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Research Article

On the Existence of the Solutions for Some Nonlinear Volterra Integral Equations

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We present a theorem which gives sufficient conditions for existence of at least one solution for some nonlinear functional integral equations in the space of continuous functions on the interval $[0, a]$. To do this, we will use Darbo's fixed-point theorem associated with the measure of noncompactness. We give also an example satisfying the conditions of our main theorem but not satisfying the conditions described by Maleknejad et al. (2009).

1. Introduction

As it is known, nonlinear integral equations constitute an important branch of nonlinear analysis. Particularly integral equations are often used in the characterization of several problems of engineering, mechanics, physics, economics, and so on. Some authors have given some results for solvability of some functional integral equations such as Mureşan in [1], Banaś and Sadarangani in [2], and Djebali and Hammache in [3]. The following equation has been considered in [4]:

$$x(t) = f(t, x(\alpha(t))) \int_0^1 u(t, s, x(s))\, ds, \tag{1}$$

for $t \in [0, 1]$. Maleknejad et al. in [5] studied the existence of solutions of the following equation:

$$x(t) = f(t, x(\alpha(t))) \int_0^t u(t, s, x(s))\, ds, \quad t \in [0, 1] \tag{2}$$

under the following conditions.

(K1) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist nonnegative constants μ and k such that

$$\begin{gathered} |f(t, 0)| \le \mu \\ |f(t, y_1) - f(t, y_2)| \le k |y_1 - y_2| \end{gathered} \tag{3}$$

for all $y_1, y_2 \in \mathbb{R}$ and $t \in [0, 1]$.

(K2) $u : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the sublinearity condition, so there exist constants α and β such that

$$|u(t, s, x)| \le \alpha + \beta |x| \tag{4}$$

for all $t, s \in [0, 1]$ and $x \in \mathbb{R}$.

(K3) $(\alpha' - \beta')^2 > 2(\alpha' + \beta') - 1$ for $\alpha' = k\alpha$ and $\beta' = \mu\beta$.

(K4) $kk' < 1$ for $k' = \sup\{|u(t, s, x(t))| : t, s \in [0, 1], x \in BC([0, 1])\}$.

In this paper, we consider the following nonlinear functional integral equation:

$$\begin{aligned} x(t) &= g(t, x(\beta(t))) \\ &\quad + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))\, ds, \quad t \in [0, a] \end{aligned} \tag{5}$$

which is more general than the equation given in [5].

In Section 2, we present some definitions and preliminary results about the concept of measure of noncompactness. In Section 3, we give our main results concerning the existence of solutions of the integral equation (5) by applying Darbo's fixed-point theorem associated with the measure of noncompactness defined by Banaś and Goebel [6] and, finally, we establish an example to show that these results are applicable.

2. Definitions and Auxiliary Facts

In this section, we give some definitions and results which will be needed next section. Let $(E, \|\cdot\|)$ be an infinite Banach space with zero element θ. We write $B(x, r)$ to denote the closed ball centered at x with radius r and, especially, we write B_r in case of $x = \theta$. We write $\overline{X}$, Conv X to denote the closure X and closed convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ indicate the subfamily of all relatively compact sets. Finally, the standard algebraic operations on sets are denoted by λX and $X + Y$, respectively [2].

We use the following definition of the measure of noncompactness, given in [6].

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv } X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

(5) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $\cap_{n=1}^{\infty} X_n$ is nonempty.

Theorem 2 (see [6]). *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and let $F : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(FX) \leq k\mu(X) \tag{6}$$

for any nonempty subset X of C. Then, F has a fixed point in the set C.

As it is known, the family of all real-valued and continuous functions defined on the interval $[0, a]$ forms a Banach space with the standard norm

$$\|x\| = \max\{|x(t)| : t \in [0, a]\}. \tag{7}$$

Let X be a fixed subset of $\mathfrak{M}_{C[0,a]}$. For $\varepsilon > 0$ and $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus continuity of x defined by

$$\omega(x, \varepsilon) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, a],\ |t_1 - t_2| \leq \varepsilon\}. \tag{8}$$

Furthermore, let $\omega(X, \varepsilon)$ and $\omega_0(X)$ be defined by

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},\qquad \omega_0(X) = \lim_{\varepsilon\to 0}\omega(X, \varepsilon). \tag{9}$$

The authors have shown in [6] that the previous function ω_0 is a measure of noncompactness in the space $C[0, a]$.

3. The Main Result

First of all, we write I to denote the interval $[0, a]$ throughout this section. We study the functional integral equation (5) under the following conditions.

(a) $\alpha, \beta : I \to I$, $\varphi : I \to \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \to I$ are continuous.

Remark 3. Note that assumption (a) implies that there exists positive constant C such that

$$\varphi(t) \leq C \tag{10}$$

for all $t \in I$.

(b) Functions $f, g : I \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist positive constants k and l such that

$$|f(t, x_1) - f(t, x_2)| \leq l|x_1 - x_2|,\qquad |g(t, x_1) - g(t, x_2)| \leq k|x_1 - x_2| \tag{11}$$

for all $t \in I$ and $x_1, x_2 \in \mathbb{R}$.

Remark 4. Note that assumption (b) implies that there exist positive constants M and N such that

$$|f(t, 0)| \leq N,\qquad |g(t, 0)| \leq M \tag{12}$$

for all $t \in I$.

(c) $u : I \times [0, C] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive constants m, n, and p such that

$$|u(t, s, x)| \leq m + n|x|^p \tag{13}$$

for all $t \in I$, $s \in [0, C]$ and $x \in \mathbb{R}$.

(d) The inequality

$$M + C(m + n)(l + N) + k < 1 \tag{14}$$

holds.

Theorem 5. *Under assumptions (a)–(d), there exists at least one $r_0 \in (0, 1)$ such that (5) has at least one solution $x = x(t)$ which belongs to $B_{r_0} \subset C[0, a]$.*

Proof. We define the continuous function $h : [0, 1] \to \mathbb{R}$ such that

$$h(r) = (k-1)r + M + C\left(lrm + nlr^{p+1} + Nm + Nnr^p\right), \tag{15}$$

where p is the constant given in assumption (c). Then $h(0) > 0$ and $h(1) < 0$ by assumption (d). The continuity of h guarantees that there exists the number r_0 such that $r_0 \in (0, 1)$ and $h(r_0) = 0$. Now, we will prove that (5) has at least one solution $x = x(t)$ which belongs to $B_{r_0} \subset C[0, a]$. Note that we will use Theorem 2 as our main tool. We define operator F by

$$(Fx)(t) = g(t, x(\beta(t))) + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))\, ds, \tag{16}$$

for any $x \in C[0, a]$. Using the conditions of Theorem 5, we infer that Fx is continuous on I. For any $x \in B_{r_0}$,

$$\begin{aligned} &|(Fx)(t)| \\ &= \left| g(t, x(\beta(t))) + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))\, ds \right| \\ &\le |g(t, x(\beta(t))) - g(t, 0)| + |g(t, 0)| \\ &\quad + \{|f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)|\} \\ &\qquad \times \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))\, ds \\ &\le k\|x\| + M + C(l\|x\| + N)(m + n\|x\|^p) \\ &\le kr_0 + M + C(lr_0 + N)(m + n(r_0)^p) \\ &= h(r_0) + r_0 \\ &= r_0. \end{aligned} \tag{17}$$

This result shows that $Fx \in B_{r_0}$. Now we will prove that operator $F : B_{r_0} \to B_{r_0}$ is continuous. To do this, consider $\varepsilon > 0$ and any $x, y \in B_{r_0}$ such that $\|x - y\| \le \varepsilon$. Then, taking into account the equality

$$\begin{aligned} &(Fx)(t) - (Fy)(t) \\ &\quad = g(t, x(\beta(t))) - g(t, y(\beta(t))) \\ &\qquad + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))\, ds \\ &\qquad - f(t, y(\alpha(t))) \int_0^{\varphi(t)} u(t, s, y(\gamma(s)))\, ds \\ &\qquad + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, y(\gamma(s)))\, ds \\ &\qquad - f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, y(\gamma(s)))\, ds, \end{aligned} \tag{18}$$

we have by (18) that the inequality

$$\begin{aligned} &|(Fx)(t) - (Fy)(t)| \\ &\quad \le k|x(\beta(t)) - y(\beta(t))| \\ &\qquad + \{|f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)|\} \\ &\qquad\quad \times \int_0^{\varphi(t)} |u(t, s, x(\gamma(s))) - u(t, s, y(\gamma(s)))|\, ds \\ &\qquad + |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \\ &\qquad\quad \times \int_0^{\varphi(t)} |u(t, s, y(\gamma(s)))|\, ds \\ &\quad \le k\|x - y\| + C(lr_0 + N)\,\omega_{u_3}(I, \varepsilon) \\ &\qquad + Cl\|x - y\|(m + n\|y\|^p) \\ &\quad \le k\varepsilon + C(lr_0 + N)\,\omega_{u_3}(I, \varepsilon) + Cl\varepsilon(m + n(r_0)^p) \end{aligned} \tag{19}$$

holds, where

$$\begin{aligned} &\omega_{u_3}(I, \varepsilon) \\ &\quad = \sup\{|u(t, s, x) - u(t, s, y)| : t \in I, \\ &\qquad\quad s \in [0, C],\ x, y \in [-r_0, r_0],\ |x - y| \le \varepsilon\}. \end{aligned} \tag{20}$$

On the other hand, since function $u = u(t, s, x)$ is uniformly continuous on $I \times [0, C] \times [-r_0, r_0]$, we infer that $\omega_{u_3}(I, \varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, the previous estimate (19) proves that operator F is continuous on ball B_{r_0}. Now, we will show that operator F satisfies (6) with respect to measure of noncompactness ω_0 given by (9). To do this, we choose a fixed arbitrary $\varepsilon > 0$. Let us take $x \in X$ and $t_1, t_2 \in I$ with $|t_1 - t_2| \le \varepsilon$, for any nonempty subset X of B_{r_0}. Since

$$\begin{aligned} &(Fx)(t_1) - (Fx)(t_2) \\ &\quad = g(t_1, x(\beta(t_1))) \\ &\qquad + f(t_1, x(\alpha(t_1))) \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s)))\, ds \\ &\qquad - g(t_2, x(\beta(t_2))) \\ &\qquad - f(t_2, x(\alpha(t_2))) \int_0^{\varphi(t_2)} u(t_2, s, x(\gamma(s)))\, ds, \end{aligned} \tag{21}$$

we get

$$\begin{aligned} &|(Fx)(t_1) - (Fx)(t_2)| \\ &\le |g(t_1, x(\beta(t_1))) - g(t_2, x(\beta(t_1)))| \\ &\quad + |g(t_2, x(\beta(t_1))) - g(t_2, x(\beta(t_2)))| \\ &\quad + \left| f(t_1, x(\alpha(t_1))) \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s)))\, ds \right. \\ &\qquad \left. - f(t_1, x(\alpha(t_2))) \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s)))\, ds \right| \\ &\quad + \left| f(t_1, x(\alpha(t_2))) \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s)))\, ds \right. \\ &\qquad \left. - f(t_2, x(\alpha(t_2))) \int_0^{\varphi(t_1)} u(t_1, s, x(\gamma(s)))\, ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\left|f\left(t_2, x\left(\alpha\left(t_2\right)\right)\right) \int_0^{\varphi(t_1)} u\left(t_1, s, x\left(\gamma(s)\right)\right) ds\right.\\
&\qquad \left.- f\left(t_2, x\left(\alpha\left(t_2\right)\right)\right) \int_0^{\varphi(t_2)} u\left(t_2, s, x\left(\gamma(s)\right)\right) ds\right|\\
&\leq \omega_g(I,\varepsilon) + k\left|x\left(\beta\left(t_1\right)\right) - x\left(\beta\left(t_2\right)\right)\right|\\
&\quad + A\left|x\left(\alpha\left(t_1\right)\right) - x\left(\alpha\left(t_2\right)\right)\right| + C\left(m + n(r_0)^p\right)\omega_f(I,\varepsilon)\\
&\quad + \left\{\left|f\left(t_2, x\left(\alpha\left(t_2\right)\right)\right) - f\left(t_2, 0\right)\right| + \left|f\left(t_2, 0\right)\right|\right\}\\
&\quad \times \left(\int_0^{\varphi(t_2)} \left|u\left(t_1, s, x\left(\gamma(s)\right)\right) - u\left(t_2, s, x\left(\gamma(s)\right)\right)\right| ds\right.\\
&\qquad\qquad \left. + \left|\int_{\varphi(t_2)}^{\varphi(t_1)} u\left(t_2, s, x\left(\gamma(s)\right)\right) ds\right|\right)\\
&\leq \omega_g(I,\varepsilon) + k\omega\left(x, \omega(\beta,\varepsilon)\right) + A\omega\left(x, \omega(\alpha,\varepsilon)\right)\\
&\quad + C\left(m + n(r_0)^p\right)\omega_f(I,\varepsilon)\\
&\quad + \left(lr_0 + N\right)\left[C\omega_{u_1}(I,\varepsilon) + \left(m + n(r_0)^p\right)\omega(\varphi,\varepsilon)\right]
\end{aligned}
\tag{22}
$$

from (21), where

$$
\begin{aligned}
A &= Cl\left(m + n(r_0)^p\right),\\
\omega_{f_i}(I,\varepsilon) &= \sup\left\{\left|f_i(t,x) - f_i\left(t',x\right)\right| : t,t' \in I,\right.\\
&\qquad\qquad \left. x \in \left[-r_0, r_0\right],\ \left|t - t'\right| \leq \varepsilon\right\},
\end{aligned}
\tag{23}
$$

for $i = 1, 2$ such that $f_1 = f$ and $f_2 = g$. Also,

$$
\begin{aligned}
&\omega_{u_1}(I,\varepsilon)\\
&\quad = \sup\left\{\left|u(t,x,y) - u\left(t',x,y\right)\right| : t,t' \in I,\right.\\
&\qquad\qquad \left. x \in [0,C],\ y \in \left[-r_0, r_0\right],\ \left|t - t'\right| \leq \varepsilon\right\},\\
&\omega\left(\alpha_j, \varepsilon\right)\\
&\quad = \sup\left\{\left|\alpha_j(t) - \alpha_j\left(t'\right)\right| : t,t' \in I,\ \left|t - t'\right| \leq \varepsilon\right\},
\end{aligned}
\tag{24}
$$

for $j = 1, 2, 3$ such that $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and $\alpha_3 = \varphi$. Thus, by using the previous estimate (22), we can write

$$
\begin{aligned}
&\omega(FX,\varepsilon)\\
&\quad \leq \omega_g(I,\varepsilon) + k\omega\left(X, \omega(\beta,\varepsilon)\right)\\
&\qquad + A\omega\left(X, \omega(\alpha,\varepsilon)\right) + C\left(m + n(r_0)^p\right)\omega_f(I,\varepsilon)\\
&\qquad + \left(lr_0 + N\right)\left[C\omega_{u_1}(I,\varepsilon) + \left(m + n(r_0)^p\right)\omega(\varphi,\varepsilon)\right].
\end{aligned}
\tag{25}
$$

We have that $\omega(\beta,\varepsilon) \to 0$, $\omega(\alpha,\varepsilon) \to 0$, and $\omega(\varphi,\varepsilon) \to 0$ as $\varepsilon \to 0$ since functions β, α, and φ are uniformly continuous on I. Similarly, we get $\omega_g(I,\varepsilon) \to 0$, $\omega_f(I,\varepsilon) \to 0$, and $\omega_{u_1}(I,\varepsilon) \to 0$ as $\varepsilon \to 0$ since functions g, f, and u are uniformly continuous on $I\times[-r_0,r_0]$, $I\times[-r_0,r_0]$, and $I\times[0,C]\times[-r_0,r_0]$, respectively. Hence, we obtain that

$$
\omega_0(FX) \leq (k + A)\,\omega_0(X).
\tag{26}
$$

Thus, since $A < Cl(m+n)$ from (23) and $k + A < 1$ from condition (14), we derive that operator F is a contraction on ball B_{r_0} with respect to measure of noncompactness ω_0. Therefore, from Theorem 2 we get that F has at least one fixed point in B_{r_0}. Consequently, nonlinear functional integral equation (5) has at least one continuous solution in $B_{r_0} \subset C[0,a]$. This completes the proof. □

Example 6. Consider the following nonlinear functional integral equation in $C[0,1]$:

$$
\begin{aligned}
x(t) &= \frac{x(t) + \ln(t+1)}{6 + t^2}\\
&\quad + \frac{3x\left(t^2\right) + t^3}{18}\int_0^{\sqrt{t}} \frac{s\cos(tx(s)) + x^3(s)}{\exp(t) + s^2t^3} ds.
\end{aligned}
\tag{27}
$$

Put

$$
\begin{gathered}
\alpha(t) = t^2, \qquad \beta(t) = t,\\
\gamma(s) = s, \qquad \varphi(t) = \sqrt{t},\\
g(t,x) = \frac{x + \ln(t+1)}{6 + t^2}, \qquad f(t,x) = \frac{3x + t^3}{18},\\
u(t,s,x) = \frac{s\cos(tx) + x^3}{\exp(t) + s^2t^3},\\
C = m = n = 1, \qquad k = l = M = \frac{1}{6},\\
N = \frac{1}{18}, \qquad p = 3.
\end{gathered}
\tag{28}
$$

It is easy to prove that the assumptions of Theorem 5 hold. Therefore, Theorem 5 guarantees that (27) has at least one solution $x = x(t) \in B_{r_0} \subset C[0,1]$. Since there exist no constants α and β such that

$$
|u(t,s,x)| \leq \alpha + \beta|x|,
\tag{29}
$$

for all $t, s \in [0,1]$ and $x \in \mathbb{R}$, the results presented in [5] are inapplicable to integral equation (27) with

$$
\begin{gathered}
g(t,x) = 0, \qquad \gamma(s) = s,\\
\varphi(t) = t, \qquad f(t,x) = \frac{3x + t^3}{18},\\
u(t,s,x) = \frac{s\cos(tx) + x^3}{\exp(t) + s^2t^3}.
\end{gathered}
\tag{30}
$$

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