Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 715258, 6 pages
http://dx.doi.org/10.1155/2013/715258

Research Article

Explicit Solutions of Singular Differential Equation by Means of Fractional Calculus Operators

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Received 8 July 2013; Revised 6 September 2013; Accepted 10 September 2013

Academic Editor: Juan J. Trujillo

Recently, several authors demonstrated the usefulness of fractional calculus operators in the derivation of particular solutions of a considerably large number of linear ordinary and partial differential equations of the second and higher orders. By means of fractional calculus techniques, we find explicit solutions of second-order linear ordinary differential equations.

1. Introduction, Definitions, and Preliminaries

The widely investigated subject of fractional calculus (i.e., calculus of derivatives and integrals of any arbitrary real or complex order) has gained considerable importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details, [1–6]). The fractional calculus provides a set of axioms and methods to extend the coordinate and corresponding derivative definitions from integer n to arbitrary order α, $\{x^n, \partial^n/\partial x^n\} \to \{x^\alpha, \partial^\alpha/\partial x^\alpha\}$ in a reasonable way. The first question was already raised by Leibniz (1646–1716): can we define a derivative of the order $1/2$, that is, so that a double action of that derivative gives the ordinary one? We can mention that the fractional differential equations are playing an important role in fluid dynamics, traffic model with fractional derivative, measurement of viscoelastic material properties, modeling of viscoplasticity, control theory, economy, nuclear magnetic resonance, geometric mechanics, mechanics, optics, signal processing, and so on.

The differintegration operators and their generalizations [7–16] have been used to solve some classes of differential equations and fractional differential equations.

Some of most obvious formulations based on the fundamental definitions of Riemann-Liouville fractional integration and fractional differentiation are, respectively,

$$\begin{gathered} {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau \\ (t > a, \alpha > 0), \\ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \left(\frac{d}{dt}\right)^k \int_a^t f(\tau)(t-\tau)^{k-\alpha-1} d\tau \\ (k-1 \le \alpha < k), \end{gathered} \tag{1}$$

where $k \in N$, N being the set of positive integers and Γ stands for Euler's function gamma.

Definition 1 (cf. [10–14, 17]). If the function $f(z)$ is analytic (regular) inside and on C, where $C = \{C^-, C^+\}$, C^- is a contour along the cut joining the points z and $-\infty + i\,\mathrm{Im}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, and C^+ is a contour along the cut joining the points z and $\infty + i\,\mathrm{Im}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$\begin{aligned} f_\mu(z) &= (f(z))_\mu \\ &= \frac{\Gamma(\mu+1)}{2\pi i}\int_C \frac{f(t)}{(t-z)^{\mu+1}}dt \quad (\mu \neq -1,-2,\ldots), \\ f_{-n}(z) &= \lim_{\mu\to -n} f_\mu(z) \quad (n \in \mathbb{Z}^+), \end{aligned} \tag{2}$$

where $t \neq z$,

$$\begin{aligned} -\pi \le \arg(t-z) \le \pi \quad &\text{for } C^-, \\ 0 \le \arg(t-z) \le 2\pi \quad &\text{for } C^+, \end{aligned} \tag{3}$$

then $f_\mu(z)$ ($\mu > 0$) is said to be the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$ ($\mu < 0$) is said to be the fractional integral of $f(z)$ of order $-\mu$, provided (in each case) that

$$\left|f_\mu(z)\right| < \infty \quad (\mu \in \mathbb{R}). \tag{4}$$

We find it to be worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration which is defined above (cf., e.g., [10–14, 18]).

Lemma 2 (Linearity). *Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_μ and g_μ exist, then*

$$\begin{aligned} (i)\ (h_1 f(z))_\mu &= h_1 f_\mu(z) \\ (ii)\ (h_1 f(z) + h_2 g(z))_\mu &= h_1 f_\mu(z) + h_2 g_\mu(z), \end{aligned} \tag{5}$$

where h_1 and h_2 are constants and $\mu \in \mathbb{R}; z \in \mathbb{C}$.

Lemma 3 (Index Law). *Let $f(z)$ be an analytic and single-valued function. If $(f_\rho)_\mu$ and $(f_\mu)_\rho$ exist, then*

$$(f_\rho(z))_\mu = f_{\rho+\mu}(z) = (f_\mu(z))_\rho, \tag{6}$$

where $\rho, \mu \in \mathbb{R}$ and $z \in \mathbb{C}$, and $|\Gamma(\rho+\mu+1)/\Gamma(\rho+1)\Gamma(\mu+1)| < \infty$.

Lemma 4 (Generalized Leibniz Rule). *Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_μ and g_μ exist, then*

$$\begin{aligned} &(f(z)\cdot g(z))_\mu \\ &\quad = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\Gamma(n+1)} f_{\mu-n}(z)\cdot g_n(z), \end{aligned} \tag{7}$$

where $\mu \in \mathbb{R}; z \in \mathbb{C}$ and $|\Gamma(\mu+1)/\Gamma(\mu-n+1)\Gamma(n+1)| < \infty$.

Property 1. For a constant λ,

$$\left(e^{\lambda z}\right)_\mu = \lambda^\mu e^{\lambda z} \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{8}$$

Property 2. For a constant λ,

$$\left(e^{-\lambda z}\right)_\mu = e^{-i\pi\mu}\lambda^\mu e^{-\lambda z} \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{9}$$

Property 3. For a constant λ,

$$\begin{aligned} \left(z^\lambda\right)_\mu &= e^{-i\pi\mu}\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\mu} \\ &\left(\mu \in \mathbb{R}; z \in \mathbb{C}; \left|\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \end{aligned} \tag{10}$$

Some of the most recent contributions on the subject of particular solutions of linear ordinary and partial fractional differintegral equations are those given by Tu et al. [19] who presented unification and generalization of a significantly large number of widely scattered results on this subject. We begin by recalling here one of the main results of Tu et al. [19], involving a family of linear ordinary fractional differintegral equations, as Theorem 5 below.

Theorem 5 (Tu et al. [19, p. 295, Theorem 1; p. 296, Theorem 2]). *Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$\begin{aligned} P(z;p) &:= \sum_{k=0}^{p} a_k z^{p-k} \\ &= a_0 \prod_{j=1}^{p}(z-z_j) \quad (a_0 \neq 0, p \in \mathbb{N}), \\ Q(z;q) &:= \sum_{k=0}^{q} b_k z^{q-k} \quad (b_0 \neq 0, q \in \mathbb{N}). \end{aligned} \tag{11}$$

Suppose also that $f_{-\nu} \neq 0$ exists for a given function f.

Then, the nonhomogeneous linear ordinary fractional differintegral equation

$$\begin{aligned} P(z;p)\,\phi_\mu(z) &+ \left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right] \\ &\times \phi_{\mu-k}(z) + \binom{\nu}{q} q! b_0 \phi_{\mu-q-1}(z) = f(z) \\ &(\mu,\nu \in \mathbb{R},\ p,q \in \mathbb{N}) \end{aligned} \tag{12}$$

has a particular solution of the form

$$\begin{aligned} \phi(z) &= \left(\left(\frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)}\right)_{-1} e^{-H(z;p,q)}\right)_{\nu-\mu+1} \\ &\quad \left(z \in \mathbb{C}\setminus\{z_1,\ldots,z_p\}\right), \end{aligned} \tag{13}$$

where, for convenience,

$$H(z;p,q) := \int^{z} \frac{Q(\zeta;q)}{P(\zeta;p)}d\zeta \quad \left(z \in \mathbb{C}\setminus\{z_1,\ldots,z_p\}\right), \tag{14}$$

provided that the second member of (13) *exists.*

Furthermore, the homogeneous linear ordinary fractional differintegral equation

$$P(z;p)\phi_\mu(z) + \left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p) + \sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right] \times \phi_{\mu-k}(z) + \binom{\nu}{q}q!b_0\phi_{\mu-q-1}(z) = 0 \quad (\mu,\nu\in\mathbb{R},\ p,q\in\mathbb{N}) \tag{15}$$

has solutions of the form

$$\phi(z) = K\left(e^{-H(z;p,q)}\right)_{\nu-\mu+1}, \tag{16}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (14), *provided that the second member of* (16) *exists.*

2. Schrödinger Equation

In this stud, the main aim is to investigate the Schrödinger equation in a given α-dimensional fractional space with a Coulomb potential depending on a parameter.

The Schrödinger equation to start with is given by

$$\left[-\frac{\hbar^2}{2mr^{\alpha-1}}\frac{\partial}{\partial r}\left(r^{\alpha-1}\frac{\partial}{\partial r}\right) + \frac{\ell^2}{2mr^2} - e^2\frac{\kappa}{r^{\delta-2}}\right]\varphi(r,\theta) = E\varphi(r,\theta), \tag{17}$$

where ℓ^2 corresponds to the angular momentum operator given by

$$\ell^2\varphi(r,\theta) = \left[-\frac{\hbar^2}{\sin^{\alpha-2}}\frac{\partial}{\partial\theta}\left(\sin^{\alpha-1}\frac{\partial}{\partial\theta}\right)\right]\varphi(r,\theta) = \ell(\ell+\alpha-2)\varphi(r,\theta), \tag{18}$$

where α is the dimension of a solid ($1 \le \alpha \le 3$) and the radial distance r ($0 \le r \le \infty$) and related angle θ ($0 \le \theta \le \pi$) measured relative to an axis passing through the origin are two coordinates describing r in the α-dimensional (αD) space. The constant κ has the value of $1/4\pi\epsilon_0$ for $\delta = 3$ and is generally defined as [20]

$$\kappa = \frac{\Gamma(\delta/2)}{2\pi^{\delta/2}(\delta-2)\epsilon_0} \quad (\delta > 2). \tag{19}$$

Looking for solutions of (17) in the form

$$\varphi(r,\theta) = R(r)\Phi(\theta), \tag{20}$$

we easily find that

$$R''(r) + \frac{\alpha-1}{r}R'(r) + \left[\frac{2m}{\hbar^2}\left(E + e^2\frac{\kappa}{r^{\delta-2}}\right) - \frac{\ell(\ell-\alpha-2)}{r^2}\right]R(r) = 0, \tag{21}$$

$$\Phi''(\theta) + (\alpha-2)\cot\theta\Phi'(\theta) + \ell(\ell-\alpha-2)\Phi(\theta) = 0.$$

The angular equation (18) has solutions in terms of Gegenbauer polynomials $C_\ell^{\alpha/2-1}(\cos\theta)$ as follows:

$$\Phi_\ell(\theta) = H_\ell(\alpha)C_\ell^{\alpha/2-1}(\cos\theta) \quad (\ell = 0,1,2,\ldots,n-1), \tag{22}$$

where H_ℓ is the normalization factor and is given by [21],

$$H_\ell(\alpha) = \begin{cases} \Gamma\left(\dfrac{\alpha}{2}-1\right)\left[\dfrac{\ell!\,(\ell+\alpha/2-1)}{2^{3-\alpha}\pi\Gamma(\ell+\alpha-2)}\right]^{1/2} & (\alpha\neq 2), \\ \dfrac{1}{(2\pi)^{1/2}} \quad (\ell\neq 0) & \text{or} \\ \dfrac{1}{2\pi^{1/2}} \quad (\ell = 0) & (\alpha = 2). \end{cases} \tag{23}$$

To solve the radial equation $R(r)$, let us use the substitutions

$$R(r) = r^\ell e^{-kr}\phi(r), \tag{24}$$

where $k^2 = -2mE/\hbar^2$. We arrive at the following differential equation:

$$z\phi''(z) + [(2\ell+\alpha-1)-z]\phi'(z) + \left[\frac{b}{2^{3-\delta}k^{4-\delta}}z^{3-\delta} - \frac{2\ell+\alpha-1}{2}\right]\phi(z) = 0, \tag{25}$$

where we use the substitutions

$$z = 2kr, \qquad b = \frac{me^2\kappa}{\hbar^2}. \tag{26}$$

It is worthwhile to mention that for $3D$ ($\delta = 3$), we arrive at the special case as given in reference [21].

Consider the differential equation

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\sigma z^{3-\delta} - \frac{\tau}{2}\right)\phi(z) = 0, \tag{27}$$

where

$$\tau = 2\ell+\alpha-1, \qquad \sigma = \frac{b}{2^{3-\delta}k^{4-\delta}}. \tag{28}$$

(i) Let $\delta = 2$. For this δ (27) becomes the differential equation

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\sigma z - \frac{\tau}{2}\right)\phi(z) = 0. \tag{29}$$

For (29), we use the substitution

$$\phi(z) = e^{z/2}z^{-\tau/2}w(z). \tag{30}$$

Thus, we have

$$\frac{d\phi}{dz} = e^{z/2} z^{-\tau/2-1}\left[z\frac{dw}{dz} + \frac{1}{2}(z-\tau)\,w(z)\right], \tag{31}$$

$$\begin{aligned}\frac{d^2\phi}{dz^2}(z) = e^{z/2} z^{-\tau/2-2} &\times\left\{z^2\frac{d^2w}{dz^2} + z(z-\tau)\frac{dw}{dz}\right. \\ &\left. + \frac{1}{4}\left[(z-\tau)^2 + 2\tau\right] w(z)\right\}.\end{aligned} \tag{32}$$

After substituting (30), (31), and (32) into (29) and doing some simplifications, we obtain at the differential equation

$$z^2\frac{d^2w}{dz^2} + \left[\frac{2\tau-\tau^2}{4} + \left(\sigma - \frac{1}{4}\right) z^2\right] w(z) = 0. \tag{33}$$

The transformation

$$w(z) = z^{1/2}\varphi(z) \tag{34}$$

has first and second derivative

$$\frac{dw}{dz} = z^{1/2}\left[\frac{d\varphi}{dz} + \frac{1}{2z}\varphi(z)\right], \tag{35}$$

$$\frac{d^2w}{dz^2} = z^{1/2}\left[\frac{d^2\varphi}{dz^2} + \frac{1}{z}\frac{d\varphi}{dz} - \frac{1}{4z^2}\varphi(z)\right]. \tag{36}$$

Finally, substituting (34) and (36) into (33) and doing simplifications we arrived at the equation

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left[\left(\frac{\sqrt{4\sigma-1}}{2}z\right)^2 - \left(\frac{\tau-1}{2}\right)^2\right]\varphi(z) = 0. \tag{37}$$

(ii) Let $\delta = 4$. For this δ (27) becomes the following differential equation:

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\frac{\sigma}{z} - \frac{\tau}{2}\right)\phi(z) = 0. \tag{38}$$

For (38), we use the substitution

$$\phi(z) = e^{z/2} z^{-\tau/2} u(z). \tag{39}$$

Therefore, we obtain

$$\begin{aligned}\frac{d\phi}{dz} &= e^{z/2} z^{-\tau/2-1}\left[z\frac{du}{dz} + \frac{1}{2}(z-\tau)\,u(z)\right], \\ \frac{d^2\phi}{dz^2} &= e^{z/2} z^{-\tau/2-2} \\ &\quad\times\left\{z^2\frac{d^2u}{dz^2} + z(z-\tau)\frac{du}{dz}\right. \\ &\quad\left. + \frac{1}{4}\left[(z-\tau)^2 + 2\tau\right] u(z)\right\}.\end{aligned} \tag{40}$$

After substituting (39) and (40) into (38) and doing simplifications, we arrived at the equation

$$4z^2\frac{d^2u}{dz^2} + \left[4\sigma - \tau^2 + 2\tau - z^2\right] u(z) = 0. \tag{41}$$

Similarly, for (41), we use the transformation

$$u(z) = z^{1/2}\varphi(z). \tag{42}$$

Thus, we have

$$\frac{du}{dz} = z^{1/2}\left[\frac{d\varphi}{dz} + \frac{1}{2z}\varphi(z)\right], \tag{43}$$

$$\frac{d^2u}{dz^2} = z^{1/2}\left[\frac{d^2\varphi}{dz^2} + \frac{1}{z}\frac{d\varphi}{dz} - \frac{1}{4z^2}\varphi(z)\right]. \tag{44}$$

Finally, substitute (42) and (44) into (41) and do simplifications to obtain the equation

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} - \left[\left(\frac{z}{2}\right)^2 + \left(\frac{\sqrt{(\tau-1)^2 - 4\sigma}}{2}\right)^2\right]\varphi(z) = 0. \tag{45}$$

Our aim is to obtain explicit solutions of (37) and (45), by means of (27), according to different δ.

3. Applications of Theorem 5 to a Class of Ordinary Second-Order Equations

In order to apply Theorem 5 to the following class of ordinary homogeneous differential equations:

$$\begin{aligned}&Az^2\frac{d^2\varphi}{dz^2} + (Bz + C)\frac{d\varphi}{dz} \\ &\quad + \left(Dz^2 + Ez + F\right)\varphi(z) = 0 \quad (z \in \mathbb{C}\setminus\{0\}),\end{aligned} \tag{46}$$

If the first two lines and last lines of (47) substitute into (46) $(A, D \neq 0)$, (37) and (45) is obtained, respectively.

$$\begin{gathered}A = B = 1, \qquad D = \frac{4\sigma-1}{4}, \\ C = E = 0, \qquad F = -\left(\frac{\tau-1}{2}\right)^2 \\ \left(A = B = 1, D = \frac{-1}{4}, C = E = 0,\right. \\ \left. F = \sigma - \left(\frac{\tau-1}{2}\right)^2\right).\end{gathered} \tag{47}$$

Indeed, by applying Theorem 5 in order to find explicit solutions of the homogeneous differential equation (46), Lin et al. [22] deduced the following result.

Theorem 6 (see [22, Theorem 3, p. 39]). *If the given function f satisfies the constraint* (4) *and $f_{-\nu} \neq 0$, then the nonhomogeneous linear ordinary differential equation*

$$Az^2 \frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2 + Ez + F\right)\varphi(z) = f(z) \quad (A \neq 0, D \neq 0) \tag{48}$$

has a particular solution in the form

$$\varphi(z) = z^\rho e^{\lambda z}\left[\left(A^{-1}z^{-\nu-1+(2A\rho+B)/A}\; e^{2\lambda z} \times \left(z^{-\rho-1}e^{-\lambda z}f(z)\right)_{-\nu}\right)_{-1} \times z^{\nu-(2A\rho+B)/A}e^{-2\lambda z}\right]_{\nu-1} \quad (A \neq 0, D \neq 0, z \in \mathbb{C} \setminus \{0\}), \tag{49}$$

where ρ and λ are given by

$$\rho = \frac{A - B \pm \sqrt{(A-B)^2 - 4AF}}{2A}, \qquad \lambda = \pm i\sqrt{\frac{D}{A}}, \tag{50}$$

$$\nu = \frac{(2A\rho + B)\lambda + E}{2A\lambda}, \tag{51}$$

provided that the second member of (49) *exists.*

Furthermore, the homogeneous linear ordinary differential equation

$$Az^2 \frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2 + Ez + F\right)\varphi(z) = 0 \tag{52}$$

has solutions of the form

$$\varphi(z) = Kz^\rho e^{\lambda z}\left(z^{\nu-(2A\rho+B)/A}e^{-2\lambda z}\right)_{\nu-1} \quad (A \neq 0, D \neq 0, z \in \mathbb{C} \setminus \{0\}), \tag{53}$$

where K is an arbitrary constant, ρ and λ are given by (50), and ν is given by (51), provided that the second member of (53) exists.

Theorem 7. *Under the hypotheses of Theorem 6, the homogeneous linear ordinary differential equation*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left[\left(\frac{\sqrt{4\sigma-1}}{2}z\right)^2 - \left(\frac{\tau-1}{2}\right)^2\right]\varphi(z) = 0 \tag{54}$$

has a particular solituon in the form

$$\varphi(z) = Nz^{\nu-1/2}e^{\lambda z}\left(z^{-\nu}e^{-2\lambda z}\right)_{\nu-1} \quad (\nu \in \mathbb{R}, z \in \mathbb{C} \setminus \{0\}), \tag{55}$$

where N is an arbitrary constant and ρ and λ are given by

$$\rho = \frac{\pm(\tau-1)}{2}, \qquad \lambda = \pm i\frac{\sqrt{4\sigma-1}}{2}, \qquad \nu = \frac{2\rho+1}{2}, \tag{56}$$

provided that the second member of (55) *exists.*

Theorem 8. *Under the hypotheses of Theorem 6, the homogeneous linear ordinary differential equation*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} - \left[\left(\frac{z}{2}\right)^2 + \left(\frac{\sqrt{(\tau-1)^2 - 4\sigma}}{2}\right)^2\right]\varphi(z) = 0 \tag{57}$$

has a particular solution of the form

$$\varphi(z) = Hz^{\nu-1/2}e^{\lambda z}\left(z^{-\nu}e^{-2\lambda z}\right)_{\nu-1} \quad (\nu \in \mathbb{R}, z \in \mathbb{C} \setminus \{0\}), \tag{58}$$

where H is an arbitrary constant and ρ and λ are given by

$$\rho = \pm\sqrt{\left(\frac{\tau-1}{2}\right)^2 - \sigma}, \qquad \lambda = \mp\frac{1}{2}, \qquad \nu = \frac{2\rho+1}{2}, \tag{59}$$

provided that the second member of (58) *exists.*

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