \leqslant -\frac{\partial^2 G(t,s)}{\partial t^2} \leqslant t, \quad t, s \in [0,1]. \tag{13}$$

For all $t, s \in [0,1]$, it follows from (9) that

$$\begin{aligned} 0 &\leqslant \frac{\partial G(t,s)}{\partial t} \\ &= \begin{cases} (1-t)s, & 0 \leqslant s \leqslant t \leqslant 1, \\ s - \frac{1}{2}s^2 - \frac{1}{2}t^2, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \leqslant \frac{1}{2}\left(1 - t^2\right). \end{aligned} \tag{14}$$

For all $t, s \in [0,1]$, if $s \leqslant t$, it yields

$$\begin{aligned} G(t,s) &= \frac{1}{6}\left(6t - 3t^2 - s^2\right)s = \frac{1}{6}\left[3 - 3(1-t)^2 - s^2\right]s \\ &\leqslant \frac{1}{6}\left(3 - s^2\right)s \leqslant \frac{1}{6}\left(3t - t^3\right). \end{aligned} \tag{15}$$

While $t \leqslant s$, we have

$$G(t,s) = \frac{1}{6}\left(6s - 3s^2 - t^2\right)t = \frac{1}{6}\left[3 - 3(1-s)^2 - t^2\right]t \leqslant \frac{1}{6}\left(3t - t^3\right). \tag{16}$$

The proof is completed. □

In what follows we will consider the Banach space $E = C^3[0,1]$ equipped with norm

$$\|u\| = \max\left\{\max_{0\leqslant t\leqslant 1}|u(t)|, \max_{0\leqslant t\leqslant 1}\left|u'(t)\right|, \max_{0\leqslant t\leqslant 1}\left|u''(t)\right|, \max_{0\leqslant t\leqslant 1}\left|u'''(t)\right|\right\}. \tag{17}$$

Define the cone $K \subset C^3[0,1]$ by

$$K = \left\{u \in C^3[0,1] : u(t) \geqslant 0, u'(t) \geqslant 0, u''(t) \leqslant 0, u'''(t) \leqslant 0, t \in [0,1]\right\}. \tag{18}$$

For $u \in K$, we give an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)\,q(s)\,f\left(s, u(s), u'(s), u''(s), u'''(s)\right)ds. \tag{19}$$

Obviously, the fixed points of T are monotone and nonnegative solutions of the problem (1)-(2).

Lemma 4. *Consider that $T : K \to K$ is completely continuous.*

Proof. Suppose that $u \in K$. Using $(A1)$, $(A2)$, (19), and Lemma 3, we obtain $(Tu)(t) \geqslant 0$, $(Tu)'(t) \geqslant 0$, $(Tu)''(t) \leqslant 0$, $(Tu)'''(t) \leqslant 0$, for all $t \in [0,1]$. Then $T(K) \subseteq K$. Using arguments similar to those of [21, 22], one can prove that T is a completely continuous operator. □

3. Main Results

In this section, we give the main results of this paper.

Theorem 5. *Assume that there exists $R > 0$ such that*

- $(H1)$ $f(t, x_1, y_1, z_1, w_1) \leqslant f(t, x_2, y_2, z_2, w_2)$ *for any* $0 \leqslant t \leqslant 1$, $0 \leqslant x_1 \leqslant x_2 \leqslant R$, $0 \leqslant y_1 \leqslant y_2 \leqslant R$, $0 \leqslant -z_1 \leqslant -z_2 \leqslant R$, $0 \leqslant -w_1 \leqslant -w_2 \leqslant R$;
- $(H2)$ $\max_{0\leqslant t\leqslant 1} f(t,R,R,-R,-R) \leqslant \Lambda R$, *where* $\Lambda = \left(\int_0^1 q(s)ds\right)^{-1}$;
- $(H3)$ $f(t,0,0,0,0) \not\equiv 0$.

Then problem (1)-(2) *has two monotone positive solutions w^* and v^* satisfying $\|w^*\| \leqslant R$, $\|v^*\| \leqslant R$ and $\lim_{n\to\infty}\|w_n - w^*\| = 0$, $\lim_{n\to\infty}\|v_n - v^*\| = 0$, where $w_n = T^n w_0$, $v_n = T^n v_0$, $n = 1, 2, \ldots$, $w_0(t) = (R/6)(3t - t^3)$, and $v_0(t) = 0$, for $t \in [0,1]$.*

Proof. Let $K_R = \{u \in K : \|u\| \leqslant R\}$; it follows that $T : K_R \to K_R$. In fact, if $u \in K_R$, then $\|u\| \leqslant R$; thus, for any $t \in [0,1]$,

$$\begin{aligned} 0 &\leqslant u(t) \leqslant \max_{0\leqslant t\leqslant 1} u(t) \leqslant \|u\| \leqslant R,\\ 0 &\leqslant u'(t) \leqslant \max_{0\leqslant t\leqslant 1} u'(t) \leqslant \|u\| \leqslant R,\\ 0 &\leqslant -u''(t) \leqslant \max_{0\leqslant t\leqslant 1}\left(-u''(t)\right) \leqslant \|u\| \leqslant R,\\ 0 &\leqslant -u'''(t) \leqslant \max_{0\leqslant t\leqslant 1}\left(-u'''(t)\right) \leqslant \|u\| \leqslant R, \end{aligned} \tag{20}$$

which together with the conditions $(H1)$ and $(H2)$ implies that

$$\begin{aligned} 0 &\leqslant (Tu)(t)\\ &= \int_0^1 G(t,s)\,q(s)\,f\left(s, u(s), u'(s), u''(s), u'''(s)\right)ds\\ &\leqslant \frac{1}{6}\left(3t - t^3\right)\int_0^1 q(s)\,f(s,R,R,-R,-R)\,ds\\ &\leqslant \frac{1}{2}\Lambda R\int_0^1 q(s)\,ds = \frac{1}{2}R, \quad t \in [0,1],\\ 0 &\leqslant (Tu)'(t)\\ &= \int_0^1 \frac{\partial G(t,s)}{\partial t}q(s)\,f\left(s, u(s), u'(s), u''(s), u'''(s)\right)ds\\ &\leqslant \frac{1}{2}\left(1 - t^2\right)\int_0^1 q(s)\,f(s,R,R,-R,-R)\,ds\\ &\leqslant \frac{1}{2}\Lambda R\int_0^1 q(s)\,ds = \frac{1}{2}R, \quad t \in [0,1],\\ 0 &\leqslant -(Tu)''(t)\\ &= -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2}q(s)\,f\left(s, u(s), u'(s), u''(s), u'''(s)\right)ds\\ &\leqslant t\int_0^1 q(s)\,f(s,R,R,-R,-R)\,ds\\ &\leqslant \Lambda R\int_0^1 q(s)\,ds = R, \quad t \in [0,1],\\ 0 &\leqslant -(Tu)'''(t)\\ &= -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3}q(s)\,f\left(s, u(s), u'(s), u''(s), u'''(s)\right)ds\\ &\leqslant \int_0^1 q(s)\,f(s,R,R,-R,-R)\,ds\\ &\leqslant \Lambda R\int_0^1 q(s)\,ds = R, \quad t \in [0,1]. \end{aligned} \tag{21}$$

Hence, we have shown that $\|Tu\| \leqslant R$, that is, $T : K_R \to K_R$.

(1) Let $w_0(t) = (R/6)(3t - t^3)$, $t \in [0,1]$; then $\|w_0\| = R$ and $w_0 \in K_R$. Denote $w_n = T^n w_0$, $n = 1, 2, 3, \ldots$. Since

$T : K_R \to K_R$, we have $w_n \in T(K_R) \subseteq K_R$, $n = 1, 2, 3, \ldots$. Since T is completely continuous, we assert that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $w_1 = Tw_0 \in K_R$, by Lemma 3, $(H1)$, and $(H2)$, we obtain

$$
\begin{aligned}
&w_1(t) \\
&= (Tw_0)(t) \\
&= \int_0^1 G(t,s)\, q(s)\, f\left(s, w_0(s), w_0'(s), w_0''(s), w_0'''(s)\right) ds \\
&= \int_0^1 G(t,s)\, q(s)\, f\left(s, \frac{R}{6}\left(3s - s^3\right), \right. \\
&\qquad \left. \frac{R}{2}\left(1 - s^2\right), -Rs, -R\right) ds \\
&\leqslant \frac{1}{6}\left(3t - t^3\right) \int_0^1 q(s)\, f(s, R, R, -R, -R)\, ds \\
&\leqslant \frac{R}{6}\left(3t - t^3\right) \Lambda \int_0^1 q(s)\, ds = \frac{R}{6}\left(3t - t^3\right) = w_0(t), \\
&\qquad t \in [0,1], \\
&w_1'(t) \\
&= (Tw_0)'(t) \\
&= \int_0^1 \frac{\partial G(t,s)}{\partial t} q(s)\, f\left(s, w_0(s), w_0'(s), w_0''(s), w_0'''(s)\right) ds \\
&= \int_0^1 \frac{\partial G(t,s)}{\partial t} q(s)\, f\left(s, \frac{R}{6}\left(3s - s^3\right), \right. \\
&\qquad \left. \frac{R}{2}\left(1 - s^2\right), -Rs, -R\right) ds \\
&\leqslant \frac{1}{2}\left(1 - t^2\right) \int_0^1 q(s)\, f(s, R, R, -R, -R)\, ds \\
&\leqslant \frac{R}{2}\left(1 - t^2\right) \Lambda \int_0^1 q(s)\, ds = \frac{R}{2}\left(1 - t^2\right) = w_0'(t), \\
&\qquad t \in [0,1], \\
&-w_1''(t) \\
&= -(Tw_0)''(t) \\
&= -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2} q(s)\, f\left(s, w_0(s), w_0'(s), w_0''(s), w_0'''(s)\right) ds \\
&= -\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2} q(s)\, f\left(s, \frac{R}{6}\left(3s - s^3\right), \right. \\
&\qquad \left. \frac{R}{2}\left(1 - s^2\right), -Rs, -R\right) ds \\
&\leqslant t \int_0^1 q(s)\, f(s, R, R, -R, -R)\, ds \\
&\leqslant t\Lambda R \int_0^1 q(s)\, ds = Rt = -w_0''(t), \quad t \in [0,1], \\
&-w_1'''(t) \\
&= -(Tw_0)'''(t) \\
&= -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3} q(s)\, f\left(s, w_0(s), w_0'(s), w_0''(s), w_0'''(s)\right) ds \\
&= -\int_0^1 \frac{\partial^3 G(t,s)}{\partial t^3} q(s)\, f\left(s, \frac{R}{6}\left(3s - s^3\right), \right. \\
&\qquad \left. \frac{R}{2}\left(1 - s^2\right), -Rs, -R\right) ds \\
&\leqslant \int_0^1 q(s)\, f(s, R, R, -R, -R)\, ds \\
&\leqslant \Lambda R \int_0^1 q(s)\, ds = R = -w_0'''(t), \quad t \in [0,1].
\end{aligned}
\tag{22}
$$

Thus,

$$
\begin{aligned}
&w_1(t) \leqslant w_0(t), \quad w_1'(t) \leqslant w_0'(t), \quad -w_1''(t) \leqslant -w_0''(t), \\
&-w_1'''(t) \leqslant -w_0'''(t), \quad t \in [0,1].
\end{aligned}
\tag{23}
$$

Furthermore, it follows from $(H1)$ that

$$
\begin{aligned}
&w_2(t) = (Tw_1)(t) \leqslant (Tw_0)(t) = w_1(t), \quad t \in [0,1], \\
&w_2'(t) = (Tw_1)'(t) \leqslant (Tw_0)'(t) = w_1'(t), \quad t \in [0,1], \\
&-w_2''(t) = -(Tw_1)''(t) \leqslant -(Tw_0)''(t) \\
&\qquad = -w_1''(t), \quad t \in [0,1], \\
&-w_2'''(t) = -(Tw_1)'''(t) \leqslant -(Tw_0)'''(t) \\
&\qquad = -w_1'''(t), \quad t \in [0,1].
\end{aligned}
\tag{24}
$$

By the induction, we obtain

$$
\begin{aligned}
&w_{n+1}(t) \leqslant w_n(t), \quad w_{n+1}'(t) \leqslant w_n'(t), \\
&-w_{n+1}''(t) \leqslant -w_n''(t), \quad -w_{n+1}'''(t) \leqslant -w_n'''(t), \\
&\qquad t \in [0,1], \ n = 0, 1, 2, \ldots.
\end{aligned}
\tag{25}
$$

Hence, there exists $w^* \in K_R$ such that $\lim_{n\to\infty} \|w_n - w^*\| = 0$. Applying the continuity of T and using $w_{n+1} = Tw_n$, we obtain $Tw^* = w^*$, which implies that w^* is a nonnegative solution of problem (1)-(2). Furthermore, by $(H3)$ we know that the zero function is not a solution of problem (1)-(2). Thus, $\max_{0\leqslant t\leqslant 1} w^*(t) > 0$. From the definition of the cone K,

it yields $w^*(t) \geqslant t\|w^*\| > 0$, $(w^*)'(t) \geqslant 0$, $t \in (0,1)$; that is, $w^*(t)$ is a monotone positive solution of problem (1)-(2).

(2) Let $v_0(t) = 0$, $t \in [0,1]$; then $v_0 \in K_R$. Let $v_n = T^n v_0$, $n = 1,2,3,\ldots$. With respect to $T : K_R \to K_R$, we obtain $v_n \in T(K_R) \subseteq K_R$, $n = 1,2,3,\ldots$. Since T is completely continuous, we assert that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $v_1 = Tv_0 = T0 \in K_R$, we obtain

$$\begin{aligned} R &\geqslant v_1(t) = (Tv_0)(t) = (T0)(t) \geqslant 0 = v_0(t), \quad t \in [0,1], \\ R &\geqslant v_1'(t) = (Tv_0)'(t) = (T0)'(t) \geqslant 0 = v_0'(t), \quad t \in [0,1], \\ R &\geqslant -v_1''(t) = -(Tv_0)''(t) = -(T0)''(t) \\ &\geqslant 0 = -v_0''(t), \quad t \in [0,1], \\ R &\geqslant -v_1'''(t) = -(Tv_0)'''(t) = -(T0)'''(t) \\ &\geqslant 0 = -v_0'''(t), \quad t \in [0,1]. \end{aligned} \tag{26}$$

So by $(H1)$, we have

$$\begin{aligned} v_2(t) &= (Tv_1)(t) \geqslant (Tv_0)(t) = v_1(t), \quad t \in [0,1], \\ v_2'(t) &= (Tv_1)'(t) \geqslant (Tv_0)'(t) = v_1'(t), \quad t \in [0,1], \\ -v_2''(t) &= -(Tv_1)''(t) \geqslant -(Tv_0)''(t) \\ &= -v_1''(t), \quad t \in [0,1], \\ -v_2'''(t) &= -(Tv_1)'''(t) \geqslant -(Tv_0)'''(t) \\ &= -v_1'''(t), \quad t \in [0,1]. \end{aligned} \tag{27}$$

By the induction, we further obtain

$$\begin{aligned} &v_{n+1}(t) \geqslant v_n(t), \quad v_{n+1}'(t) \geqslant v_n'(t), \quad -v_{n+1}''(t) \geqslant -v_n''(t), \\ &-v_{n+1}'''(t) \geqslant -v_n'''(t), \quad t \in [0,1], \ n = 0,1,2,\ldots. \end{aligned} \tag{28}$$

Hence, there exists $v^* \in K_R$ such that $\lim_{n\to\infty}\|v_n - v^*\| = 0$, and v^* is a monotone positive solution of problem (1)-(2). The proof is completed. □

Corollary 6. *Suppose that (H_1) and (H_3) hold. Assume*

$$(H4)\ \lim_{x\to+\infty}\max_{0\leqslant t\leqslant 1} f(t,x,x,-x,-x)/x < \Lambda.$$

Then problem (1)-(2) *has at least two monotone positive solutions w^* and v^* satisfying the conclusion of Theorem 5.*

Remark 7. w^* and v^* may coincide with each other and then the problem (1)-(2) has only one monotone positive solution in K_R.

Remark 8. By the proof of Theorem 5, $\{w_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge in $C^3[0,1]$ to the monotone positive solution w^* and v^* of the problem (1)-(2), respectively. In addition, the sequences $\{w_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{w_n'\}_{n=1}^{\infty}$, and $\{v_n'\}_{n=1}^{\infty}$ are increasing, while the sequences $\{w_n''\}_{n=1}^{\infty}$, $\{v_n''\}_{n=1}^{\infty}$, $\{w_n'''\}_{n=1}^{\infty}$, and $\{v_n'''\}_{n=1}^{\infty}$ are decreasing.

Remark 9. The iterative schemes in Theorem 5 start off with a known cubic function or the zero function, which is feasible for the computational purpose.

4. Examples

In this section we provide two examples to illustrate the application of our results.

Example 1. Consider the problem

$$\begin{aligned} u''''(t) &= \frac{1}{2}t + \frac{1}{18}u^2(t) + \frac{1}{6}u'(t) - \frac{1}{6}u''(t) \\ &\quad - \frac{1}{6}u'''(t), \quad t \in [0,1], \\ u(0) &= u'(1) = u''(0) = u'''(1) = 0. \end{aligned} \tag{29}$$

For this problem, $f(t,x,y,z,w) = t + (1/9)x^2 + (1/3)y - (1/3)z - (1/3)w$ and $q(t) = 1/2$. By the calculations, the conditions $(A1)$, $(A2)$, and $(H1)$–$(H3)$ of Theorem 5 hold for $R = 6$. From Theorem 5, we have that problem (29) has two monotone positive solutions w^* and v^* satisfying $\|w^*\| \leqslant 6$, $\|v^*\| \leqslant 6$ and $\lim_{n\to\infty}\|w_n - w^*\| = 0$, $\lim_{n\to\infty}\|v_n - v^*\| = 0$.

Moreover, the two iterative schemes are

$$\begin{aligned} w_0(t) &= 3t - t^3, \quad t \in [0,1], \\ w_{n+1}(t) &= \frac{1}{12}\int_0^t (t-s)^3 \\ &\quad \times \left(s + \frac{1}{9}w_n^2(s) + \frac{1}{3}w_n'(s) - \frac{1}{3}w_n''(s) - \frac{1}{3}w_n'''(s)\right) ds \\ &\quad + \frac{t}{12}\int_0^1 \left(6s - 3s^2 - t^2\right) \\ &\quad \times \left(s + \frac{1}{9}w_n^2(s) + \frac{1}{3}w_n'(s) - \frac{1}{3}w_n''(s) \right. \\ &\quad \left. - \frac{1}{3}w_n'''(s)\right) ds, \quad t \in [0,1], \ n = 0,1,2,\ldots, \\ v_0(t) &= 0, \quad t \in [0,1], \end{aligned}$$

$$v_{n+1}(t) = \frac{1}{12}\int_0^t (t-s)^3 \times \left(s + \frac{1}{9}v_n^2(s) + \frac{1}{3}v_n'(s) - \frac{1}{3}v_n''(s) - \frac{1}{3}v_n'''(s)\right) ds + \frac{t}{12}\int_0^1 \left(6s - 3s^2 - t^2\right) \times \left(s + \frac{1}{9}v_n^2(s) + \frac{1}{3}v_n'(s) - \frac{1}{3}v_n''(s) - \frac{1}{3}v_n'''(s)\right) ds, \quad t \in [0,1],\ n = 0,1,2,\ldots. \tag{30}$$

Using Matlab 7.5, the second and third terms of the two schemes are as follows:

$$w_1(t) = \frac{3559}{4536}t - \frac{2761}{7560}t^3 + \frac{1}{16}t^4 + \frac{1}{80}t^5 - \frac{1}{5040}t^8 + \frac{1}{90720}t^{10}, \quad t \in [0,1],$$

$$w_2(t) = \frac{5}{48}t - \frac{33627935}{1111065984}t^3 - \frac{134954777}{11110659840}t^5 + \frac{28810105}{8888527872}t^6 + \frac{75364613}{9258883200}t^7 - \frac{22814857}{7407106560}t^8 - \frac{31759981}{24690355200}t^9 + \frac{3292141}{4115059200}t^{10} - \frac{1777}{32659200}t^{11} - \frac{735019}{35554111488}t^{12} + \frac{48893}{49380710400}t^{13} - \frac{1227799}{888852787200}t^{14} + \frac{280289}{444426393600}t^{15} + \frac{10099}{148142131200}t^{16} - \frac{223}{5290790400}t^{17} + \frac{1}{1567641600}t^{18} + \frac{19}{98761420800}t^{19} + \frac{1}{10973491200}t^{20} + \frac{19}{444426393600}t^{21} - \frac{1}{98761420800}t^{22} - \frac{1}{888852787200}t^{23} + \frac{1}{3555411148800}t^{24}, \quad t \in [0,1],$$

$$v_1(t) = \frac{5}{48}t - \frac{1}{24}t^3 + \frac{1}{240}t^5, \quad t \in [0,1],$$

$$v_2(t) = \frac{5}{48}t - \frac{5159}{124416}t^3 + \frac{4859}{1244160}t^5 + \frac{25}{995328}t^6 + \frac{1}{7776}t^7 - \frac{5}{248832}t^8 - \frac{19}{622080}t^9 + \frac{1}{165888}t^{10} + \frac{11}{3110400}t^{11} - \frac{1}{1244160}t^{12} - \frac{1}{6220800}t^{13} + \frac{1}{24883200}t^{14}, \quad t \in [0,1]. \tag{31}$$

Example 2 (see [22]). Consider the problem

$$u''''(t) = \frac{\min\{t, 1-t\}}{48\sqrt{|1-2t|}} \times \left\{2 + \left[u(t) + u'(t)\right]^3 - \left[u''(t) + u'''(t)\right]^3\right\}, \quad t \in \left[0, \frac{1}{2}\right) \bigcup \left(\frac{1}{2}, 1\right],$$
$$u(0) = u'(1) = u''(0) = u'''(1) = 0. \tag{32}$$

In this problem, $f(t,x,y,z,w) = (1/48)[2+(x+y)^3-(z+w)^3]$ and $q(t) = (\min\{t, 1-t\})/\sqrt{|1-2t|}$. Obviously, $q(t)$ and $f(t,x,y,z,w)$ satisfy conditions $(A1)$ and $(A2)$. In addition, $f(t,x,y,z,w)$ is increasing with regard to x and y and is decreasing with z and w, and $f(t,0,0,0,0) \not\equiv 0$. Moreover,

$$\max_{0\leqslant t\leqslant 1} f\left(t, \frac{1}{30}, \frac{1}{30}, -\frac{1}{30}, -\frac{1}{30}\right) = \frac{1}{48}\left[2 + \frac{2}{3375}\right] < \frac{3}{2} \times \frac{1}{30}, \qquad \int_0^1 q(t)\,dt = \int_0^1 \frac{\min\{t, 1-t\}}{\sqrt{|1-2t|}}dt = \frac{2}{3}. \tag{33}$$

Therefore, assumptions $(H1)$–$(H3)$ of Theorem 5 are satisfied with $R = 1/30$. From Theorem 5, problem (32) has two monotone positive solutions w^* and v^* satisfying $\|w^*\| \leqslant 1/30$, $\|v^*\| \leqslant 1/30$ and $\lim_{n\to\infty}\|w_n - w^*\| = 0$, $\lim_{n\to\infty}\|v_n - v^*\| = 0$.

Moreover, the two iterative schemes are

$$w_0(t) = \frac{1}{180}\left(3t - t^3\right), \quad t \in [0,1],$$

$$w_{n+1}(t) = \frac{1}{144}\int_0^t (t-s)^3 \frac{\min\{s, 1-s\}}{\sqrt{|1-2s|}} \times \left\{2 + \left[w_n(s) + w_n'(s)\right]^3 - \left[w_n''(s) + w_n'''(s)\right]^3\right\} ds$$

$$+\frac{t}{144}\int_0^1\left(6s-3s^2-t^2\right)\frac{\min\{s,1-s\}}{\sqrt{|1-2s|}}\times\left\{\left[2+w_n(s)+w_n'(s)\right]^3-\left[w_n''(s)+w_n'''(s)\right]^3\right\}ds,\quad t\in[0,1],\ n=0,1,2,\ldots,$$

$$v_0(t)=0,\quad t\in[0,1],$$

$$v_{n+1}(t)=\frac{1}{144}\int_0^t(t-s)^3\frac{\min\{s,1-s\}}{\sqrt{|1-2s|}}\times\left\{2+\left[v_n(s)+v_n'(s)\right]^3-\left[v_n''(s)+v_n'''(s)\right]^3\right\}ds+\frac{t}{144}\int_0^1\left(6s-3s^2-t^2\right)\frac{\min\{s,1-s\}}{\sqrt{|1-2s|}}\times\left\{2+\left[v_n(s)+v_n'(s)\right]^3-\left[v_n''(s)+v_n'''(s)\right]^3\right\}ds,\quad t\in[0,1],\ n=0,1,2,\ldots.\tag{34}$$

Using Mathlab 7.5, the second terms of the two schemes are given as follows:

$$w_1(t)=-\frac{8449074379661}{23951656146600000}+\frac{441468132942481}{19516164267600000}t-\frac{10842948935033}{1951616426760000}t^2-\frac{785738110303}{169705776240000}t^3+\left(\frac{8449074379661}{23951656146600000}+\frac{122149319510627}{60221306882880000}t+\frac{2602305979889593}{702581913633600000}t^2-\frac{265540119476783}{150553267207200000}t^3-\frac{13518333667229}{19161324917280000}t^4-\frac{1855160149}{117096985605600000}t^5-\frac{74353511}{17564547840840000}t^6-\frac{26984453}{175645478408400000}t^7+\frac{1291391}{117096985605600000}t^8-\frac{4131727}{38322649834560000}t^9-\frac{31823}{2105640100800000}t^{10}+\frac{3953}{491316023520000}t^{11}+\frac{509}{216757069200000}t^{12}+\frac{1}{5704133400000}t^{13}\right)\times\sqrt{1-2t},\quad \text{for } t\in\left[0,\frac{1}{2}\right],$$

$$w_1(t)=-\frac{8449074379661}{23951656146600000}+\frac{441468132942481}{19516164267600000}t-\frac{10842948935033}{1951616426760000}t^2-\frac{785738110303}{169705776240000}t^3+\left(-\frac{929396364017203}{2107745740900800000}+\frac{62633084321047}{22910279792400000}t-\frac{817863130365013}{140516382726720000}t^2+\frac{2416385432114987}{526936435225200000}t^3-\frac{10620690488597}{15055326720720000}t^4-\frac{568554781}{117096985605600000}t^5-\frac{686043443}{175645478408400000}t^6-\frac{546781}{3054703972320000}t^7+\frac{46120147}{234193971211200000}t^8-\frac{1594511}{19161324917280000}t^9-\frac{23767}{867028276800000}t^{10}+\frac{11287}{2456580117600000}t^{11}+\frac{13}{6193059120000}t^{12}+\frac{1}{5704133400000}t^{13}\right)\times\sqrt{2t-1},\quad \text{for } t\in\left[\frac{1}{2},1\right],$$

$$v_1(t)=-\frac{1}{2835}+\frac{19}{840}t-\frac{1}{180}t^2-\frac{1}{216}t^3+\left(\frac{1}{2835}-\frac{23}{11340}t+\frac{1}{270}t^2-\frac{1}{576}t^3-\frac{2}{2835}t^4\right)\sqrt{1-2t},\quad \text{for } t\in\left[0,\frac{1}{2}\right],$$

$$v_1(t)=-\frac{1}{2835}+\frac{19}{840}t-\frac{1}{180}t^2-\frac{1}{216}t^3+\left(-\frac{1}{2268}+\frac{31}{11340}t-\frac{11}{1890}t^2+\frac{13}{2835}t^3-\frac{2}{2835}t^4\right)\sqrt{2t-1},\quad \text{for } t\in\left[\frac{1}{2},1\right].\tag{35}$$

Remark 10. This example was carried out firstly in [22], in which the author showed that the solution $u^*\in C^3[0,1]$ is nontrivial, $\|u^*\|\leqslant 1$, and the iterative scheme starts off with

$$u_0(t)=-\frac{7}{24}\int_0^1 G(t,s)\frac{\min\{s,1-s\}}{\sqrt{|1-2s|}}ds.\tag{36}$$

In the present paper, we proved that the solution $u^*\in C^3[0,1]$ is monotone positive, $\|u^*\|\leqslant 1/30$, and the iterative scheme starts off with $u_0(t)=(1/180)(3t-t^3)$ or $u_0(t)=0$, which is much more simple than that of [22].

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