Research Article

The Jacobian Consistency of a One-Parametric Class of Smoothing Functions for SOCCP

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Second-order cone (SOC) complementarity functions and their smoothing functions have been much studied in the solution of second-order cone complementarity problems (SOCCP). In this paper, we study the directional derivative and B-subdifferential of the one-parametric class of SOC complementarity functions, propose its smoothing function, and derive the computable formula for the Jacobian of the smoothing function. Based on these results, we prove the Jacobian consistency of the one-parametric class of smoothing functions, which will play an important role for achieving the rapid convergence of smoothing methods. Moreover, we estimate the distance between the subgradient of the one-parametric class of the SOC complementarity functions and the gradient of its smoothing function, which will help to adjust a parameter appropriately in smoothing methods.

1. Introduction

The second-order cone complementarity problem (SOCCP) [1] is to find $(x, y, \zeta) \in R^n \times R^n \times R^l$ such that

$$x \in K, \quad y \in K, \quad x^T y = 0, \quad F(x, y, \zeta) = 0, \qquad (1)$$

where $F : R^n \times R^n \times R^l \to R^n \times R^l$ is a continuously differentiable mapping, $K \subset R^n$ is the Cartesian product of second-order cones (SOC), that is, $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $n = n_1 + n_2 + \cdots + n_m$ and the n_i-dimensional SOC defined by

$$K^{n_i} = \left\{ x^i = \left(x_1^i; x_2^i \right) \in R \times R^{n_i-1} : x_1^i - \left\| x_2^i \right\| \geq 0 \right\}. \qquad (2)$$

Without loss of generality, we may assume that $m = 1$ and $K = K^n$ in the following analysis, since our analysis can be easily extended to the general case.

The SOCCP contains a wide class of problems, such as nonlinear complementarity problems [2], second-order cone programming [1, 3, 4], and has a variety of engineering and management applications, such as filter design, antenna array weight design, truss design, and grasping force optimization in robotics [5, 6].

Recently, great attention has been paid to smoothing methods, partially due to their superior theoretical and numerical performances [7–10]. Smoothing methods usually reformulate the SOCCP as a system of equations by using smoothing functions of SOC complementarity functions [10, 11]. The smoothing parameter involved in smoothing functions may be treated as a variable [9] or a parameter with an appropriate parameter control [7]. In the latter case, the Jacobian consistency plays a key role for achieving a rapid convergence of Newton the methods or the Newton-like methods. Hayashi et al. [7] propose a combined smoothing and regularized method for monotone SOCCP and show its global and quadratic convergence based on the Jacobian consistency of the smoothing natural residual function. Ogasawara and Narushima [12] show the Jacobian consistency of a smoothed Fischer-Burmeister (FB) function. Chen et al. [13] present a smoothing function of a generalized FB function in the context of nonlinear complementarity programming and study some of its favorable properties, including the Jacobian consistency property. Based on the results, they [13] propose a smoothing algorithm for the mixed complementarity problem, which is shown to possess

global convergence and local superlinear (or quadratic) convergence.

In this paper, we are concerned with the one-parametric class of SOC complementarity functions $\phi_\tau : R^n \to R^n$ defined by [14]

$$\phi_\tau(x, y) := x + y - \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)}, \tag{3}$$

where $\tau \in (0, 4)$ is an arbitrary but fixed parameter. When $\tau = 2$, ϕ_τ reduces to the vector-valued Fischer-Burmeister function given by

$$\phi_{FB}(x, y) := x + y - \sqrt{x^2 + y^2}, \tag{4}$$

and as $\tau \to 0$, it becomes a multiple of the following vector-valued residual function:

$$\phi_{NR}(x, y) := x - \Pi_{K^n}(x - y), \tag{5}$$

where $\Pi_{K^n}(\cdot)$ denotes the metric projection on the second-order cone K^n. Thus, the one-parametric class of vector-valued functions (3) cover two popular second-order cone complementarity functions.

In this paper, we aim to show the Jacobian consistency of smoothing functions of the one-parametric class of SOC complementarity functions, which will play an important role for achieving the rapid convergence of smoothing methods. Moreover, we estimate the distance between the subgradient of the one-parametric class of the SOC complementarity functions and the gradient of the smoothing functions, which will help to adjust a parameter appropriately in smoothing methods.

The organization of this paper is as follows. In Section 2, we review some preliminaries including the Euclidean Jordan algebra associated with SOC and subdifferentials. In Section 3, we derive the computable formula for the Jacobian of the one-parametric class of smoothing functions in the SOCCP. In Section 4, we prove the Jacobian consistency of the one-parametric class of smoothing functions and estimate the distance between the gradient of the smoothing functions and the subgradient of the one-parametric class of the SOC complementarity functions. In Section 5, we study the directional derivative and B-subdifferential of the one-parametric class of SOC complementarity functions and then present an alternative way to prove the Jacobian consistency of the one-parametric class of smoothing functions. Finally, we close this paper with some conclusions in Section 6.

In what follows, we denote the nonnegative orthant of R by R_+. We use the symbol $\| \cdot \|$ to denote the Euclidean norm defined by $\|x\| := \sqrt{x^T x}$ for a vector x or the corresponding induced matrix norm. For simplicity, we often use $x = (x_1; x_2)$ for the column vector $x = (x_1, x_2^T)^T$. For the SOC K^n, int K^n and bd K^n mean the topological interior and the boundary of K^n, respectively. For a given set $S \subset R^{m \times n}$, conv S denotes the convex hull of S in $R^{m \times n}$, and dist(X, S) denotes inf$\{\|X - Y\| : Y \in S\}$ for a matrix $X \in R^{m \times n}$.

2. Preliminaries

In this section, we recall some concepts and results, which include the Euclidean Jordan algebra [3, 15] associated with the SOC K^n and subdifferentials [16].

First, we recall the Euclidean Jordan algebra associated with the SOC and some useful definitions. The Euclidean Jordan algebra for the SOC K^n is the algebra defined by

$$x \circ y = (x^T y; x_1 y_2 + y_1 x_2), \quad \forall x, y \in R^n, \tag{6}$$

with $e = (1, 0, \ldots, 0) \in R^n$ being its unit element. Given an element $x = (x_1; x_2) \in R \times R^{n-1}$, we define

$$L(x) = \begin{pmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{pmatrix}, \tag{7}$$

where I represents the $(n-1) \times (n-1)$ identity matrix. It is easy to verify that $x \circ y = L(x)y$ for any $y \in R^n$. Moreover, $L(x)$ is symmetric positive definite (and hence invertible) if and only if $x \in \text{int } K^n$.

Now, we give the spectral factorization of vectors in R^n associated with the SOC K^n. Let $x = (x_1; x_2) \in R \times R^{n-1}$. Then x can be decomposed as

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{8}$$

where λ_1, λ_2, and $u^{(1)}$, $u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|,$$

$$u^{(i)} = \begin{cases} \dfrac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0, \\ \dfrac{1}{2}\left(1, (-1)^i \omega\right) & \text{otherwise,} \end{cases} \tag{9}$$

for $i = 1, 2$, with any $\omega \in R^{n-1}$ such that $\|\omega\| = 1$. By the spectral factorization, a scalar function can be extended to a function for the SOC. For any $x \in R^n$, we define

$$x^2 = \lambda_1^2 u^{(1)} + \lambda_2^2 u^{(2)}. \tag{10}$$

Since both eigenvalues of any $x \in K^n$ are nonnegative, we define

$$\sqrt{x} = \sqrt{\lambda_1} u^{(1)} + \sqrt{\lambda_2} u^{(2)}. \tag{11}$$

For any $x = (x_1; x_2) \in R \times R^{n-1}$, we define [12] $x' = (x_1; -x_2)$. Obviously, $x'' = x$, $(x + y)' = x' + y'$, and $(cx)' = cx'$ for any $c \in R$. Moreover, $x \circ x' = x_1^2 - \|x_2\|^2 = 0$ if $x \in \text{bd} K^n$.

Let $G : R^m \to R^n$ be a locally Lipschitzian function. Then, G is differentiable almost everywhere by Rademacher's theorem [17]. The Bouligand- (B-) subdifferential and the Clarke subdifferential of G at z are defined by

$$\partial G_B(z) := \left\{ \lim_{\hat{z} \to z} \nabla G(\hat{z}) : \hat{z} \in D_G \right\},$$

$$\partial G(z) = \text{conv} \, \partial G_B(z), \tag{12}$$

respectively, where D_G denotes the set of points at which G is differentiable. It is obvious that $\partial G(z) = \{\nabla G(z)\}$ if G is continuously differentiable at z.

By using the concepts of subdifferentials, we give the definition of the Jacobian consistency, which was first introduced by Chen et al. [16], which is a concept relating the generalized Jacobian of a nonsmooth function with the Jacobian of a smoothing function [7].

Definition 1 (see [16]). Let $G : R^m \rightarrow R^n$ be a locally Lipschitzian function. Let $G_\varepsilon : R^m \rightarrow R^n$ be a continuously differentiable function for any $\varepsilon > 0$ such that $\lim_{\varepsilon \downarrow 0} G_\varepsilon(z) = G(z)$ for any $z \in R^m$. We say that G_ε satisfies the Jacobian consistency property if for any $z \in R^m$, $\lim_{\varepsilon \downarrow 0} \text{dist}(\nabla G_\varepsilon(z), \partial G(z)) = 0$.

It should be noted that the "inf" appearing in the definition of $\text{dist}(\nabla G_\varepsilon(z), \partial G(z))$ can be replaced by "min," since the set $\partial G(z)$ is compact at all $z \in R^m$ [17].

3. Smoothing Function

In this section, we propose a smoothing function of the one-parametric class of SOC complementarity functions and derive the computable formula for its Jacobian.

Since the one-parametric class of SOC complementarity functions ϕ_τ defined by (3) is nonsmooth, we consider the function $\phi_{\tau,\varepsilon}$ defined by

$$\phi_{\tau,\varepsilon}(x, y) := x + y - \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y) + 2\varepsilon^2 e}, \tag{13}$$

where the smoothing parameter $\varepsilon \in R$.

Definition 2 (see [7]). For a nondifferentiable function $h : R^m \rightarrow R^n$, one considers a function $h_\varepsilon : R^m \rightarrow R^n$ with a parameter $\varepsilon > 0$ that has the following properties:

(i) h_ε is differentiable for any $\varepsilon > 0$;

(ii) $\lim_{\varepsilon \downarrow 0} h_\varepsilon(x) = h(x)$ for any $x \in R^m$.

Such a function h_ε is called a smoothing function of h.

In the following, we will show that the function $\phi_{\tau,\varepsilon}$ given by (13) is a smoothing function of ϕ_τ. Thus, we can solve a family of smoothing subproblems $\phi_{\tau,\varepsilon}(x, y) = 0$ for $\varepsilon > 0$ and obtain a solution of $\phi_\tau(x, y) = 0$ by letting $\varepsilon \downarrow 0$.

For convenience, we give some notations. For any $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$, and any $\varepsilon \in R$, we define the mapping $z^\varepsilon : R^{2n} \rightarrow R \times R^{n-1}$ by

$$z^\varepsilon = (z_1^\varepsilon; z_2^\varepsilon)$$
$$= z^\varepsilon(x, y) := x^2 + y^2 + (\tau - 2)(x \circ y) + 2\varepsilon^2 e, \tag{14}$$

and drop the subscript for simplicity for $\varepsilon = 0$, and thus,

$$z = (z_1; z_2)$$
$$= z(x, y) := x^2 + y^2 + (\tau - 2)(x \circ y). \tag{15}$$

By direct calculations, we obtain

$$z_1^\varepsilon = \|x\|^2 + \|y\|^2$$
$$+ (\tau - 2) x^T y + 2\varepsilon^2 = 2\varepsilon^2 + z_1,$$
$$z_2^\varepsilon = 2x_1 x_2 + 2y_1 y_2$$
$$+ (\tau - 2)(x_1 y_2 + y_1 x_2) = z_2, \tag{16}$$

and therefore $z^\varepsilon = (z_1^\varepsilon; z_2)$. Then, the spectral factorization of z^ε is

$$z^\varepsilon = \lambda_1(z^\varepsilon) u_1(z) + \lambda_2(z^\varepsilon) u_2(z), \tag{17}$$

where $\lambda_1(z^\varepsilon)$, $\lambda_2(z^\varepsilon)$, and $u_1(z)$, $u_2(z)$ are the spectral values and the associated spectral vectors of z^ε given by

$$\lambda_i(z^\varepsilon)$$
$$= 2\varepsilon^2 + \|x\|^2$$
$$+ \|y\|^2 + (\tau - 2) x^T y \tag{18}$$
$$+ (-1)^i \| 2x_1 x_2 + 2y_1 y_2$$
$$+ (\tau - 2)(x_1 y_2 + y_1 x_2) \|,$$

$$u_i(z) = \frac{1}{2} \left(1; (-1)^i \bar{z}_2 \right), \tag{19}$$

for $i = 1, 2$. Here,

$$\bar{z}_2 := \frac{z_2}{\|z_2\|} \tag{20}$$
$$= \frac{2x_1 x_2 + 2y_1 y_2 + (\tau - 2)(x_1 y_2 + y_1 x_2)}{\| 2x_1 x_2 + 2y_1 y_2 + (\tau - 2)(x_1 y_2 + y_1 x_2) \|},$$

if $z_2 \neq 0$, and otherwise, $\bar{z}_2$ is any vector in R^{n-1} such that $\|\bar{z}_2\| = 1$.

For any $(x, y) \in R^n \times R^n$, it is not difficult to verify that

$$z^\varepsilon = x^2 + y^2 + (\tau - 2)(x \circ y) + 2\varepsilon^2 e$$
$$= \left(x + \frac{\tau - 2}{2} y \right)^2 + \frac{\tau(4 - \tau)}{4} y^2 + 2\varepsilon^2 e \tag{21}$$
$$= \left(y + \frac{\tau - 2}{2} x \right)^2 + \frac{\tau(4 - \tau)}{4} x^2 + 2\varepsilon^2 e \in \text{int } K^n,$$

for any $\varepsilon \neq 0$, and

$$z = x^2 + y^2 + (\tau - 2)(x \circ y)$$
$$= \left(x + \frac{\tau - 2}{2} y \right)^2 + \frac{\tau(4 - \tau)}{4} y^2 \tag{22}$$
$$= \left(y + \frac{\tau - 2}{2} x \right)^2 + \frac{\tau(4 - \tau)}{4} x^2 \in K^n.$$

Therefore, for any $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$, and any $\varepsilon \in R$, we can also define

$$
\begin{aligned}
w^\varepsilon &= (w_1^\varepsilon; w_2^\varepsilon) \\
&= w^\varepsilon(x, y) \\
&:= \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y) + 2\varepsilon^2 e},
\end{aligned}
\tag{23}
$$

and for $\varepsilon = 0$,

$$
\begin{aligned}
w &= (w_1; w_2) \\
&= w(x, y) \\
&:= \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)}.
\end{aligned}
\tag{24}
$$

The spectral factorization of w^ε and w is given by, respectively,

$$
\begin{aligned}
w^\varepsilon &= \sqrt{\lambda_1(z^\varepsilon)} u_1(z) + \sqrt{\lambda_2(z^\varepsilon)} u_2(z), \\
w &= \sqrt{\lambda_1(z)} u_1(z) + \sqrt{\lambda_2(z)} u_2(z).
\end{aligned}
\tag{25}
$$

By (22), we can partition R^{2n} as $R^{2n} = S_1 \cup S_2 \cup \{(0,0)\}$, where

$$
\begin{aligned}
S_1 &:= \left\{ (x, y) \in R^{2n} : z \in \text{int } K^n \right\} \\
&= \left\{ (x, y) \in R^{2n} : \lambda_2(z) \geq \lambda_1(z) > 0 \right\}, \\
S_2 &:= \left\{ (x, y) \in R^{2n} : z \in \text{bd } K^n \setminus \{0\} \right\} \\
&= \left\{ (x, y) \in R^{2n} : 2z_1 = \lambda_2(z) > \lambda_1(z) = 0 \right\}.
\end{aligned}
\tag{26}
$$

Theorem 3. *For any $(x, y) \in R^n \times R^n$ and $\varepsilon \neq 0$, let ϕ_τ and $\phi_{\tau,\varepsilon}$ be, respectively, defined by (3) and (13). Then, the following results hold.*

(i) *The function $\phi_{\tau,\varepsilon}$ is continuously differentiable everywhere, and its Jacobian is given by*

$$
\begin{aligned}
&\nabla \phi_{\tau,\varepsilon}(x, y) \\
&= \begin{pmatrix} I - L\left(x + \dfrac{\tau - 2}{2} y \right) L^{-1}(w^\varepsilon) \\ I - L\left(y + \dfrac{\tau - 2}{2} x \right) L^{-1}(w^\varepsilon) \end{pmatrix},
\end{aligned}
\tag{27}
$$

where $L^{-1}(w^\varepsilon) = (1/\sqrt{z_1^\varepsilon}) I$ if $z_2 = 0$, and otherwise,

$$
\begin{aligned}
L^{-1}(w^\varepsilon) &= L_1(z^\varepsilon) + L_2(z^\varepsilon) \\
&= \begin{pmatrix} b_\varepsilon & c_\varepsilon \bar{z}_2^T \\ c_\varepsilon \bar{z}_2 & a_\varepsilon I + (b_\varepsilon - a_\varepsilon) \bar{z}_2 \bar{z}_2^T \end{pmatrix},
\end{aligned}
\tag{28}
$$

with

$$
L_1(z^\varepsilon) = \frac{1}{2\sqrt{\lambda_1(z^\varepsilon)}} \begin{pmatrix} 1 & -\bar{z}_2^T \\ -\bar{z}_2 & \bar{z}_2 \bar{z}_2^T \end{pmatrix},
$$

$$
\begin{aligned}
L_2(z^\varepsilon) &= \frac{1}{2\sqrt{\lambda_2(z^\varepsilon)}} \begin{pmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & \bar{z}_2 \bar{z}_2^T \end{pmatrix} \\
&\quad + a_\varepsilon \begin{pmatrix} 0 & 0^T \\ 0 & I - \bar{z}_2 \bar{z}_2^T \end{pmatrix},
\end{aligned}
\tag{29}
$$

$$
a_\varepsilon = \frac{2}{\sqrt{\lambda_1(z^\varepsilon)} + \sqrt{\lambda_2(z^\varepsilon)}},
$$

$$
b_\varepsilon = \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_1(z^\varepsilon)}} + \frac{1}{\sqrt{\lambda_2(z^\varepsilon)}} \right),
\tag{30}
$$

$$
c_\varepsilon = \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_2(z^\varepsilon)}} - \frac{1}{\sqrt{\lambda_1(z^\varepsilon)}} \right).
$$

(ii) *For any $(x, y) \in R^n \times R^n$, $\lim_{\varepsilon \downarrow 0} \phi_{\tau,\varepsilon}(x, y) = \phi_\tau(x, y)$. Thus, $\phi_{\tau,\varepsilon}$ is a smoothing function of ϕ_τ.*

Proof. (i) For any $(x, y) \in R^n \times R^n$ and any $\varepsilon \neq 0$, it follows from Corollary 5.4 in [1], the chain rule for differentiation, and (21) that formula in (27) holds. Formula (28) is due to Proposition 5.2 and its proof in [1].

(ii) Fix any $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$. For any $\varepsilon > 0$, it follows from the spectral factorization of z and z^ε that

$$
\begin{aligned}
\phi_\tau(x, y) &= x + y \\
&\quad - \left(\sqrt{\lambda_1(z)} u_1(z) + \sqrt{\lambda_2(z)} u_2(z) \right), \\
\phi_{\tau,\varepsilon}(x, y) &= x + y \\
&\quad - \left(\sqrt{\lambda_1(z^\varepsilon)} u_1(z) + \sqrt{\lambda_2(z^\varepsilon)} u_2(z) \right),
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\lambda_i(z) &= \|x\|^2 + \|y\|^2 + (\tau - 2) x^T y \\
&\quad + (-1)^i \big\| 2x_1 x_2 + 2y_1 y_2 \\
&\quad + (\tau - 2)(x_1 y_2 + y_1 x_2) \big\|,
\end{aligned}
\tag{32}
$$

and $\lambda_i(z^\varepsilon)$ and $u_i(z)$ are, respectively, given by (18) and (19) for $i = 1, 2$. It is obvious that $\lambda_i(z^\varepsilon) = \lambda_i(z) + 2\varepsilon^2$ for $i = 1, 2$. Then,

$$
\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \left(\sqrt{\lambda_1(z^\varepsilon)} u_1(z) + \sqrt{\lambda_2(z^\varepsilon)} u_2(z) \right) \\
&= \lim_{\varepsilon \downarrow 0} \left(\sqrt{\lambda_1(z) + 2\varepsilon^2} u_1(z) + \sqrt{\lambda_2(z) + 2\varepsilon^2} u_2(z) \right) \\
&= \sqrt{\lambda_1(z)} u_1(z) + \sqrt{\lambda_2(z)} u_2(z),
\end{aligned}
\tag{33}
$$

and hence, $\lim_{\varepsilon \downarrow 0} \phi_{\tau,\varepsilon}(x, y) = \phi_\tau(x, y)$. Therefore, it follows from (i) and Definition 2 that $\phi_{\tau,\varepsilon}$ is a smoothing function of ϕ_τ. $\qquad \square$

Next, we give some properties of ϕ_τ [14], which will be used in the subsequent analysis.

Lemma 4. *For any $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$, let $z = (z_1; z_2) = x^2 + y^2 + (\tau - 2)(x \circ y) \in \text{bd}\, K^n$. Then one has*

$$\left(x + \frac{\tau-2}{2}y\right) \circ \left(x + \frac{\tau-2}{2}y\right)' = 0, \quad y \circ y' = 0, \quad (34)$$

$$\left(x + \frac{\tau-2}{2}y\right) \circ z' = 0, \quad y \circ z' = 0, \quad (35)$$

$$\left(x + \frac{\tau-2}{2}y\right) \circ y' = 0,$$

$$\left(y + \frac{\tau-2}{2}x\right) \circ x' = 0, \quad (36)$$

$$\left(y + \frac{\tau-2}{2}x\right) \circ \left(y + \frac{\tau-2}{2}x\right)' = 0, \quad x \circ x' = 0, \quad (37)$$

$$\left(y + \frac{\tau-2}{2}x\right) \circ z' = 0, \quad x \circ z' = 0, \quad (38)$$

$$\left(x + \frac{\tau-2}{2}y\right) \circ \left(y + \frac{\tau-2}{2}x\right)' = 0, \quad x \circ y' = 0. \quad (39)$$

Moreover, the following equivalence holds:

$$z_1 = 0 \iff z_2 = 0 \iff z = 0$$
$$\iff x_1 = y_1 = 0 \iff x_2 = y_2 = 0 \quad (40)$$
$$\iff (x, y) = (0, 0).$$

Proof. From Lemma 3.3 and its proof in [14], it is not difficult to see that relations (34)–(39) hold. The equivalence is also true, since

$$z = x^2 + y^2 + (\tau - 2)(x \circ y)$$

$$= \left(x + \frac{\tau-2}{2}y\right)^2 + \frac{\tau(4-\tau)}{4}y^2 \quad (41)$$

$$= \left(y + \frac{\tau-2}{2}x\right)^2 + \frac{\tau(4-\tau)}{4}x^2 \in \text{bd}\, K^n.$$

This completes the proof. $\square$

4. The Jacobian Consistency

In this section, we show the Jacobian consistency of the smoothing function $\phi_{\tau,\varepsilon}$, which will play an important role for establishing the rapid convergence of smoothing methods. Moreover, we estimate the distance between the gradient of the smoothing functions and the subgradient of the one-parametric class of the SOC complementarity functions, which will help to adjust a parameter appropriately in smoothing methods.

It has been shown in Proposition 3.1 in [14] that the function ϕ_τ with any $\tau \in (0, 4)$ satisfies

$$\phi_\tau(x, y) = 0 \iff x \in K^n, \quad y \in K^n, \quad \langle x, y \rangle = 0. \quad (42)$$

Let $(x, y, \zeta) \in R^n \times R^n \times R^l$ and define

$$\Phi_{\tau,\varepsilon}(x, y, \zeta) := \begin{pmatrix} F(x, y, \zeta) \\ \phi_{\tau,\varepsilon}(x, y) \end{pmatrix},$$

$$\Phi_\tau(x, y, \zeta) := \begin{pmatrix} F(x, y, \zeta) \\ \phi_\tau(x, y) \end{pmatrix}. \quad (43)$$

It is easy to see that $\Phi_{\tau,\varepsilon}(x, y, \zeta) = 0$ is the perturbation of the system of equations $\Phi_\tau(x, y, \zeta) = 0$. On account of (1), (42), and (43), we have

$$\Phi_\tau(x, y, \zeta) = 0 \iff (x, y, \zeta) \text{ solves } (1). \quad (44)$$

Since $\Phi_\tau(x, y, \zeta)$ is typically nonsmooth, we can solve approximately the smooth system $\Phi_{\tau,\varepsilon}(x, y, \zeta) = 0$ by using Newton's method at each iteration, and then obtain a solution of $\Phi_\tau(x, y, \zeta) = 0$ by reducing the parameter ε to zero.

First, we show that the function $\Phi_{\tau,\varepsilon}(x, y, \zeta)$ satisfies the Jacobian consistency.

Lemma 5. *For any $(x, y) \in R^n \times R^n$, one has*

$$J^0_{\phi_\tau}(x, y) := \lim_{\varepsilon \to 0} \nabla \phi_{\tau,\varepsilon}(x, y)$$

$$= \begin{pmatrix} I - L\left(x + \frac{\tau-2}{2}y\right)J \\ I - L\left(y + \frac{\tau-2}{2}x\right)J \end{pmatrix}, \quad (45)$$

where

$$J := \begin{cases} L^{-1}(w) & \text{if } (x, y) \in S_1, \\ \dfrac{1}{2\sqrt{2z_1}}\begin{pmatrix} 1 & \bar{z}_2^T \\ \bar{z}_2 & 4I - 3\bar{z}_2\bar{z}_2^T \end{pmatrix} & \text{if } (x, y) \in S_2, \\ O & \text{if } (x, y) = (0, 0). \end{cases} \quad (46)$$

Proof. By (27) and the symmetry of x and y, it suffices to show that

$$\lim_{\varepsilon \to 0} L\left(x + \frac{\tau-2}{2}y\right)L^{-1}(w^\varepsilon) = L\left(x + \frac{\tau-2}{2}y\right)J. \quad (47)$$

Case (i). If $(x, y) \in S_1$, we have from (18) that

$$\lim_{\varepsilon \to 0} \lambda_i(z^\varepsilon) = \lim_{\varepsilon \to 0} \left[\lambda_i(z) + 2\varepsilon^2\right] = \lambda_i(z) > 0, \quad (48)$$

for $i = 1, 2$, and then,

$$\lim_{\varepsilon \to 0} w^\varepsilon = \lim_{\varepsilon \to 0} \left[\sqrt{\lambda_1(z^\varepsilon)}u_1(z) + \sqrt{\lambda_2(z^\varepsilon)}u_2(z)\right]$$

$$= \sqrt{\lambda_1(z)}u_1(z) + \sqrt{\lambda_2(z)}u_2(z) \quad (49)$$

$$= w \in \text{int}\, K^n.$$

Therefore,

$$\lim_{\varepsilon \to 0} L\left(x + \frac{\tau-2}{2}y\right)L^{-1}(w^\varepsilon)$$

$$= L\left(x + \frac{\tau-2}{2}y\right)L^{-1}(w). \quad (50)$$

Case (ii). If $(x, y) \in S_2$, we obtain $(x, y) \neq (0, 0)$, and (41) holds, and thus,

$$\|z_2\| = z_1 = \left\|x + \frac{\tau - 2}{2}y\right\|^2 + \frac{\tau(4 - \tau)}{4}\|y\|^2 > 0,$$

$$2z_1 = \lambda_2(z) > \lambda_1(z) = 0. \tag{51}$$

Then we have from (18) that

$$\lambda_1(z^\varepsilon) = \lambda_1(z) + 2\varepsilon^2 = 2\varepsilon^2 > 0,$$

$$\lambda_2(z^\varepsilon) = \lambda_2(z) + 2\varepsilon^2 = 2z_1 + 2\varepsilon^2 > 0. \tag{52}$$

For any $\varepsilon \neq 0$, it follows from (28) that $L^{-1}(w^\varepsilon) = L_1(z^\varepsilon) + L_2(z^\varepsilon)$. We first show that $L(x + ((\tau - 2)/2)y)L_1(z^\varepsilon) = O$ for any $\varepsilon \neq 0$. Let

$$\omega := (1; \overline{z}_2) = \frac{1}{\|z_2\|}(z_1; z_2) = \frac{z}{z_1}. \tag{53}$$

By (35), we have

$$\left(x + \frac{\tau - 2}{2}y\right) \circ \omega' = \left(x + \frac{\tau - 2}{2}y\right) \circ \frac{z'}{z_1} = 0, \tag{54}$$

and therefore,

$$L\left(x + \frac{\tau - 2}{2}y\right)L_1(z^\varepsilon)$$

$$= \frac{1}{2\sqrt{\lambda_1(z^\varepsilon)}}L\left(x + \frac{\tau - 2}{2}y\right)\omega'\omega'^T$$

$$= \frac{1}{2\sqrt{2}|\varepsilon|}\left(x + \frac{\tau - 2}{2}y\right) \circ \omega'\omega'^T \tag{55}$$

$$= O.$$

We next show that $\lim_{\varepsilon \to 0} L_2(z^\varepsilon) = J$. In fact, we obtain from (52)

$$\lim_{\varepsilon \to 0} L_2(z^\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\sqrt{2z_1 + 2\varepsilon^2}}\begin{pmatrix} 1 & \overline{z}_2^T \\ \overline{z}_2 & \overline{z}_2\overline{z}_2^T \end{pmatrix}$$

$$+ \lim_{\varepsilon \to 0} \frac{2}{\sqrt{2\varepsilon^2} + \sqrt{2z_1 + 2\varepsilon^2}}\begin{pmatrix} 0 & 0^T \\ 0 & I - \overline{z}_2\overline{z}_2^T \end{pmatrix}, \tag{56}$$

$$= \frac{1}{2\sqrt{2z_1}}\begin{pmatrix} 1 & \overline{z}_2^T \\ \overline{z}_2 & \overline{z}_2\overline{z}_2^T \end{pmatrix} + \frac{2}{\sqrt{2z_1}}\begin{pmatrix} 0 & 0^T \\ 0 & I - \overline{z}_2\overline{z}_2^T \end{pmatrix},$$

$$= \frac{1}{2\sqrt{2z_1}}\begin{pmatrix} 1 & \overline{z}_2^T \\ \overline{z}_2 & 4I - 3\overline{z}_2\overline{z}_2^T \end{pmatrix} = J.$$

Combining (55) and (56) yields

$$\lim_{\varepsilon \to 0} L\left(x + \frac{\tau - 2}{2}y\right)L^{-1}(w^\varepsilon)$$

$$= \lim_{\varepsilon \to 0} L\left(x + \frac{\tau - 2}{2}y\right)L_2(z^\varepsilon) \tag{57}$$

$$= L\left(x + \frac{\tau - 2}{2}y\right)J.$$

Case (iii). If $(x, y) = (0, 0)$, then $w^\varepsilon = \sqrt{z^\varepsilon} = \sqrt{2}|\varepsilon|e \in \text{int } K^n$, and

$$\lim_{\varepsilon \to 0} L\left(x + \frac{\tau - 2}{2}y\right)L^{-1}(w^\varepsilon)$$

$$= \lim_{\varepsilon \to 0} O \cdot \frac{1}{\sqrt{2}|\varepsilon|}e \tag{58}$$

$$= O = L\left(x + \frac{\tau - 2}{2}y\right)J.$$

This completes the proof. □

Lemma 6. *For any $(x, y) \in R^n \times R^n$, one has*

$$\begin{pmatrix} I - U_x \\ I - U_y \end{pmatrix} \in \partial_B \phi_\tau(x, y), \tag{59}$$

where

$$U_x = \pm Z + L\left(x + \frac{\tau - 2}{2}y\right)J,$$

$$U_y = \pm \frac{\tau - 2}{2}Z + L\left(y + \frac{\tau - 2}{2}x\right)J, \tag{60}$$

$$Z = \begin{cases} O & \text{if } (x, y) \in S_1, \\ \frac{1}{2}\begin{pmatrix} 1 & -\overline{z}_2^T \\ -\overline{z}_2 & \overline{z}_2\overline{z}_2^T \end{pmatrix} & \text{if } (x, y) \in S_2, \\ I & \text{if } (x, y) = (0, 0), \end{cases} \tag{61}$$

$$J = \begin{cases} L^{-1}(w) & \text{if } (x, y) \in S_1, \\ \frac{1}{2\sqrt{2z_1}}\begin{pmatrix} 1 & \overline{z}_2^T \\ \overline{z}_2 & 4I - 3\overline{z}_2\overline{z}_2^T \end{pmatrix} & \text{if } (x, y) \in S_2, \\ O & \text{if } (x, y) = (0, 0). \end{cases}$$

Proof. It follows from Proposition 5.2 [1] and the chain rule for differentiation that the one-parametric class of SOC complementarity functions ϕ_τ is continuously differentiable at any $(x, y) \in S_1$ with

$$\nabla\phi_\tau(x, y)$$

$$= \begin{pmatrix} I - L\left(x + \frac{\tau - 2}{2}y\right)L^{-1}(w) \\ I - L\left(y + \frac{\tau - 2}{2}x\right)L^{-1}(w) \end{pmatrix} \in \partial_B \phi_\tau(x, y). \tag{62}$$

Therefore, it suffices to consider the two cases: $(x, y) \in S_2$ and $(x, y) = (0, 0)$.

For any $(x, y) \in S_2$ or $(x, y) = (0, 0)$, let $(\widehat{x}, y) = (x + \varepsilon e, y)$ with sufficiently small $\varepsilon \neq 0$, and define

$$\widehat{z} = (\widehat{z}_1; \widehat{z}_2) := \widehat{x}^2 + y^2 + (\tau - 2)(\widehat{x} \circ y),$$

$$\widehat{w} = (\widehat{w}_1; \widehat{w}_2) := \sqrt{\widehat{z}}, \quad \widetilde{z}_2 := \frac{\widehat{z}_2}{\|\widehat{z}_2\|}, \tag{63}$$

$$\widehat{\lambda}_i = \lambda_i(\widehat{z}) := \widehat{z}_1 + (-1)^i \|\widehat{z}_2\|, \quad i = 1, 2.$$

Direct calculations yield

$$
\begin{aligned}
\widehat{z} &= (x + \varepsilon e)^2 + y^2 + (\tau - 2)(x + \varepsilon e) \circ y \\
&= x^2 + y^2 + (\tau - 2)(x \circ y) \\
&\quad + 2\varepsilon x + \varepsilon(\tau - 2) y + \varepsilon^2 e \\
&= z + 2\varepsilon x + \varepsilon(\tau - 2) y + \varepsilon^2 e.
\end{aligned}
\tag{64}
$$

Therefore, we obtain

$$
\widehat{z}_1 = z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1 + \varepsilon^2,
\tag{65}
$$
$$
\widehat{z}_2 = z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2,
$$

$$
\widehat{\lambda}_i = z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1 + \varepsilon^2 + (-1)^i
$$
$$
\times \|z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2\|, \quad i = 1, 2.
\tag{66}
$$

It is obvious that as $\varepsilon \to 0$, we have $(\widehat{x}, y) \to (x, y)$, $\widehat{z} \to z$, $\widehat{w} \to w$, and $\widehat{\lambda}_i \to \lambda_i(z)$ for $i = 1, 2$.

By the definition of B-subdifferential and (3), it suffices to show that

$$
\lim_{\varepsilon \to 0} L\left(\widehat{x} + \frac{\tau - 2}{2} y\right) L^{-1}(\widehat{w}) = U_x,
$$
$$
\lim_{\varepsilon \to 0} L\left(y + \frac{\tau - 2}{2} \widehat{x}\right) L^{-1}(\widehat{w}) = U_y,
\tag{67}
$$

if ϕ_τ is differentiable at $(\widehat{x}, y)$.

Case (i). If $(x, y) = (0, 0)$, it is easy to see that $\widehat{z} = \varepsilon^2 e \in \operatorname{int} K^n$, $\widehat{w} = |\varepsilon| e$, and ϕ_τ is differentiable at $(\widehat{x}, y)$. Then, we have

$$
\begin{aligned}
&\lim_{\varepsilon \to \pm 0} L\left(\widehat{x} + \frac{\tau - 2}{2} y\right) L^{-1}(\widehat{w}) \\
&= \lim_{\varepsilon \to \pm 0} \varepsilon I \cdot \frac{1}{|\varepsilon|} I = \lim_{\varepsilon \to \pm 0} \operatorname{sgn}(\varepsilon) I \\
&= \pm I = U_x, \\
&\lim_{\varepsilon \to \pm 0} L\left(y + \frac{\tau - 2}{2} \widehat{x}\right) L^{-1}(\widehat{w}) \\
&= \lim_{\varepsilon \to \pm 0} \frac{\tau - 2}{2} \varepsilon I \cdot \frac{1}{|\varepsilon|} I = \lim_{\varepsilon \to \pm 0} \frac{\tau - 2}{2} \operatorname{sgn}(\varepsilon) I \\
&= \pm \frac{\tau - 2}{2} I = U_y.
\end{aligned}
\tag{68}
$$

Case (ii). If $(x, y) \in S_2$, we obtain $z \in \operatorname{bd} K^n \setminus \{0\}$, and thus, from (38) and (39),

$$
\begin{aligned}
\|\widehat{z}_2\|^2 &= \|z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2\|^2 \\
&= \|z_2\|^2 + 4\varepsilon^2 \|x_2\|^2 + \varepsilon^2(\tau - 2)^2 \|y_2\|^2 + 4\varepsilon z_2^T x_2 \\
&\quad + 2\varepsilon(\tau - 2) z_2^T y_2 + 4\varepsilon^2(\tau - 2) x_2^T y_2 \\
&= z_1^2 + 4\varepsilon^2 x_1^2 + \varepsilon^2(\tau - 2)^2 y_1^2 + 4\varepsilon z_1 x_1 \\
&\quad + 2\varepsilon(\tau - 2) z_1 y_1 + 4\varepsilon^2(\tau - 2) x_1 y_1 \\
&= [z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1]^2.
\end{aligned}
\tag{69}
$$

Since $z_1 > 0$ and ε is sufficiently small, we obtain

$$
\|\widehat{z}_2\| = z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1 > 0.
\tag{70}
$$

By (66) and (70), we have

$$
\begin{aligned}
\widehat{\lambda}_1 &= z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1 + \varepsilon^2 - \|\widehat{z}_2\| \\
&= \varepsilon^2 > 0,
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
\widehat{\lambda}_2 &= z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1 + \varepsilon^2 + \|\widehat{z}_2\| \\
&= 2[z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1] + \varepsilon^2 \\
&> 0.
\end{aligned}
\tag{72}
$$

Relations (71) and (72) imply that $\widehat{z} \in \operatorname{int} K^n$, and thus, ϕ_τ is differentiable at $(\widehat{x}, y)$.

Now we will show that

$$
\lim_{\varepsilon \to 0} L\left(\widehat{x} + \frac{\tau - 2}{2} y\right) L^{-1}(\widehat{w}) = U_x.
\tag{73}
$$

By Proposition 5.2 in [1], we have $L^{-1}(\widehat{w}) = L_1(\widehat{z}) + L_2(\widehat{z})$, where $L_1(\widehat{z})$ and $L_2(\widehat{z})$ are given by (29) with $\widehat{z}$ and $\widetilde{z}_2$ replacing z^ε and $\widetilde{z}_2$, respectively. By (70),

$$
\begin{aligned}
\widehat{w} &:= (1; \widetilde{z}_2) = \frac{1}{\|\widehat{z}_2\|} (\|\widehat{z}_2\|; \widehat{z}_2) \\
&= \frac{1}{\|\widehat{z}_2\|} (\|z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2\|; \\
&\qquad\qquad z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2) \\
&= \frac{1}{\|\widehat{z}_2\|} (z_1 + 2\varepsilon x_1 + \varepsilon(\tau - 2) y_1; \\
&\qquad\qquad z_2 + 2\varepsilon x_2 + \varepsilon(\tau - 2) y_2) \\
&= \frac{1}{\|\widehat{z}_2\|} [z + 2\varepsilon x + \varepsilon(\tau - 2) y].
\end{aligned}
\tag{74}
$$

According to (35), (39), (71), and (74), we have

$$\left(x + \frac{\tau - 2}{2}y\right) \circ \widehat{\omega}'$$

$$= \frac{1}{\|\widehat{z}_2\|}\left(x + \frac{\tau - 2}{2}y\right)$$

$$\circ \left[z + 2\varepsilon x + \varepsilon(\tau - 2)y\right]'$$

$$= \frac{1}{\|\widehat{z}_2\|}\left[\left(x + \frac{\tau - 2}{2}y\right) \circ z'\right.$$

$$\left. + 2\varepsilon\left(x + \frac{\tau - 2}{2}y\right) \circ \left(x + \frac{\tau - 2}{2}y\right)'\right] \tag{75}$$

$$= 0,$$

and thus,

$$L\left(\widehat{x} + \frac{\tau - 2}{2}y\right)L_1(\widehat{z})$$

$$= \frac{1}{2\sqrt{\widehat{\lambda}_1}}\left(x + \varepsilon e + \frac{\tau - 2}{2}y\right) \circ \widehat{\omega}'\widehat{\omega}'^T$$

$$= \frac{1}{2|\varepsilon|}\left[\left(x + \frac{\tau - 2}{2}y\right) \circ \widehat{\omega}'\widehat{\omega}'^T + \varepsilon\widehat{\omega}'\widehat{\omega}'^T\right] \tag{76}$$

$$= \frac{\operatorname{sgn}(\varepsilon)}{2}\widehat{\omega}'\widehat{\omega}'^T$$

$$= \frac{\operatorname{sgn}(\varepsilon)}{2}\begin{pmatrix} 1 & -\widetilde{z}_2^T \\ -\widetilde{z}_2 & \widetilde{z}_2\widetilde{z}_2^T \end{pmatrix}.$$

It follows from (70), (71), (72), and (74) that $\widehat{\lambda}_1 \to \lambda_1(z) = 0$, $\widehat{\lambda}_2 \to \lambda_2(z) = 2z_1$, and $\widetilde{z}_2 \to \overline{z}_2$ as $\varepsilon \to 0$. Then, by following the proof of Case (ii) in Lemma 5, we obtain

$$\lim_{\varepsilon \to 0} L_2(\widehat{z}) = \frac{1}{2\sqrt{2z_1}}\begin{pmatrix} 1 & \overline{z}_2^T \\ \overline{z}_2 & 4I - 3\overline{z}_2\overline{z}_2^T \end{pmatrix} = J. \tag{77}$$

Hence, we have from (76) and (77) that

$$\lim_{\varepsilon \to \pm 0} L\left(\widehat{x} + \frac{\tau - 2}{2}y\right)L^{-1}(\widehat{w})$$

$$= \lim_{\varepsilon \to \pm 0} L\left(\widehat{x} + \frac{\tau - 2}{2}y\right)L_1(\widehat{z})$$

$$+ \lim_{\varepsilon \to \pm 0} L\left(\widehat{x} + \frac{\tau - 2}{2}y\right)L_2(\widehat{z}) \tag{78}$$

$$= \lim_{\varepsilon \to \pm 0} \frac{\operatorname{sgn}(\varepsilon)}{2}\widehat{\omega}'\widehat{\omega}'^T$$

$$+ \lim_{\varepsilon \to \pm 0} L\left(\widehat{x} + \frac{\tau - 2}{2}y\right)J$$

$$= \pm Z + L\left(x + \frac{\tau - 2}{2}y\right)J = U_x.$$

Next, we will show that

$$\lim_{\varepsilon \to 0} L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)L^{-1}(\widehat{w}) = U_y. \tag{79}$$

By (38), (39), and (74), we have

$$\left(y + \frac{\tau - 2}{2}x\right) \circ \widehat{\omega}'$$

$$= \frac{1}{\|\widehat{z}_2\|}\left(y + \frac{\tau - 2}{2}x\right)$$

$$\circ \left[z + 2\varepsilon x + \varepsilon(\tau - 2)y\right]'$$

$$= \frac{1}{\|\widehat{z}_2\|}\left[\left(y + \frac{\tau - 2}{2}x\right) \circ z' + 2\varepsilon\left(y + \frac{\tau - 2}{2}x\right)\right. \tag{80}$$

$$\left. \circ \left(x + \frac{\tau - 2}{2}y\right)'\right]$$

$$= 0,$$

and then,

$$L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)L_1(\widehat{z})$$

$$= \frac{1}{2\sqrt{\widehat{\lambda}_1}}\left(y + \frac{\tau - 2}{2}x + \frac{\tau - 2}{2}\varepsilon e\right) \circ \widehat{\omega}'\widehat{\omega}'^T$$

$$= \frac{1}{2|\varepsilon|}\left[\left(y + \frac{\tau - 2}{2}x\right) \circ \widehat{\omega}'\widehat{\omega}'^T + \frac{\tau - 2}{2}\varepsilon\widehat{\omega}'\widehat{\omega}'^T\right] \tag{81}$$

$$= \frac{\tau - 2}{4}\operatorname{sgn}(\varepsilon)\widehat{\omega}'\widehat{\omega}'^T$$

$$= \frac{\tau - 2}{4}\operatorname{sgn}(\varepsilon)\begin{pmatrix} 1 & -\widetilde{z}_2^T \\ -\widetilde{z}_2 & \widetilde{z}_2\widetilde{z}_2^T \end{pmatrix}.$$

Hence, we have from (77) and (81) that

$$\lim_{\varepsilon \to \pm 0} L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)L^{-1}(\widehat{w})$$

$$= \lim_{\varepsilon \to \pm 0} L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)L_1(\widehat{z})$$

$$+ \lim_{\varepsilon \to \pm 0} L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)L_2(\widehat{z}) \tag{82}$$

$$= \lim_{\varepsilon \to \pm 0} \frac{\tau - 2}{4}\operatorname{sgn}(\varepsilon)\widehat{\omega}'\widehat{\omega}'^T$$

$$+ \lim_{\varepsilon \to \pm 0} L\left(y + \frac{\tau - 2}{2}\widehat{x}\right)J$$

$$= \pm\frac{\tau - 2}{2}Z + L\left(y + \frac{\tau - 2}{2}x\right)J = U_y.$$

This completes the proof. $\qquad \square$

Theorem 7. *The function* $\Phi_{\tau,\varepsilon}$ *defined by (43) with* $\varepsilon > 0$ *satisfies the Jacobian consistency.*

Proof. By (43), it suffices to show the Jacobian consistency of the function $\phi_{\tau,\varepsilon}$ with $\varepsilon > 0$. Let

$$V^i := \begin{pmatrix} I - U_x^i \\ I - U_y^i \end{pmatrix}, \tag{83}$$

where

$$U_x^i = (-1)^i Z + L\left(x + \frac{\tau - 2}{2}y\right)J,$$

$$U_y^i = (-1)^i \frac{\tau - 2}{2}Z + L\left(y + \frac{\tau - 2}{2}x\right)J, \tag{84}$$

for $i = 1, 2$, and Z and J are given by (61). Define

$$V := \frac{1}{2}\left(V^1 + V^2\right)$$

$$= \begin{pmatrix} I - L\left(x + \dfrac{\tau - 2}{2}y\right)J \\ I - L\left(y + \dfrac{\tau - 2}{2}x\right)J \end{pmatrix}. \tag{85}$$

On the one hand, we obtain from Lemma 5 that $V = J^0_{\phi_\tau}(x, y) = \lim_{\varepsilon \to 0} \nabla\phi_{\tau,\varepsilon}(x, y)$. On the other hand, we have by Lemma 6 that $V^1, V^2 \in \partial_B\phi_\tau(x, y)$, and therefore, $V = (1/2)(V^1 + V^2) \in \partial_B\phi_\tau(x, y)$. This together with Theorem 3 and Definition 1 implies the Jacobian consistency of $\phi_{\tau,\varepsilon}$ with $\varepsilon > 0$. $\qquad\square$

Now, we are in a position to estimate an upper bound of the parameter $\varepsilon > 0$ for the predicted accuracy of the distance between the gradient of $\Phi_{\tau,\varepsilon}$ and the subgradient of Φ_τ.

Theorem 8. *Let $\tau \in (0, 4)$, $\delta > 0$ be given, and $(x, y, \zeta) \in R^{2n+l}$ be any point. Let $\theta_\tau(x, y)$ be any function such that*

$$\theta_\tau(x, y) \geq \left\| \begin{matrix} L\left(x + \dfrac{\tau - 2}{2}y\right)J \\ L\left(y + \dfrac{\tau - 2}{2}x\right)J \end{matrix} \right\|, \tag{86}$$

and let $\bar{\varepsilon} : R^{2n} \times R_+ \to R_+ \cup \{+\infty\}$ be defined by

$$\bar{\varepsilon}(x, y, \delta) := \begin{cases} \dfrac{\lambda_1(z)\,\delta}{\sqrt{2\left(\theta_\tau(x, y)^2 - \lambda_1(z)\,\delta^2\right)}} \\ \qquad\text{if } (x, y) \in S_1, \ \delta < \dfrac{\theta_\tau(x, y)}{\sqrt{\lambda_1(z)}}, \\[4pt] \dfrac{z_1\delta}{2\sqrt{\theta_\tau(x, y)\left(2\theta_\tau(x, y) - \delta\sqrt{2z_1}\right)}} \\ \qquad\text{if } (x, y) \in S_2, \ \delta < \dfrac{2\theta_\tau(x, y)}{\sqrt{2z_1}}, \\[4pt] +\infty \qquad\text{otherwise.} \end{cases} \tag{87}$$

Then, for any $\varepsilon \in R$ such that $0 < |\varepsilon| \leq \bar{\varepsilon}(x, y, \delta)$, we have

$$\mathrm{dist}\left(\nabla\Phi_{\tau,\varepsilon}(x, y, \zeta), \partial\Phi_\tau(x, y, \zeta)\right) < \delta. \tag{88}$$

Proof. Since it follows from the proof of Theorem 7 that $J^0_{\phi_\tau}(x, y) = V \in \partial\phi_\tau(x, y)$ for any $(x, y, \zeta) \in R^{2n+l}$, we obtain

$$J^0_{\Phi_\tau}(x, y, \zeta)$$
$$:= \begin{pmatrix} J^0_{\phi_\tau}(x, y) & \nabla_{x,y}F(x, y, \zeta) \\ O & \nabla_\zeta F(x, y, \zeta) \end{pmatrix} \in \partial\Phi_\tau(x, y, \zeta). \tag{89}$$

Therefore, we have from (27) and (85) that

$$\mathrm{dist}\left(\nabla\Phi_{\tau,\varepsilon}(x, y, \zeta), \partial\Phi_\tau(x, y, \zeta)\right)$$
$$= \min\left\{\left\|\nabla\Phi_{\tau,\varepsilon}(x, y, \zeta) - W\right\| : W \in \partial\Phi_\tau(x, y, \zeta)\right\}$$
$$\leq \left\|\nabla\Phi_{\tau,\varepsilon}(x, y, \zeta) - J^0_{\Phi_\tau}(x, y, \zeta)\right\|$$

$$= \left\|\nabla\phi_{\tau,\varepsilon}(x, y, \zeta) - J^0_{\phi_\tau}(x, y, \zeta)\right\|$$

$$= \left\| \begin{matrix} L\left(x + \dfrac{\tau - 2}{2}y\right)\left(L^{-1}(\widetilde{w}) - J\right) \\ L\left(y + \dfrac{\tau - 2}{2}x\right)\left(L^{-1}(\widetilde{w}) - J\right) \end{matrix} \right\|. \tag{90}$$

Then, by following the proof of Proposition 4.1 in [12], we obtain

$$\mathrm{dist}\left(\nabla\Phi_{\tau,\varepsilon}(x, y, \zeta), \partial\Phi_\tau(x, y, \zeta)\right)$$
$$\leq \left[h_{\tau,0}(x, y) - h_{\tau,\varepsilon}(x, y)\right]$$
$$\times \left\| \begin{matrix} L\left(x + \dfrac{\tau - 2}{2}y\right)J \\ L\left(y + \dfrac{\tau - 2}{2}x\right)J \end{matrix} \right\|, \tag{91}$$

where $h_{\tau,\varepsilon} : R^{2n} \to R_+$ is defined as

$$h_{\tau,\varepsilon}(x, y) := \begin{cases} \dfrac{1}{\sqrt{\lambda_1(z) + 2\varepsilon^2}} & \text{if } (x, y) \in S_1, \\[4pt] \dfrac{\sqrt{2}}{\sqrt{z_1 + \varepsilon^2} + |\varepsilon|} & \text{if } (x, y) \in S_2, \\[4pt] 0 & \text{if } (x, y) = (0, 0). \end{cases} \tag{92}$$

Hence, similar to the proof of Theorem 4.1 in [12], we have the desired result. $\qquad\square$

5. An Alternative Proof

In this section, we study the directional derivative and B-subdifferential of the one-parametric class of SOC complementarity functions ϕ_τ. Based on these results, we present an alternative way to prove the Jacobian consistency of the one-parametric class of smoothing functions $\phi_{\tau,\varepsilon}$.

By Corollary 3.3 in [18], it is not difficult to see that the function ϕ_τ given as (3) is directionally differentiable everywhere. However, as far as we know, the expression of its directional derivative is not given in the available literature. In this section, we derive its expression and prove that the B-subdifferential of $\phi_\tau(x, y)$ at a general point coincides with that of its directional derivative function at the origin.

In light of the B-subdifferential of $\phi_{\mathrm{FB}}(x, y)$ [4, 10, 19], we obtain the following four results, which can be shown by following the proofs of Proposition 9, Lemma 11, Lemma 12, and Proposition 13 in [4], respectively.

Proposition 9. *For any given $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$, the directional derivative $\phi'_\tau((x, y); (g, h))$ of ϕ_τ at (x, y) has the following form.*

(i) *If $(x, y) = (0, 0)$, then $\phi'_\tau((x, y); (g, h)) = \phi_\tau(g, h)$.*

(ii) *If $(x, y) \in S_1$, then*

$$\phi'_\tau((x, y); (g, h)) = \left[I - L^{-1}(w)L\left(x + \frac{\tau - 2}{2}y\right)\right]g$$
$$+ \left[I - L^{-1}(w)L\left(y + \frac{\tau - 2}{2}x\right)\right]h.$$

$$\tag{93}$$

(iii) *If $(x, y) \in S_2$, then*

$$
\phi_\tau'\left((x,y);(g,h)\right) = g + h - \frac{\varphi(g,h)}{2}\begin{pmatrix} 1 \\ -\bar{z}_2 \end{pmatrix}
$$

$$
+ \frac{\left(x_2 + ((\tau-2)/2)y_2\right)^T g_2 + \left(y_2 + ((\tau-2)/2)x_2\right)^T h_2}{2\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}}\begin{pmatrix} 0 \\ \bar{z}_2 \end{pmatrix} - \frac{1}{2\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}}
$$

$$
\times \left(\begin{array}{c} \left(x + \dfrac{\tau-2}{2}y\right)^T g + \left(y + \dfrac{\tau-2}{2}x\right)^T h \\[4mm] 2\left(x_1 + \dfrac{\tau-2}{2}y_1\right)g_2 + g_1\left(x_2 + \dfrac{\tau-2}{2}y_2\right) + 2\left(y_1 + \dfrac{\tau-2}{2}x_1\right)h_2 + h_1\left(y_2 + \dfrac{\tau-2}{2}x_2\right) \end{array} \right),
$$

$$(94)$$

where $g = (g_1; g_2)$, $h = (h_1; h_2) \in R \times R^{n-1}$, $\bar{z}_2$ *is given as (20), and the function* $\varphi : R^n \times R^n \to R$ *is defined by*

$$
\varphi(g,h) = \frac{\sqrt{(\theta_1(g,h))^2 + \|\theta_2(g,h)\|^2}}{\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}}, \tag{95}
$$

with

$$
\theta_1(g,h) = \left(x_1 + \frac{\tau-2}{2}y_1\right)g_1 - \left(x_2 + \frac{\tau-2}{2}y_2\right)^T g_2
$$

$$
+ \left(y_1 + \frac{\tau-2}{2}x_1\right)h_1 - \left(y_2 + \frac{\tau-2}{2}x_2\right)^T h_2,
$$

$$
\theta_2(g,h) = \left(x_1 + \frac{\tau-2}{2}y_1\right)h_2 - h_1\left(x_2 + \frac{\tau-2}{2}y_2\right)
$$

$$
+ g_1\left(y_2 + \frac{\tau-2}{2}x_2\right) - \left(y_1 + \frac{\tau-2}{2}x_1\right)g_2.
$$

$$(96)$$

Lemma 10. *For any given* $(x,y) \in S_2$, *let* φ *be defined by* (95). *Then, the B-subdifferential of the function* φ *at* $(0,0)$ *is*

$$
\partial_B \varphi(0,0)
$$

$$
= \left\{ \left[\begin{array}{c} \left(\dfrac{\zeta_2^T(y_2 + ((\tau-2)/2)x_2) + \zeta_1(x_1 + ((\tau-2)/2)y_1)}{\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}}; \dfrac{-(y_1 + ((\tau-2)/2)x_1)\zeta_2 - \zeta_1(x_2 + ((\tau-2)/2)y_2)}{\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}} \right) \\[5mm] \left(\dfrac{-\zeta_2^T(x_2 + ((\tau-2)/2)y_2) + \zeta_1(y_1 + ((\tau-2)/2)x_1)}{\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}}; \dfrac{(x_1 + ((\tau-2)/2)y_1)\zeta_2 - \zeta_1(y_2 + ((\tau-2)/2)x_2)}{\sqrt{x_1^2 + y_1^2 + (\tau-2)x_1 y_1}} \right) \end{array} \right] \right|
$$

$$
\zeta = (\zeta_1; \zeta_2) \in R \times R^{n-1} \text{ satisfying } \zeta_1^2 + \|\zeta_2\|^2 = 1 \left. \vphantom{\begin{array}{c} a \\ b \\ c \end{array}} \right\}.
$$

$$(97)$$

Lemma 11. *For any given* $(x,y) \in R^n \times R^n$, *let* $\psi_\tau(\cdot,\cdot) \equiv \phi_\tau'((x,y);(\cdot,\cdot))$. *Then,*

$$
\partial_B \phi_\tau(x,y) = \partial_B \psi_\tau(0,0). \tag{98}
$$

Proposition 12. *For any given* $(x,y) \in S_2$, *we have*

$$
\partial_B \phi_\tau(x,y)
$$

$$
= \left\{ \left. \begin{pmatrix} I - X - pu_1(z)^T \\ I - Y - qu_1(z)^T \end{pmatrix} \right| p = (p_1; p_2), \right.
$$

$q = (q_1; q_2)$ *satisfying*

$$p = \left(\frac{\zeta_2^T (y_2 + ((\tau-2)/2) x_2) + \zeta_1 (x_1 + ((\tau-2)/2) y_1)}{\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}}; \right.$$

$$\left. \frac{-(y_1 + ((\tau-2)/2) x_1) \zeta_2 - \zeta_1 (x_2 + ((\tau-2)/2) y_2)}{\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}} \right),$$

$$q = \left(\frac{-\zeta_2^T (x_2 + ((\tau-2)/2) y_2) + \zeta_1 (y_1 + ((\tau-2)/2) x_1)}{\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}}; \right.$$

$$\left. \frac{\left(x_1 + \left(\frac{\tau-2}{2} \right) y_1 \right) \zeta_2 - \zeta_1 \left(y_2 + \left(\frac{\tau-2}{2} \right) x_2 \right)}{\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}} \right),$$

$$\left. \text{for some } \zeta = (\zeta_1; \zeta_2) \text{ such that } \zeta_1^2 + \|\zeta_2\|^2 = 1 \right\},$$

$$(99)$$

where $u_1(z)$ and $\bar{z}_2$ are defined by (19) and (20), respectively, and

$$X = \frac{1}{2\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}}$$

$$\times \begin{pmatrix} x_1 + \frac{\tau-2}{2} y_1 & x_2^T + \frac{\tau-2}{2} y_2^T \\ x_2 + \frac{\tau-2}{2} y_2 & 2\left(x_1 + \frac{\tau-2}{2} y_1 \right) I - \left(x_2 + \frac{\tau-2}{2} y_2 \right) \bar{z}_2^T \end{pmatrix},$$

$$Y = \frac{1}{2\sqrt{x_1^2 + y_1^2 + (\tau - 2) x_1 y_1}}$$

$$\times \begin{pmatrix} y_1 + \frac{\tau-2}{2} x_1 & y_2^T + \frac{\tau-2}{2} x_2^T \\ y_2 + \frac{\tau-2}{2} x_2 & 2\left(y_1 + \frac{\tau-2}{2} x_1 \right) I - \left(y_2 + \frac{\tau-2}{2} x_2 \right) \bar{z}_2^T \end{pmatrix}.$$

$$(100)$$

Theorem 13. *The function $\phi_{\tau,\varepsilon}$ defined by (13) with $\varepsilon > 0$ satisfies the Jacobian consistency.*

Proof. By the proof of Theorem 7, it suffices to show that $V \in \partial\phi_\tau(x, y)$, where V is defined by (85). From Proposition 9 and Lemma 11, it is obvious that the result holds if $(x, y) \in S_1$ or $(x, y) = (0, 0)$. Now, we prove $V \in \partial\phi_\tau(x, y)$ if $(x, y) \in S_2$. From Proposition 12, it is not difficult to see that

$$\begin{pmatrix} I - X \\ I - Y \end{pmatrix} \in \partial\phi_\tau(x, y), \tag{101}$$

for any given $(x, y) \in S_2$. Direct calculations together with Lemma 4 yield

$$X = L\left(x + \frac{\tau-2}{2} y \right) J,$$
$$Y = L\left(y + \frac{\tau-2}{2} x \right) J, \tag{102}$$

if $(x, y) \in S_2$. Therefore, it follows from (101) that for any given $(x, y) \in S_2$,

$$V \equiv \begin{pmatrix} I - X \\ I - Y \end{pmatrix} \in \partial\phi_\tau(x, y), \tag{103}$$

which completes the proof. $\square$

6. Conclusions

In this paper, we show the Jacobian consistency of the one-parametric class of smoothing functions $\phi_{\tau,\varepsilon}$, which will play an important role for establishing the rapid convergence of smoothing methods. Moreover, we estimate the distance between the gradient of the smoothing functions $\phi_{\tau,\varepsilon}$ and the subgradient of the one-parametric class of the SOC complementarity functions ϕ_τ, which will help to adjust a parameter appropriately in smoothing methods. The related topics for further study are about smoothing methods for solving the SOCCP via the Jacobian consistency of the one-parametric class of smoothing functions.

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