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Research Article

Weighted Differentiation Composition Operators to Bloch-Type Spaces

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We characterized the boundedness and compactness of weighted differentiation composition operators from BMOA and the Bloch space to Bloch-type spaces. Moreover, we obtain new characterizations of boundedness and compactness of weighted differentiation composition operators.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all functions holomorphic on $\mathbb{D}$, $dA(z) = (1/\pi)dxdy$ the normalized area measure on $\mathbb{D}$, and H^∞ the space of all bounded holomorphic functions with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

Let $\alpha > 0$. The *α-Bloch space* $\mathscr{B}^\alpha$ on $\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \left|f'(z)\right| < \infty. \tag{1}$$

The *little α-Bloch space* $\mathscr{B}_0^\alpha$ consists of all $f \in \mathscr{B}^\alpha$ such that

$$\lim_{|z|\to 1}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| = 0. \tag{2}$$

Both spaces $\mathscr{B}^\alpha$ and $\mathscr{B}_0^\alpha$ are Banach spaces with the norm

$$\|f\|_{\mathscr{B}^\alpha} = \left|f(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \left|f'(z)\right|, \tag{3}$$

and $\mathscr{B}_0^\alpha$ is a closed subspace of $\mathscr{B}^\alpha$. If $\alpha = 1$, they become the classical Bloch space $\mathscr{B}$ and little Bloch space $\mathscr{B}_0$, respectively. For any $\alpha > 0$, the space $\mathscr{A}_\infty^\alpha$ consists of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{A}_\infty^\alpha} = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|f(z)\right| < \infty. \tag{4}$$

For information of such spaces, see, for example, [1–4].

For $a \in \mathbb{D}$, let $\sigma_a(z) = (a-z)/(1-\overline{a}z)$ be the automorphism of $\mathbb{D}$ that interchanges 0 and a. Let the Green function in $\mathbb{D}$ with logarithmic singularity at a be given by

$$g(z,a) = \log\left|\frac{1-\overline{a}z}{a-z}\right| = \log\frac{1}{\left|\sigma_a(z)\right|}. \tag{5}$$

The space BMOA consists of all f in the Hardy space H^2 such that

$$\sup_{a\in\mathbb{D}}\|f\circ\sigma_a - f(a)\|_{H^2} < \infty. \tag{6}$$

BMOA is a Banach space under following norm (see, e.g., [5]):

$$\|f\|_{\mathrm{BMOA}} = \left|f(0)\right| + \sup_{a\in\mathbb{D}}\|f\circ\sigma_a - f(a)\|_{H^2}. \tag{7}$$

Let φ and ψ be holomorphic maps on the open unit disk $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. For a nonnegative integer n, we define a linear operator $D_{\varphi,\psi}^n$ as follows:

$$D_{\varphi,\psi}^n f = \psi\cdot\left(f^{(n)}\circ\varphi\right), \quad f \in H(\mathbb{D}). \tag{8}$$

We call it weighted differentiation composition operators, which was defined in [6, 7]. If $n = 0$ and $\psi \equiv 1$, $D_{\varphi,\psi}^n$ becomes C_φ induced by φ, defined as $C_\varphi f = f\circ\varphi$, $f \in H(\mathbb{D})$. If $\psi = 1$ and $\varphi(z) = z$, then $D_{\varphi,\psi}^n$ is the differentiation operator defined as $D^n f = f^{(n)}$. If $n = 0$, then we get the weighted composition operator ψC_φ defined as $\psi C_\varphi f = \psi \cdot (f \circ \varphi)$. If $n = 1$ and $\psi(z) = \varphi'(z)$, then $D^n_{\varphi,\psi}$ reduces to DC_φ. When $\psi \equiv 1$, then $D^n_{\varphi,\psi}$ reduces to differentiation composition operator $C_\varphi D^n$ (also named as product of differentiation and composition operator). If we put $\varphi(z) = z$, then $D^n_{\varphi,\psi} = M_\psi D^n$, the product of multiplication and differentiation operator.

The boundedness and compactness of differentiation composition operator between spaces of holomorphic functions have been studied extensively. For example, Hibschweiler; Portnoy and Ohno studied differentiation composition operator $C_\varphi D$ on Hardy and Bergman spaces in [8, 9]; Li; Stević and Ohno studied $C_\varphi D$ on Bloch type spaces in [10–12]; Wu and Wulan gave a new compactness criterion of $C_\varphi D^m$ on the Bloch space in [13]. Recently, the weighted differentiation composition operator between different function spaces has also been investigated by several authors (see, for example, [14–21]).

Boundedness, compactness, and essential norm of weighted composition operator ψC_φ between Bloch-type spaces have been studied in [22–24]. Recently, Manhas and Zhao [25] and Hyvärinen and Lindström [26] gave a new characterization of boundedness and compactness of ψC_φ in terms of the norm of φ^n (for the compactness of composition operator, see [27, 28]).

Motivated by [13, 25, 26], we study the operator $D^n_{\varphi,\psi}$ $(n \geq 1)$ from BMOA and Bloch space to Bloch-type spaces.

Throughout this paper, constants are denoted by C; they are positive and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. When $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

2. Some Lemmas

It is well known that $H^\infty \subset \mathrm{BMOA} \subset \mathcal{B}$. From the definition of the norm, we know

$$\|f\|_{\mathrm{BMOA}} \lesssim \|f\|_\infty, \quad f \in H^\infty. \tag{9}$$

Indeed, Girela proved that

$$\|f\|_{\mathcal{B}} \leq \|f\|_{\mathrm{BMOA}_1} \tag{10}$$

in Corollary 5.2 of [5]. The following lemma is from Lemma 5 in [29] (see also Lemma 4.12 of [4]).

Lemma 1. *If $f \in H(\mathbb{D})$, then*

$$|f(0)|^2 \leq 2\int_{\mathbb{D}} |f(z)|^2 \log\frac{1}{|z|} dA(z). \tag{11}$$

The following lemma may be known, but we fail to find its reference; so we give a proof for the completeness of the paper.

Lemma 2. *Let $f \in H(\mathbb{D})$. Then,*

$$\|f\|_{\mathcal{B}} \leq \|f\|_{BMOA}. \tag{12}$$

Proof. Applying Littlewood-Paley identity

$$\|f\|^2_{H^2} = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \tag{13}$$

and Lemma 1, we have

$$\begin{aligned} \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2} &= \sup_{a\in\mathbb{D}} \left(2\int_{\mathbb{D}} \left|f'(\sigma_a(z))\sigma_a'(z)\right|^2 \log\frac{1}{|z|} dA(z)\right)^{1/2} \\ &\geq \sup_{a\in\mathbb{D}} \left(1 - |a|^2\right)\left|f'(a)\right|. \end{aligned} \tag{14}$$

It follows from the definitions of Bloch space and BMOA space that

$$\|f\|_{\mathcal{B}} \leq \|f\|_{\mathrm{BMOA}}. \tag{15}$$

□

By Theorem 6.2 of [5] and the proof of Theorem 1 of [30], we have the following lemma.

Lemma 3. *Let n be a fixed positive integer and $f \in \mathcal{B}$ with $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. If*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^2 \left(1 - |z|^2\right)^{2n-2} \left(1 - |\sigma_a(z)|^2\right) dA(z) \lesssim 1, \tag{16}$$

then $\|f\|_{BMOA} \lesssim 1$.

Lemma 4. *Suppose that n is a fixed positive integer. Let $k \in \mathbb{N}^+$, $0 \leq x \leq 1$, and*

$$H^n_k(x) = \begin{cases} k(k-1)\cdots(k-n+1)(1-x)^n x^{k-n} & \text{if } k > n \\ n!(1-x)^n & \text{if } k = n. \end{cases} \tag{17}$$

If $k \geq n$, then there are two positive constants c_n and C_n, depending only on n, such that

$$c_n \leq H^n_k(x) \leq C_n, \quad \text{for } \frac{k-n}{k} \leq x \leq \frac{k-n+1}{k+1}. \tag{18}$$

Proof. The proof is similar to that of Lemma 2.2 of [13] and is so omitted. □

3. Boundedness of $D^n_{\varphi,\psi}$

In this section, we characterize the boundedness of $D^n_{\varphi,\psi}$ from BMOA and the Bloch space to Bloch-type spaces.

Theorem 5. *Let $\alpha > 0$, $\psi \in H(\mathbb{D})$, $n \in \mathbb{N}^+$, and φ a holomorphic self-map of $\mathbb{D}$. Then, the following statements are equivalent:*

(a) *$D^n_{\varphi,\psi} : BMOA \to \mathcal{B}^\alpha$ is bounded.*

(b) *$D^n_{\varphi,\psi'} : BMOA \to \mathcal{A}^\alpha_\infty$ and $D^{n+1}_{\varphi,\psi\varphi'} : BMOA \to \mathcal{A}^\alpha_\infty$ are bounded.*

(c) *$D^n_{\varphi,\psi} : \mathcal{B}_0 \to \mathcal{B}^\alpha$ is bounded.*

(d) $D^n_{\varphi,\psi'} : \mathcal{B}_0 \to \mathcal{A}^\alpha_\infty$ *and* $D^{n+1}_{\varphi,\psi\varphi'} : \mathcal{B}_0 \to \mathcal{A}^\alpha_\infty$ *are bounded.*

(e) $D^n_{\varphi,\psi} : \mathcal{B} \to \mathcal{B}^\alpha$ *is bounded.*

(f) $D^n_{\varphi,\psi'} : \mathcal{B} \to \mathcal{A}^\alpha_\infty$ *and* $D^{n+1}_{\varphi,\psi\varphi'} : \mathcal{B} \to \mathcal{A}^\alpha_\infty$ *are bounded.*

(g) $\sup_{z\in\mathbb{D}}((1-|z|^2)^\alpha/(1-|\varphi(z)|^2)^n)|\psi'(z)| < \infty$ *and* $\sup_{z\in\mathbb{D}}((1-|z|^2)^\alpha/(1-|\varphi(z)|^2)^{n+1})|\psi(z)\varphi'(z)| < \infty$.

(h) $\sup_{k\in\mathbb{N}}\|D^n_{\varphi,\psi'}(z^k)\|_{\mathcal{A}^\alpha_\infty} < \infty$ *and* $\sup_{k\in\mathbb{N}}\|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\|_{\mathcal{A}^\alpha_\infty} < \infty$.

Proof. It is obvious that (f) ⇒ (b), (f) ⇒ (d), (e) ⇒ (c), and (e) ⇒ (a). Thus, we will prove the theorem according to the following steps. (I): (a) ⇒ (g), (c) ⇒ (g). (II): (b) ⇒ (g), (d) ⇒ (g). (III): (g) ⇒ (e), (g) ⇒ (f). (IV): (f) ⇔ (h).

(I): (a) ⇒ (g), (c) ⇒ (g). Suppose that (a) or (c) holds. We choose the test function $g_1(z) = z^n$. By Lemma 2, we get

$$\|g_1\|_{\mathcal{B}} \le \|g_1\|_{\mathrm{BMOA}} \lesssim \|g_1\|_\infty = 1. \tag{19}$$

So

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|\psi'(z)\right| \le \left\|D^n_{\varphi,\psi}g_1\right\|_{\mathcal{B}^\alpha} < \infty. \tag{20}$$

Taking $g_2(z) = z^{n+1}$ and using the fact that $|\varphi(z)| < 1$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}&\left(1-|z|^2\right)^\alpha \left|\psi(z)\varphi'(z)\right| \\ &\le \left\|D^n_{\varphi,\psi}g_2\right\|_{\mathcal{B}^\alpha} + \left\|D^n_{\varphi,\psi}g_1\right\|_{\mathcal{B}^\alpha} < \infty.\end{aligned} \tag{21}$$

We now consider the function

$$f_\lambda(z) = (n+1)\frac{1-|\varphi(\lambda)|^2}{1-\overline{\varphi(\lambda)}z} - \frac{\left(1-|\varphi(\lambda)|^2\right)^2}{\left(1-\overline{\varphi(\lambda)}z\right)^2}, \quad \lambda\in\mathbb{D}. \tag{22}$$

It is easy to check that $f_\lambda \in \mathcal{B}_0 \cap \mathrm{BMOA}$ and $\|f_\lambda\|_{\mathrm{BMOA}} \lesssim \|f_\lambda\|_\infty \lesssim 1$. Moreover,

$$\begin{aligned}f^{(n)}_\lambda(z) =\ &(n+1)!\left(\overline{\varphi(\lambda)}\right)^n \\ &\times\left[\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{n+1}} - \frac{\left(1-|\varphi(\lambda)|^2\right)^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{n+2}}\right].\end{aligned} \tag{23}$$

Thus, $f^{(n)}_\lambda(\varphi(\lambda)) = 0$ and

$$f^{(n+1)}_\lambda(\varphi(\lambda)) = \frac{-(n+1)!\left(\overline{\varphi(\lambda)}\right)^{n+1}}{\left(1-|\varphi(\lambda)|^2\right)^{n+1}}. \tag{24}$$

We obtain

$$\begin{aligned}\left\|D^n_{\varphi,\psi}\right\| &\gtrsim \left\|D^n_{\varphi,\psi}f_\lambda\right\|_{\mathcal{B}^\alpha} \\ &\gtrsim \left(1-|\lambda|^2\right)^\alpha\Big|\psi'(\lambda)f^{(n)}_\lambda(\varphi(\lambda)) \\ &\qquad + \psi(\lambda)\varphi'(\lambda)f^{(n+1)}_\lambda(\varphi(\lambda))\Big| \\ &\gtrsim (n+1)!\frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^{n+1}}|\varphi(\lambda)|^{n+1}\left|\psi(\lambda)\varphi'(\lambda)\right|.\end{aligned} \tag{25}$$

Thus, for any $r_0 \in (0,1)$, we have

$$\sup_{r_0<|\varphi(\lambda)|<1}\frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^{n+1}}\left|\psi(\lambda)\varphi'(\lambda)\right| < \infty. \tag{26}$$

Using (21) yields

$$\begin{aligned}\sup_{|\varphi(\lambda)|\le r_0}&\frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^{n+1}}\left|\psi(\lambda)\varphi'(\lambda)\right| \\ &\lesssim \frac{1}{\left(1-r_0^2\right)^{n+1}}\sup_{\lambda\in\mathbb{D}}\left(1-|\lambda|^2\right)^\alpha\left|\psi(\lambda)\varphi'(\lambda)\right| \\ &< \infty.\end{aligned} \tag{27}$$

Combining (26) with (27), we get

$$\sup_{\lambda\in\mathbb{D}}\frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^{n+1}}\left|\psi(\lambda)\varphi'(\lambda)\right| < \infty. \tag{28}$$

We next consider the function

$$g_\lambda(z) = (n+2)\frac{1-|\varphi(\lambda)|^2}{1-\overline{\varphi(\lambda)}z} - \frac{\left(1-|\varphi(\lambda)|^2\right)^2}{\left(1-\overline{\varphi(\lambda)}z\right)^2}, \quad \lambda\in\mathbb{D}. \tag{29}$$

Similarly, we get $g_\lambda \in \mathcal{B}_0 \cap \mathrm{BMOA}$ and

$$\|g_\lambda\|_{\mathrm{BMOA}} \lesssim \|g_\lambda\|_\infty \lesssim 1. \tag{30}$$

Moreover,

$$\begin{aligned}g^{(n)}_\lambda(z) = n!\left(\overline{\varphi(\lambda)}\right)^n&\Bigg[(n+2)\frac{1-|\varphi(\lambda)|^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{n+1}} \\ &\quad -(n+1)\frac{\left(1-|\varphi(\lambda)|^2\right)^2}{\left(1-\overline{\varphi(\lambda)}z\right)^{n+2}}\Bigg].\end{aligned} \tag{31}$$

So

$$g^{(n)}_\lambda(\varphi(\lambda)) = \frac{n!\left(\overline{\varphi(\lambda)}\right)^n}{\left(1-|\varphi(\lambda)|^2\right)^n} \tag{32}$$

and $g_\lambda^{(n+1)}(\varphi(\lambda)) = 0$. We have, as above,

$$\begin{aligned}\left\|D^n_{\varphi,\psi}\right\| &\gtrsim \left\|D^n_{\varphi,\psi} g_\lambda\right\|_{\mathscr{B}^\alpha} \\ &\gtrsim n! \frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^n} |\varphi(\lambda)|^n \left|\psi'(\lambda)\right|.\end{aligned} \tag{33}$$

Thus, for any $s_0 \in (0,1)$,

$$\sup_{s_0<|\varphi(\lambda)|<1} \frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^n} \left|\psi'(\lambda)\right| < \infty. \tag{34}$$

Applying (20), we get

$$\sup_{|\varphi(\lambda)|\le s_0} \frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^n} \left|\psi'(\lambda)\right| < \infty. \tag{35}$$

Combining (34) with (35) yields

$$\sup_{\lambda\in\mathbb{D}} \frac{\left(1-|\lambda|^2\right)^\alpha}{\left(1-|\varphi(\lambda)|^2\right)^n} \left|\psi'(\lambda)\right| < \infty. \tag{36}$$

(II): (b) $\Rightarrow$ (g) and (d) $\Rightarrow$ (g). Suppose that $D^n_{\varphi,\psi'}$: BMOA $\rightarrow \mathscr{A}^\alpha_\infty$ is bounded or $D^n_{\varphi,\psi'} : \mathscr{B}_0 \rightarrow \mathscr{A}^\alpha_\infty$ is bounded. Set

$$\lambda = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^n} \left|\psi'(z)\right|. \tag{37}$$

If $\lambda = \infty$, then for any positive integer N, we can find $b \in \mathbb{D}$ such that

$$\frac{\left(1-|b|^2\right)^\alpha}{\left(1-|\varphi(b)|^2\right)^n} \left|\psi'(b)\right| > N. \tag{38}$$

If $\varphi(b) = 0$, then choose the test function $g(z) = z^n$. It is clear that $g \in \mathscr{B}_0$. From Lemma 2, we have

$$\|g\|_{\mathscr{B}} \le \|g\|_{\mathrm{BMOA}} \lesssim \|g\|_\infty = 1. \tag{39}$$

So

$$\left\|D^n_{\varphi,\psi'}\right\| \gtrsim \left\|D^n_{\varphi,\psi'} g\right\|_{\mathscr{A}^\alpha_\infty} > \left(1-|b|^2\right)^\alpha \left|\psi'(b)\right| > N. \tag{40}$$

If $\varphi(b) \neq 0$, consider the function

$$g(z) = \frac{1}{\bar{a}^n} \frac{\left(1-|a|^2\right)^n}{(1-\bar{a}z)^n} \triangleq \sum_{j=0}^\infty c_j z^j, \tag{41}$$

where $a = \varphi(b)$. Let $F(z) = \sum_{j=n}^\infty c_j z^j$. Then, $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$ and

$$F^{(n)}(z) = \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^n. \tag{42}$$

It is easy to see that

$$\left(1-|z|^2\right)^n \left|F^{(n)}(z)\right| = \left(1-\left|\sigma_a(z)\right|^2\right)^n \le 1. \tag{43}$$

So, by Theorems 5.4 and 5.13 of [4], we have $F \in \mathscr{B}_0$ and $\|F\|_{\mathscr{B}} \lesssim 1$. By Lemma 1 of [31] and Lemma 3, we get $\|F\|_{\mathrm{BMOA}} \lesssim 1$. We have

$$\left\|D^n_{\varphi,\psi'}\right\| \gtrsim \left\|D^n_{\varphi,\psi'} F\right\|_{\mathscr{A}^\alpha_\infty} > \frac{\left(1-|b|^2\right)^\alpha}{\left(1-|\varphi(b)|^2\right)^n} \left|\psi'(b)\right| > N. \tag{44}$$

Since N is arbitrary, we get $\| D^n_{\varphi,\psi'} \| = \infty$. This contradicts the boundedness of $D^n_{\varphi,\psi'}$: BMOA $\rightarrow \mathscr{A}^\alpha_\infty$ and that of $D^n_{\varphi,\psi'} : \mathscr{B}_0 \rightarrow \mathscr{A}^\alpha_\infty$.

Now, suppose that $D^{n+1}_{\varphi,\psi\varphi'}$: BMOA $\rightarrow \mathscr{A}^\alpha_\infty$ is bounded or $D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B}_0 \rightarrow \mathscr{A}^\alpha_\infty$ is bounded. Set

$$\eta = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n+1}} \left|\psi(z)\varphi'(z)\right|. \tag{45}$$

If $\eta = \infty$, then for any positive integer M, exists $u \in \mathbb{D}$ such that

$$\frac{\left(1-|u|^2\right)^\alpha}{\left(1-|\varphi(u)|^2\right)^{n+1}} \left|\psi(u)\varphi'(u)\right| > M. \tag{46}$$

If $\varphi(u) = 0$, then set $g(z) = z^{n+1}$. The process as above gives

$$\left\|D^{n+1}_{\varphi,\psi\varphi'}\right\| \gtrsim \left\|D^{n+1}_{\varphi,\psi\varphi'} g\right\|_{\mathscr{A}^\alpha_\infty} > M. \tag{47}$$

If $\varphi(u) \neq 0$, consider the function

$$g(z) = \frac{1}{\bar{a}^{n+1}} \frac{\left(1-|a|^2\right)^{n+1}}{(1-\bar{a}z)^{n+1}} \triangleq \sum_{j=0}^\infty c_j z^j, \tag{48}$$

where $a = \varphi(u)$. Let $F(z) = \sum_{j=n+1}^\infty c_j z^j$. Then, $F(0) = F'(0) = \cdots = F^{(n)}(0) = 0$ and

$$\begin{gathered} F^{(n+1)}(z) = \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^{n+1}, \\ \left(1-|z|^2\right)^{n+1} \left|F^{(n+1)}(z)\right| = \left(1-\left|\sigma_a(z)\right|^2\right)^{n+1} \le 1. \end{gathered} \tag{49}$$

Applying Theorems 5.4 and 5.13 of [4] again yields $F \in \mathscr{B}_0$ and $\|F\|_{\mathscr{B}} \lesssim 1$. We get $\|F\|_{\mathrm{BMOA}} \lesssim 1$ and

$$\begin{aligned}\left\|D^{n+1}_{\varphi,\psi\varphi'}\right\| &\gtrsim \left\|D^{n+1}_{\varphi,\psi\varphi'} F\right\|_{\mathscr{A}^\alpha_\infty} \\ &> \frac{\left(1-|u|^2\right)^\alpha}{\left(1-|\varphi(u)|^2\right)^{n+1}} \left|\psi(u)\varphi'(u)\right| > M.\end{aligned} \tag{50}$$

Since M is arbitrary, we have $\| D^{n+1}_{\varphi,\psi\varphi'} \| = \infty$. This contradicts the boundedness of $D^{n+1}_{\varphi,\psi\varphi'}$.

(III): (g) $\Rightarrow$ (e), (g) $\Rightarrow$ (f). Note that

$$\begin{aligned}
\left\|D^n_{\varphi,\psi} f\right\|_{\mathscr{B}^\alpha} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \\
&\quad\times\left|\psi(z)\varphi'(z) f^{(n+1)}(\varphi(z)) \right. \\
&\quad\left. +\psi'(z) f^{(n)}(\varphi(z))\right| \\
&\lesssim \|f\|_{\mathscr{B}}\left[\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n+1}}\left|\psi(z)\varphi'(z)\right|\right. \\
&\quad\left.+\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n}}\left|\psi'(z)\right|\right], \\
\left\|D^{n+1}_{\varphi,\psi\varphi'} f\right\|_{\mathscr{A}^\alpha_\infty} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \\
&\quad\times\left|\psi(z)\varphi'(z) f^{(n+1)}(\varphi(z))\right| \\
&\lesssim \|f\|_{\mathscr{B}}\left[\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n+1}}\left|\psi(z)\varphi'(z)\right|\right], \\
\left\|D^n_{\varphi,\psi'} f\right\|_{\mathscr{A}^\alpha_\infty} &= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi'(z) f^{(n)}(\varphi(z))\right| \\
&\lesssim \|f\|_{\mathscr{B}}\left[\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n}}\left|\psi'(z)\right|\right].
\end{aligned} \tag{51}$$

The desired results follow.

(IV): (f) $\Leftrightarrow$ (h). Suppose that (f) is true. It follows from Proposition 5.1 of [4] that $\|z^k\|_{\mathscr{B}} \le \|z^k\|_\infty = 1$ ($k \in \mathbb{N}$). So,

$$\begin{aligned}
\sup_{k\in\mathbb{N}}\left\|D^n_{\varphi,\psi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &\le \left\|D^n_{\varphi,\psi'}\right\| < \infty, \\
\sup_{k\in\mathbb{N}}\left\|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &\le \left\|D^{n+1}_{\varphi,\psi\varphi'}\right\| < \infty.
\end{aligned} \tag{52}$$

Conversely, assume that (h) is true. It is easy to see that

$$\begin{aligned}
\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right| &\le \left\|D^n_{\varphi,\psi'}\left(z^n\right)\right\|_{\mathscr{A}^\alpha_\infty} \\
&\le \sup_{k\in\mathbb{N}}\left\|D^n_{\varphi,\psi'}\left(z^k\right)\right\|_{\mathscr{A}^\alpha_\infty} < \infty, \\
\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi(z)\varphi'(z)\right| &\le \left\|D^{n+1}_{\varphi,\psi\varphi'}\left(z^{n+1}\right)\right\|_{\mathscr{A}^\alpha_\infty} \\
&\le \sup_{k\in\mathbb{N}}\left\|D^{n+1}_{\varphi,\psi\varphi'}\left(z^k\right)\right\|_{\mathscr{A}^\alpha_\infty} < \infty.
\end{aligned} \tag{53}$$

If $\|\varphi\|_\infty < 1$, then

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^n}\left|\psi'(z)\right| \\
&\quad< \frac{1}{\left(1-\|\varphi\|_\infty\right)^n}\sup_{z\in\mathbb{D}}\left(1-|\varphi(z)|^2\right)^\alpha\left|\psi'(z)\right| < \infty, \\
&\sup_{z\in\mathbb{D}}\frac{\left(1-|z|^2\right)^\alpha\left|\psi(z)\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}} \\
&\quad< \frac{1}{\left(1-\|\varphi\|_\infty\right)^{n+1}}\sup_{z\in\mathbb{D}}\left(1-|\varphi(z)|^2\right)^\alpha\left|\psi(z)\varphi'(z)\right| < \infty.
\end{aligned} \tag{54}$$

Hence, (g) is true. From (g) $\Rightarrow$ (f), we obtain that (f) is also true.

From now on, we assume that $\|\varphi\|_\infty = 1$. For any integer $k \ge n$, let

$$\Delta^n_k = \left\{z \in \mathbb{D} : \frac{k-n}{k} \le |\varphi(z)| \le \frac{k-n+1}{k+1}\right\}. \tag{55}$$

Let m with $m \ge n$ be the smallest positive integer such that $\Delta^n_m \ne \varnothing$. Since Δ^n_k is not empty for every integer $k \ge m$ and $\mathbb{D} = \cup_{k=m}^\infty \Delta^n_k$. By Lemma 4, for $f \in \mathscr{B}$,

$$\begin{aligned}
&\left\|D^n_{\varphi,\psi'} f\right\|_{\mathscr{A}^\alpha_\infty} \\
&\quad= \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|f^{(n)}(\varphi(z))\right|\left|\psi'(z)\right| \\
&\quad= \sup_{k\ge m}\sup_{z\in\Delta^n_k}\left(1-|z|^2\right)^\alpha\left|f^{(n)}(\varphi(z))\psi'(z)\right| \\
&\quad= \sup_{k\ge m}\sup_{z\in\Delta^n_k}\left(1-|\varphi(z)|\right)^n\left|f^{(n)}(\varphi(z))\right| \\
&\qquad\times\frac{\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|\left(H^n_k(|\varphi(z)|)/(1-|\varphi(z)|)^n\right)}{H^n_k(|\varphi(z)|)} \\
&\quad\lesssim \frac{1}{c_n}\|f\|_{\mathscr{B}}\sup_{k\in\mathbb{N}}\left\|D^n_{\varphi,\psi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty}.
\end{aligned} \tag{56}$$

So, $D^n_{\varphi,\psi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ is bounded. Similar argument implies

$$\begin{aligned}
\left\|D^{n+1}_{\varphi,\psi\varphi'} f\right\|_{\mathscr{A}^\alpha_\infty} &= \sup_{k\ge m+1}\sup_{z\in\Delta^{n+1}_k}\left(1-|z|^2\right)^\alpha\left|f^{(n+1)}(\varphi(z))\right| \\
&\quad\times\left|\psi(z)\varphi'(z)\right| \\
&\lesssim \frac{1}{c_{n+1}}\|f\|_{\mathscr{B}}\sup_{k\in\mathbb{N}}\left\|D^{n+1}_{\varphi,\psi\varphi'}\left(z^k\right)\right\|_{\mathscr{A}^\alpha_\infty}.
\end{aligned} \tag{57}$$

Thus, $D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ is bounded. Theorem 5 is proved. $\square$

4. Compactness of $D^n_{\varphi,\psi}$

The following criterion for the compactness is a useful tool and it follows from standard arguments, for example, Proposition 3.11 of [32] or Lemma 2.10 of [33].

Lemma 6. *Let $\alpha > 0$, $n \in \mathbb{N}^+$, and $X = \mathscr{B}_0, \mathscr{B}$, or BMOA. Suppose that ψ and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then, $D^n_{\varphi,\psi} : X \to \mathscr{B}^\alpha$ is compact if and only if for any sequence $\{f_m\}$ in X with $\sup_m \|f_m\|_X < \infty$, which converges to zero locally uniformly on $\mathbb{D}$; we have $\lim_{m\to\infty} \| D^n_{\varphi,\psi} f_m\|_{\mathscr{B}^\alpha} = 0$.*

We now give the compactness of $D^n_{\varphi,\psi}$ from BMOA and the Bloch space to Bloch-type spaces.

Theorem 7. *Let $\alpha > 0$, $\psi \in H(\mathbb{D})$, $n \in \mathbb{N}^+$, and φ a holomorphic self-map of $\mathbb{D}$. Then, the following statements are equivalent:*

(a) $D^n_{\varphi,\psi} : BMOA \to \mathscr{B}^\alpha$ *is compact.*

(b) $D^n_{\varphi,\psi'} : BMOA \to \mathscr{A}^\alpha_\infty$ *is compact and* $D^{n+1}_{\varphi,\psi\varphi'} : BMOA \to \mathscr{A}^\alpha_\infty$ *is compact.*

(c) $D^n_{\varphi,\psi} : \mathscr{B}_0 \to \mathscr{B}^\alpha$ *is compact.*

(d) $D^n_{\varphi,\psi'} : \mathscr{B}_0 \to \mathscr{A}^\alpha_\infty$ *is compact and* $D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B}_0 \to \mathscr{A}^\alpha_\infty$ *is compact.*

(e) $D^n_{\varphi,\psi} : \mathscr{B} \to \mathscr{B}^\alpha$ *is compact.*

(f) $D^n_{\varphi,\psi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ *is compact and* $D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ *is compact.*

(g) $\psi \in \mathscr{B}^\alpha$, $\psi\varphi' \in \mathscr{A}^\alpha_\infty$,

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^n} \left|\psi'(z)\right| = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{n+1}} \left|\psi(z)\varphi'(z)\right| = 0. \tag{58}$$

(h) $\limsup_{k\to\infty} \|D^n_{\varphi,\psi'}(z^k)\|_{\mathscr{A}^\alpha_\infty} = 0$ *and* $\limsup_{k\to\infty} \|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\|_{\mathscr{A}^\alpha_\infty} = 0$.

Proof. The proof is a modification of that of Theorem 5; so we give a sketch of the proof. We will prove the theorem according to the following steps. (I): (a) $\Rightarrow$ (g), (c) $\Rightarrow$ (g). (II): (b) $\Rightarrow$ (g), (d) $\Rightarrow$ (g). (III): (g) $\Rightarrow$ (e), (g) $\Rightarrow$ (f). (IV): (f) $\Leftrightarrow$ (h).

(I): (a) $\Rightarrow$ (g), (c) $\Rightarrow$ (g). Suppose that (a) or (c) holds. Then by Theorem 5, we have

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\psi'(z)\right| \le \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha \left|\psi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^n} < \infty,$$
$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|\psi(z)\varphi'(z)\right| \le \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha \left|\psi(z)\varphi'(z)\right|}{\left(1-|\varphi(z)|^2\right)^{n+1}} < \infty. \tag{59}$$

That is, $\psi \in \mathscr{B}^\alpha$, $\psi\varphi' \in \mathscr{A}^\alpha_\infty$.

Let $\{z_j\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Now, we consider the function

$$f_j(z) = (n+1)\frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z} - \frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}z\right)^2}. \tag{60}$$

Simple computation shows that $f_j \in \mathscr{B}_0 \cap \text{BMOA}$ and

$$\|f_j\|_{\text{BMOA}} \lesssim \|f_j\|_\infty \lesssim 1. \tag{61}$$

It is also easy to check that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Moreover,

$$f_j^{(n)}(z) = (n+1)!\left(\overline{\varphi(z_j)}\right)^n \times \left[\frac{1-|\varphi(z_j)|^2}{\left(1-\overline{\varphi(z_j)}z\right)^{n+1}} - \frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}z\right)^{n+2}}\right]. \tag{62}$$

We have

$$\|D^n_{\varphi,\psi} f_j\|_{\mathscr{B}^\alpha} \gtrsim (n+1)!\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^{n+1}}\left|\varphi(z_j)\right|^{n+1}\left|\psi(z_j)\varphi'(z_j)\right|. \tag{63}$$

By Lemma 6, we get

$$\lim_{|\varphi(z_j)|\to 1}\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^{n+1}}\left|\psi(z_j)\varphi'(z_j)\right| = 0. \tag{64}$$

We next consider the function

$$g_j(z) = (n+2)\frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z} - \frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}z\right)^2}. \tag{65}$$

Similarly, we get $g_j \in \mathscr{B}_0 \cap \text{BMOA}$ and

$$\|g_j\|_{\text{BMOA}} \lesssim \|g_j\|_\infty \lesssim 1. \tag{66}$$

It is easy to see that g_j converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$ and

$$g_j^{(n)}(z) = n!\left(\overline{\varphi(z_j)}\right)^n \times \left[(n+2)\frac{1-|\varphi(z_j)|^2}{\left(1-\overline{\varphi(z_j)}z\right)^{n+1}} - (n+1)\frac{\left(1-|\varphi(z_j)|^2\right)^2}{\left(1-\overline{\varphi(z_j)}z\right)^{n+2}}\right]. \quad (67)$$

Thus,

$$\left\|D^n_{\varphi,\psi}g_j\right\|_{\mathscr{B}^\alpha} \gtrsim n!\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^n}|\varphi(z_j)|^n|\psi'(z_j)|. \quad (68)$$

Applying Lemma 6 again, we have

$$\lim_{|\varphi(z_j)|\to 1}\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^n}|\psi'(z_j)| = 0. \quad (69)$$

Since $z_j \in \mathbb{D}$ is arbitrary, we proved that (g) is true.

(II) (b) $\Rightarrow$ (g), (d) $\Rightarrow$ (g). Suppose that (b) or (d) holds. A similar argument to (I) shows that $\psi \in \mathscr{B}^\alpha$, $\psi\varphi' \in \mathscr{A}^\alpha_\infty$. Now, suppose that the equations in (g) are not true. Then, there exists a sequence $\{z_j\}$ in $\mathbb{D}$ and $\delta > 0$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$ and

$$\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^n}|\psi'(z_j)| > \delta,$$
$$\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^{n+1}}|\psi(z_j)\varphi'(z_j)| > \delta. \quad (70)$$

Choose a subsequence of $\{z_j\}$ if necessary and suppose that $\inf_j|\varphi(z_j)| > 1/2$. Let

$$f_j(z) = \frac{1-|\varphi(z_j)|^2}{1-\overline{\varphi(z_j)}z}, \quad z \in \mathbb{D}. \quad (71)$$

Then, it is easy to check that $f_j \in \mathscr{B}_0 \cap \mathrm{BMOA}$, $f_j \to 0$, uniformly on compact subsets of $\mathbb{D}$ and

$$f_j^{(n)}(z) = n!\frac{1-|\varphi(z_j)|^2}{\left(1-\overline{\varphi(z_j)}z\right)^{n+1}}\left(\overline{\varphi(z_j)}\right)^n. \quad (72)$$

Thus,

$$\left\|D^n_{\varphi,\psi'}f_j\right\|_{\mathscr{A}^\alpha_\infty} \geq n!\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^n}|\varphi(z_j)|^n|\psi'(z_j)| > \frac{n!\delta}{2^n},$$
$$\left\|D^{n+1}_{\varphi,\psi\varphi'}f_j\right\|_{\mathscr{A}^\alpha_\infty} \geq (n+1)!\frac{\left(1-|z_j|^2\right)^\alpha}{\left(1-|\varphi(z_j)|^2\right)^{n+1}} \times |\varphi(z_j)|^{n+1}|\psi(z_j)\varphi'(z_j)| > \frac{(n+1)!\delta}{2^{n+1}}. \quad (73)$$

Those contradict the compactness of $D^n_{\varphi,\psi'}$ and $D^{n+1}_{\varphi,\psi\varphi'}$.

(III) (g) $\Rightarrow$ (e), (g) $\Rightarrow$ (f). Let $\{f_m\}$ be a norm bounded sequence in $\mathscr{B}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_m\|f_m\|_{\mathscr{B}} < \infty$. For $\varepsilon > 0$, then there exists $r_0 \in (0,1)$ such that for $|\varphi(z)| > r_0$, we have

$$\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^n}|\psi'(z)| < \varepsilon,$$
$$\frac{\left(1-|z|^2\right)^\alpha}{\left(1-|\varphi(z)|^2\right)^{n+1}}|\psi(z)\varphi'(z)| < \varepsilon. \quad (74)$$

Thus, for $z \in \mathbb{D}$, we have

$$\begin{aligned}\left\|D^n_{\varphi,\psi}f_m\right\|_{\mathscr{B}^\alpha} \lesssim{}& \left|\psi(0)f_m^{(n)}(\varphi(0))\right| + \sup_{|\varphi(z)|\le r_0}\left(1-|z|^2\right)^\alpha\left|f_m^{(n)}(\varphi(z))\right||\psi'(z)| \\ &+ \sup_{|\varphi(z)|> r_0}\frac{\left(1-|z|^2\right)^\alpha|\psi'(z)|\,\|f_m\|_{\mathscr{B}}}{\left(1-|\varphi(z)|^2\right)^n} \\ &+ \sup_{|\varphi(z)|\le r_0}\left(1-|z|^2\right)^\alpha\left|f_m^{(n+1)}(\varphi(z))\right| \times |\psi(z)\varphi'(z)| \\ &+ \sup_{|\varphi(z)|> r_0}\frac{\left(1-|z|^2\right)^\alpha|\psi(z)\varphi'(z)|\,\|f_m\|_{\mathscr{B}}}{\left(1-|\varphi(z)|^2\right)^{n+1}} \\ \lesssim{}& \left|\psi(0)f_m^{(n)}(\varphi(0))\right| + K_1\sup_{|z|\le r_0}\left|f_m^{(n)}(z)\right| + K_2\sup_{|z|\le r_0}\left|f_m^{(n+1)}(z)\right| + 2\varepsilon M,\end{aligned} \quad (75)$$

where $K_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi'(z)|$ and $K_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|$. Since $f_m^{(n)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$, we have $\|D^n_{\varphi,\psi}f_m\|_{\mathscr{B}^\alpha} \to 0$ as $m \to \infty$. It follows from Lemma 6 that $D^n_{\varphi,\psi} : \mathscr{B} \to \mathscr{B}^\alpha$ is compact.

Similar as above, we know

$$\begin{aligned}
\left\|D^n_{\varphi,\psi'} f_m\right\|_{\mathscr{A}^\alpha_\infty} &\leq \sup_{|\varphi(z)|\leq r_0}\left(1-|z|^2\right)^\alpha\left|f^{(n)}_m(\varphi(z))\right|\left|\psi'(z)\right| \\
&\quad + \sup_{|\varphi(z)|>r_0}\frac{\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|\|f_m\|_{\mathscr{B}}}{\left(1-|\varphi(z)|^2\right)^n} \\
&\leq K_1\sup_{|z|\leq r_0}\left|f^{(n)}_m(z)\right|+\varepsilon M, \\
\left\|D^{n+1}_{\varphi,\psi\varphi'} f_m\right\|_{\mathscr{A}^\alpha_\infty} &\leq \sup_{|\varphi(z)|\leq r_0}\left(1-|z|^2\right)^\alpha\left|f^{(n+1)}_m(\varphi(z))\right| \\
&\quad \times\left|\psi(z)\varphi'(z)\right| \\
&\quad + \sup_{|\varphi(z)|>r_0}\frac{\left(1-|z|^2\right)^\alpha\left|\psi(z)\varphi'(z)\right|\|f_m\|_{\mathscr{B}}}{\left(1-|\varphi(z)|^2\right)^{n+1}} \\
&\leq K_2\sup_{|z|\leq r_0}\left|f^{(n+1)}_m(z)\right|+\varepsilon M.
\end{aligned} \tag{76}$$

From $f^{(n)}_m \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have $\|D^n_{\varphi,\psi'} f_m\|_{\mathscr{A}^\alpha_\infty} \to 0$ and $\|D^{n+1}_{\varphi,\psi\varphi'} f_m\|_{\mathscr{A}^\alpha_\infty} \to 0$ as $m \to \infty$. So, $D^n_{\varphi,\psi'}, D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ are compact.

(IV): (f) $\Leftrightarrow$ (h). Suppose that (f) is true. Note that $\|z^k\|_{\mathscr{B}} \leq \|z^k\|_\infty = 1$ and $z^k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$; by Lemma 6, we have

$$\begin{aligned}
\lim_{k\to\infty}\left\|D^n_{\varphi,\psi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &= 0, \\
\lim_{k\to\infty}\left\|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &= 0.
\end{aligned} \tag{77}$$

Conversely, assume that (h) is true. It is easy to see that

$$\begin{aligned}
\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right| &\leq \left\|D^n_{\varphi,\psi'}(z^n)\right\|_{\mathscr{A}^\alpha_\infty} \\
&\leq \sup_{k\in\mathbb{N}}\left\|D^n_{\varphi,\psi'}\left(z^k\right)\right\|_{\mathscr{A}^\alpha_\infty} < \infty, \\
\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi(z)\varphi'(z)\right| &\leq \left\|D^{n+1}_{\varphi,\psi\varphi'}\left(z^{n+1}\right)\right\|_{\mathscr{A}^\alpha_\infty} \\
&\leq \sup_{k\in\mathbb{N}}\left\|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} < \infty.
\end{aligned} \tag{78}$$

If $\|\varphi\|_\infty < 1$, from (g) $\Rightarrow$ (f), we get that (f) is true. If $\|\varphi\|_\infty = 1$, as in the proof of Theorem 5, let

$$\Delta^n_k = \left\{z\in\mathbb{D} : \frac{k-n}{k} \leq |\varphi(z)| \leq \frac{k-n+1}{k+1}\right\}. \tag{79}$$

And let m with $m \geq n$ be the smallest positive integer such that $\Delta^n_m \neq \varnothing$. For given $\varepsilon > 0$, there exists a large enough integer M_1 with $M_1 > m$ such that

$$\begin{aligned}
\left\|D^n_{\varphi,\psi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &< \varepsilon, \\
\left\|D^{n+1}_{\varphi,\psi\varphi'}(z^k)\right\|_{\mathscr{A}^\alpha_\infty} &< \varepsilon,
\end{aligned} \tag{80}$$

whenever $k > M_1$. Let $\{f_j\}$ be a norm bounded sequence in $\mathscr{B}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Denote $M = \sup_m\|f_m\|_{\mathscr{B}} < \infty$. We get

$$\begin{aligned}
&\left\|D^n_{\varphi,\psi'} f_j\right\|_{\mathscr{A}^\alpha_\infty} \\
&\quad = \sup_{k\geq m}\sup_{z\in\Delta^n_k}\left(1-|z|^2\right)^\alpha\left|f^{(n)}_j(\varphi(z))\right|\left|\psi'(z)\right| \\
&\quad = \left(\sup_{m\leq k\leq M_1} + \sup_{k>M_1}\right)\sup_{z\in\Delta^n_k}\left(1-|z|^2\right)^\alpha \\
&\qquad \times\left|f^{(n)}_j(\varphi(z))\right|\left|\psi'(z)\right| \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{81}$$

Then,

$$\begin{aligned}
I_1 &= \sup_{m\leq k\leq M_1}\sup_{z\in\Delta^n_k}\left(1-|z|^2\right)^\alpha\left|f^{(n)}_j(\varphi(z))\right|\left|\psi'(z)\right| \\
&\leq \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|\sup_{|\varphi(z)|\leq r}\left|f^{(n)}_j(\varphi(z))\right|,
\end{aligned} \tag{82}$$

where

$$r = \frac{M_1-n+1}{M_1+1}, \tag{83}$$

$$\begin{aligned}
I_2 &= \sup_{k\geq M_1}\sup_{z\in\Delta^n_k}\left(1-|z|^2\right)^\alpha\left|f^{(n)}_j(\varphi(z))\right|\left|\psi'(z)\right| \\
&= \sup_{k\geq M_1}\sup_{z\in\Delta^n_k}\left(1-|\varphi(z)|\right)^n\left|f^{(n)}_j(\varphi(z))\right| \\
&\quad \times\frac{\left(1-|z|^2\right)^\alpha\left|\psi'(z)\right|\left(H^n_k(|\varphi(z)|)/(1-|\varphi(z)|)^n\right)}{H^n_k(|\varphi(z)|)} \\
&\lesssim \frac{1}{c_n}\left\|f_j\right\|_{\mathscr{B}}\sup_{k>M_1}\left\|D^n_{\varphi,\psi'}\left(z^k\right)\right\|_{\mathscr{A}^\alpha_\infty} \\
&\lesssim \frac{1}{c_n}\varepsilon.
\end{aligned} \tag{84}$$

Since $f^{(n)}_j \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|D^n_{\varphi,\psi'} f_j\|_{\mathscr{A}^\alpha_\infty} \to 0$ as $j \to \infty$. Thus, by Lemma 6, $D^n_{\varphi,\psi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ is compact. Similar as above, we can prove that $D^{n+1}_{\varphi,\psi\varphi'} : \mathscr{B} \to \mathscr{A}^\alpha_\infty$ is compact. The proof is complete. □

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