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Research Article

Three Homoclinic Solutions for Second-Order p-Laplacian Differential System

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We consider second-order p-Laplacian differential system. By using three critical points theorem, we establish the new criterion to guarantee that this p-Laplacian differential system has at least three homoclinic solutions. An example is presented to illustrate the main result.

1. Introduction

Let us consider the following second-order p-Laplacian differential system:

$$\begin{gathered}\left(\rho(t)\Phi_p\left(u'(t)\right)\right)' - s(t)\Phi_p(u(t)) \\ + \lambda f(t,u(t)) = 0,\end{gathered} \tag{P}$$

where $\Phi_p(x) := |x|^{p-2}x$, $p > 1$, $\rho, s \in L^\infty$ with $\operatorname{ess\,inf} \rho > 0$ and $\operatorname{ess\,inf} s > 0$, $f : R \times R^n \to R^n$ is continuous, $t \in R$, and $\lambda \in [0, +\infty)$. As usual, we say that a solution $u(t)$ of (P) is nontrivial homoclinic (to 0) if $u(t) \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

In the past two decades, many authors have studied homoclinic orbits for the second-order Hamiltonian systems

$$\ddot{q}(t) + \nabla V(t, q(t)) = f(t), \tag{1}$$

and the existence and multiplicity of homoclinic solutions for (1) have been extensively investigated via critical point theory (see [1–15]). For instance, Yang et al. [5] have shown the existence of infinitely many homoclinic solutions for (1) by using fountain theorem.

Theorem A (see [5]). *Assume that f and V satisfy the following conditions:*

(H1) *$f(t) = 0$ and $\nabla V(t, q(t)) = -L(t)q(t) + \nabla W(t, q(t))$;*

(H2) *$L \in C(R, R^{n\times n})$ is a symmetric and positive definite matrix for all $t \in R$ and there is a continuous function $\alpha : R \to R$ such that $\alpha(t) > 0$ for all $t \in R$ and*

$$\begin{gathered}(L(t)u, u) \geq \alpha(t)|u|^2, \\ \alpha(t) \longrightarrow +\infty\end{gathered} \tag{2}$$

as $|t| \to \infty$;

(H3) *consider the following*

$$W(t,u) = m(t)|u|^\gamma + d|u|^p, \tag{3}$$

where $m : R \to R^+$ is a positive continuous function such that $m \in L^{2/(2-\gamma)}(R, R^+)$ and $1 < \gamma < 2$, $d \geq 0$, and $p > 2$ are constants.

Then (1) possesses infinitely many homoclinic solutions.

Moreover, Tang and Xiao [10] prove the existence of homoclinic solution of (1) as a limit of the $2kT$-periodic solutions of the following extension of system (1):

$$\ddot{q}(t) = -\nabla V(t, q(t)) + f_k(t), \tag{4}$$

and they established the following theorem.

Theorem B (see [10]). *Assume that f and V satisfy the following conditions:*

(H4) *$V, f(t) \neq 0$ and $V(t,x) = -K(t,x) + W(t,x)$, where $V \in C^1(R \times R^n, R)$ is T-periodic with respect to t, and $T > 0$;*

(H5) *$\nabla W(t,x) = o(|x|)$, as $|x| \to 0$ uniformly with respect to t;*

(H6) *there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,x) \leq (x, \nabla W(t,x)) \quad \forall (t,x) \in R \times (R^n \setminus \{0\}); \tag{5}$$

(H7) *$f : R \times R^n$ is a continuous and bounded function;*

(H8) *there exist constants $b > 0$ and $\gamma \in (1,2]$ such that*

$$K(t,0) = 0, \quad K(t,x) \geq b|x|^{\gamma} \quad \text{for } (t,x) \in [0,T] \times R^n; \tag{6}$$

(H9) *there is a constant $\varrho \in [2,\mu)$ such that*

$$(x, \nabla K(t,x)) \leq \varrho K(t,x), \quad \text{for } (t,x) \in [0,T] \times R^n; \tag{7}$$

(H10) *consider the following*

$$\int_R |f(t)|^2 dt < 2\left(\min\left\{\frac{\delta}{2}, b\delta^{\gamma-1} - M\delta^{\mu-1}\right\}\right)^2. \tag{8}$$

Then system (1) *possesses a nontrivial homoclinic solution.*

For p-Laplacian problem, Tian and Ge [16] obtained sufficient conditions that guarantee the existence of at least two positive solutions of p-Laplacian boundary value problem with impulsive effects. Two key conditions of the main results of [16] are listed as follows:

(H11) there exist $\mu > p$, $h \in C([a,b] \times [0,+\infty), [0,+\infty))$, $\eta > 0$, $r \in C([a,b] \times [0,+\infty))$, $g \in C([0,+\infty), [0,+\infty))$, and

$$\int_a^b r(s)\,ds + \eta > 0, \tag{9}$$

such that

$$\begin{gathered} f(t,x) = r(t)\,\Phi_\mu(x) + h(t,x), \\ I_i(x) = \eta \Phi_\mu(x) + g(x); \end{gathered} \tag{10}$$

(H12) there exist $c \in L^1([a,b],[0,+\infty))$, $d \in C([a,b], [0,+\infty))$, $\xi \geq 0$, such that

$$h(t,x) \leq c(t) + d(t)\,\Phi_p(x). \tag{11}$$

In [17], Ricceri established a three critical points theorem. After that, several authors used it to obtain some interesting results (see [18–22]).

Existence and multiplicity of solutions for p-Laplacian boundary value problem have been studied extensively in the literature (see [23–26]). However, to our best knowledge, the existence of at least three homoclinic solutions for p-Laplacian differential system has attracted less attention.

Motivated by the aforementioned facts, in this paper we are devoted to study the multiplicity homoclinic solutions of (P) via three critical points theorem obtained by Ricceri [17].

In order to receive the homoclinic solution of (P), similar to [10] we consider a sequence of system of differential equations as follows:

$$\begin{gathered} \left(\rho(t)\,\Phi_p\left(u'(t)\right)\right)' - s(t)\,\Phi_p(u(t)) \\ + \lambda f_{(k)}(t,u(t)) = 0, \end{gathered} \tag{P_k}$$

where $f_{(k)} : R \times R^n \to R^n$ is a $2kT$-periodic extension of restriction of f to the interval $[-kT, kT)$, $k \in N$. We will prove the existence of three homoclinic solutions of (P) as the limit of the $2kT$-periodic solutions of (P_k) as in [10]. However, many technical details in our paper are different from [10, 12].

2. Preliminaries

For each $k \in N$, let $E_{(k)} = W^{1,p}_{2kT}(R,R^n)$ denote the Sobolev space of $2kT$-periodic functions on R with values in R^n under the norm

$$\begin{aligned} \|u\| &:= \|u\|_{E_{(k)}} \\ &= \left[\int_{-kT}^{kT} \left(\rho(t)\left|u'(t)\right|^p + s(t)\,|u(t)|^p\right) dt\right]^{1/p}, \end{aligned} \tag{12}$$

which is equivalent to the usual one. We define the norm in $C([-kT,kT])$ as $\|u\|_{C_{[-kT,kT]}} = \max\{|u(t)| : t \in [-kT,kT]\}$.

Consider $J_{(k)} : E_{(k)} \times [0,+\infty) \to R^n$ defined by

$$J_{(k)}(u,\lambda) = \phi_1(u) + \lambda\phi_2(u), \tag{13}$$

where

$$\phi_1(u) = \frac{\|u\|^p}{p},$$

$$\phi_2(u) = -\int_{-kT}^{kT} F_{(k)}(t,u(t))\,dt,$$

$$F_{(k)}(t,x) = \int_0^x f_{(k)}(t,y)\,dy \quad \forall (t,x) \in [-kT,kT] \times R^n. \tag{14}$$

Using the continuity of $f_{(k)}$, one has that $J_{(k)}(u,\lambda)$ is (strongly) continuous in $E_{(k)} \times [0,+\infty)$, $J_{(k)}(\cdot,\lambda) \in C^1(E_{(k)},R^n)$ and for any $u, v \in E_{(k)}$,

$$\begin{aligned} \left\langle J'_{(k)u}(u,\lambda), v\right\rangle = &\int_{-kT}^{kT} \rho(t)\,\Phi_p\left(u'(t)\right) v'(t)\,dt \\ &+ \int_{-kT}^{kT} s(t)\,\Phi_p(u(t))\,v(t)\,dt \\ &- \lambda \int_{-kT}^{kT} f_{(k)}(t,u(t))\,v(t)\,dt. \end{aligned} \tag{15}$$

In order to prove our main result, we list some basic facts in this section.

Definition 1. A function

$$u \in \left\{u \in E_{(k)} : \rho\Phi_p\left(u'\right)(\cdot) \in W^{1,\infty}_{2kT}\left(R, R^n\right)\right\} \quad (16)$$

is said to be a $2kT$-periodic solution of (P_k) if u satisfies the equation in (P_k).

Lemma 2. *If $u \in E_{(k)}$ is a critical point of $J_{(k)}(\cdot,\lambda)$; then u is a $2kT$-periodic solution of (P_k).*

Proof. Assume that $u \in E_{(k)}$ is a critical point of $J_{(k)}(\cdot,\lambda)$; then for all $v \in E_{(k)}$, one has

$$\begin{aligned} 0 = & \int_{-kT}^{kT} \rho(t)\,\Phi_p\left(u'(t)\right)v'(t)\,dt \\ & + \int_{-kT}^{kT} s(t)\,\Phi_p(u(t))\,v(t)\,dt \\ & - \lambda \int_{-kT}^{kT} f_{(k)}(t,u(t))\,v(t)\,dt. \end{aligned} \quad (17)$$

It follows that

$$\begin{aligned} & \int_{-kT}^{kT} \rho(t)\,\Phi_p\left(u'(t)\right)v'(t)\,dt \\ & = -\int_{-kT}^{kT} s(t)\,\Phi_p(u(t))\,v(t)\,dt \\ & + \lambda \int_{-kT}^{kT} f_{(k)}(t,u(t))\,v(t)\,dt. \end{aligned} \quad (18)$$

By the definition of weak derivative, (18) implies that

$$\left(\rho(t)\,\Phi_p\left(u'(t)\right)\right)' = s(t)\,\Phi_p(u(t)) - \lambda f(t,u(t)). \quad (19)$$

Thus $\rho\Phi_p(u')(\cdot) \in W^{1,\infty}_{2kT}(R,R^n)$ and u satisfies the (P_k). Therefore, u is a solution of (P_k). □

Lemma 2 motivates us to apply three critical points theorem to discuss the multiplicity of the $2kT$-periodic solution of (P_k). Here, at the end of this section, let us recall some important facts.

Definition 3. Let X be a Banach space and $f : X \to (-\infty,+\infty]$. f is said to be sequentially weakly lower semicontinuous if $\liminf_{k\to+\infty} f(x_k) \geq f(x)$ as $x_k \rightharpoonup x$ in X.

Definition 4. Suppose E is a real Banach space. For $\phi \in C^1(E,R^n)$, we say that ϕ satisfies PS condition if any sequence $\{u_k\} \subset E$ for which $\phi(u_k)$ is bounded and $\phi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

Lemma 5 (see [16]). *For $u \in E_{(k)}$, one then has $\|u\|_{C_{[-kT,kT]}} \leq M\|u\|_{E_{(k)}}$, where*

$$\begin{aligned} M = 2^{1/q} \max & \left\{ \frac{1}{(2kT)^{1/p}\left(\operatorname{ess\,inf}_{[-kT,kT]} s\right)^{1/p}}, \right. \\ & \left. \frac{(2kT)^{1/q}}{\left(\operatorname{ess\,inf}_{[-kT,kT]} \rho\right)^{1/p}} \right\}, \\ & \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \quad (20)$$

Lemma 6 (see [27]). *Let X be a nonempty set, and Φ, Ψ are two real functions on X. Assume that there are $r > 0$, $x_0, x_1 \in X$ such that*

$$\begin{aligned} \Phi(x_0) = \Psi(x_0) = 0, \qquad & \Phi(x_1) > r, \\ \sup_{x\in\Phi^{-1}(-\infty,r]} \Psi(x) < & \; r\frac{\Psi(x_1)}{\Phi(x_1)}. \end{aligned} \quad (21)$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(-\infty,r]} \Psi(x) < \rho < r\frac{\Psi(x_1)}{\Phi(x_1)}, \quad (22)$$

one has

$$\begin{aligned} & \sup_{\lambda\geq 0} \inf_{x\in X} \left(\Phi(x) + \lambda\left(\rho - \Psi(x)\right)\right) \\ & < \inf_{x\in X} \sup_{\lambda\geq 0} \left(\Phi(x) + \lambda\left(\rho - \Psi(x)\right)\right). \end{aligned} \quad (23)$$

Lemma 7 (see [17]). *Let X be a separable and reflexive real Banach space, $I \subseteq R$ an interval, and $f : X \times I \to R$ a function satisfying the following conditions:*

(i) *for each $x \in X$, the function $f(t,\cdot)$ is continuous and concave;*

(ii) *for each $\lambda \in I$, the function $f(t,\cdot)$ is sequentially weakly lower semicontinuous and Dâteaux differentiable, and $\lim_{\|x\|\to\infty} f(x,\lambda) = +\infty$;*

(iii) *there exists a continuous concave function $h : I \to R$ such that*

$$\sup_{\lambda\in I} \inf_{x\in X} \left(f(x,\lambda) + h(\lambda)\right) < \inf_{x\in X} \sup_{\lambda\in I} \left(f(x,\lambda) + h(\lambda)\right). \quad (24)$$

Then, there exist an open interval $J \subseteq I$ and a positive real number ρ, such that, for each $\lambda \in J$, the equation

$$f'_x(x,\lambda) = 0 \quad (25)$$

has at least two solutions in X whose norms are less than ρ. If, in addition, the function f is (strongly) continuous in $X \times I$, and, for each $\lambda \in I$, the function $f(t,\cdot)$ is C^1 and satisfies the PS condition, then the above conclusion holds with "three" instead of "two."

Lemma 8. *Let $u \in W^{1,p}(R, R^n)$. Then for every $t \in R$, the following inequality holds:*

$$|u(t)| \leq \left(\int_{t-1/2}^{t+1/2} |u(s)|^p ds\right)^{1/p} + \frac{1}{2}\left(\int_{t-1/2}^{t+1/2} \left|u'(s)\right|^p ds\right)^{1/p}. \tag{26}$$

Proof. Fix $t \in R$. For every $\tau \in R$,

$$|u(t)| \leq |u(\tau)| + \left|\int_{\tau}^{t} u'(s)\, ds\right|. \tag{27}$$

Integrating (27) over $[t - 1/2, t + 1/2]$ and using the Hölder inequality, we get

$$\begin{aligned} |u(t)| &\leq \int_{t-1/2}^{t+1/2} \left[|u(\tau)| + \left|\int_{\tau}^{t} u'(s)\, ds\right|\right] d\tau \\ &\leq \int_{t-1/2}^{t+1/2} |u(\tau)|\, d\tau + \int_{t-1/2}^{t} \int_{t-1/2}^{t} \left|u'(s)\right| ds\, d\tau \\ &\quad + \int_{t}^{t+1/2} \int_{t}^{t+1/2} \left|u'(s)\right| ds\, d\tau \\ &\leq \left(\int_{t-1/2}^{t+1/2} |u(s)|^p ds\right)^{1/p} \\ &\quad + \frac{1}{2}\left(\int_{t-1/2}^{t+1/2} \left|u'(s)\right|^p ds\right)^{1/p}. \end{aligned} \tag{28}$$

□

3. Main Result

In this section, our main result of this paper is presented. First, we introduce the following three conditions:

(V1) there exist constants $c_1, \delta_1, \delta_2, \eta_1 > 0$ and $\eta_2 > 0$, with $\delta_1^2 + \delta_2^2 \neq 0$, $\eta_1 + \eta_2 < \eta_1\eta_2$ and

$$0 < \frac{c_1}{M} < (K_2)^{1/p} \tag{29}$$

such that $2kT \max_{(t,x)\in[-kT,kT]\times[-c_1,c_1]} F_{(k)}(t,x) < E\Omega$, where

$$\begin{aligned} E &= \frac{(c_1/M)^p}{K_2 + K_3^p \int_{-kT}^{kT} s(t)\, dt}, \\ \Omega &= \int_{-kT}^{-kT+2kT/\eta_1} F_{(k)}(t, g_1(t))\, dt \\ &\quad + \int_{-kT+2kT/\eta_1}^{kT-2kT/\eta_2} F_{(k)}(t, g_2(t))\, dt \\ &\quad + \int_{kT-2kT/\eta_2}^{kT} F_{(k)}(t, g_3(t))\, dt, \\ K_1 &= \frac{\delta_1\eta_2 + \delta_2\eta_1}{\eta_1 + \eta_2 - \eta_1\eta_2}, \\ K_2 &= \left|\delta_1\right|^p \int_{-kT}^{-kT+2kT/\eta_1} \rho(t)\, dt \\ &\quad + \left|K_1\right|^p \int_{-kT+2kT/\eta_1}^{kT-2kT/\eta_2} \rho(t)\, dt \\ &\quad + \left|\delta_2\right|^p \int_{kT-2kT/\eta_2}^{kT} \rho(t)\, dt, \quad k \in N, \\ K_3 &= \max\left\{\frac{2kT}{\eta_1}\left|\delta_1\right|, \frac{2kT}{\eta_2}\left|\delta_2\right|\right\}, \\ g_1(t) &= \delta_1(t + kT), \\ g_3(t) &= \delta_2(t - kT), \\ g_2(t) &= K_1\left(t + kT - \frac{2kT}{\eta_1}\right), \quad k \in N; \end{aligned} \tag{30}$$

(V2) there exist constant $\mu \in [0, p)$, and functions $\tau_1(t), \tau_2(t) \in L([-kT, kT])$ with $\operatorname{ess\,inf}_{[-kT,kT]}\tau_1 > 0$ such that

$$F_{(k)}(t,x) \leq \tau_1(t)|x|^{\mu} + \tau_2(t) \quad \forall t \in [-kT, kT] \text{ and } x \in R^n; \tag{31}$$

(V3) $\rho, s \in L^{\infty}$ and $f : R \times R^n \to R^n$ are continuous functions.

Remark 9. If there exist constant $\mu \in [0, p)$ and functions $\tau_3(t) \in C([-kT, kT])$ with $\min_{[-kT,kT]}\tau_3 > 0$ such that

$$\limsup_{|x|\to\infty} \frac{F_{(k)}(t,x)}{|x|^{\mu}} < \tau_3(t) \quad \text{uniformly } \forall t \in [-kT, kT], \tag{32}$$

then (V2) holds.

In fact, (32) implies that there exists $c_2 > 0$ such that

$$F_{(k)}(t,x) \leq \tau_3(t)|x|^{\mu} \quad \forall t \in [-kT, kT],\ |x| \geq c_2, \tag{33}$$

which combining the continuity of $F_{(k)}(t,x) - \tau_3(t)|x|^{\mu}$ on $[-kT, kT] \times [-c_2, c_2]$ yields that there exists constant $c_3 > 0$ such that

$$F_{(k)}(t,x) \leq \tau_3(t)|x|^{\mu} + c_3 \quad \forall t \in [-kT, kT],\ x \in R^n. \tag{34}$$

Lemma 10. *Assume that (V1) holds; then, for each $k \in N$, there exists a continuous concave function $h_{(k)} : [0, +\infty) \to R^n$ such that*

$$\sup_{\lambda\geq 0} \inf_{u\in E_{(k)}} \left(J_{(k)}(u,\lambda) + h(\lambda)\right) < \inf_{u\in E_{(k)}} \sup_{\lambda\geq 0} \left(J_{(k)}(u,\lambda) + h(\lambda)\right). \tag{35}$$

Proof. We define

$$r = \frac{1}{p}\left(\frac{c_1}{M}\right)^p,$$

$$u_1(t) = \begin{cases} g_1(t), & t \in \left[-kT, -kT + \dfrac{2kT}{\eta_1}\right), \\ g_2(t), & t \in \left[-kT + \dfrac{2kT}{\eta_1}, kT - \dfrac{2kT}{\eta_2}\right], \\ g_3(t), & t \in \left(kT - \dfrac{2kT}{\eta_2}, kT\right]. \end{cases} \quad (36)$$

It is clear that $u_1 \in E_{(k)}$. It follows from

$$\int_{-kT}^{kT} \rho(t)\left|u_1'(t)\right|^p dt = K_2,$$
$$0 \le \int_{-kT}^{kT} s(t)\left|u_1(t)\right|^p dt \le K_3^p \int_{-kT}^{kT} s(t)\,dt \quad (37)$$

that

$$K_2 \le \|u_1\|^p \le K_2 + K_3^p \int_{-kT}^{kT} s(t)\,dt. \quad (38)$$

Let $g(x) = (1/p)x^p$, $x \ge 0$. It is clear that $g(x)$ has the following properties: (1) $g(x)$ strictly increases for $x \ge 0$ and (2) $g(x) = w$ has unique solution $Q(w)$ for each $w > 0$.

In view of (29), (38), and (1), one has

$$\frac{1}{p}\|u_1\|^p \ge \frac{1}{p}K_2 > \frac{1}{p}\left(\frac{c_1}{M}\right)^p = r > 0, \quad (39)$$

which yields that

$$\phi_1(u_1) = \frac{1}{p}\|u_1\|^p > r > 0. \quad (40)$$

It follows from Lemma 5, (1), and (2) that

$$\begin{aligned}\phi_1^{-1}(-\infty, r] &\subseteq \left\{u \in E_{(k)} : \frac{1}{p}\|u_1\|^p \le r\right\} \\ &\subseteq \left\{u \in E_{(k)} : \|u\| \le Q(r)\right\} \\ &\subseteq \left\{u \in E_{(k)} : \max_{t\in[-kT,kT]} |u(t)| \le MQ(r)\right\}, \\ &\qquad k \in N.\end{aligned} \quad (41)$$

Let $G = MQ(r)$; then G/M is a solution of $g(x) = r$. From the definition of $g(x)$ and r, we have $g(c_1/M) = r$. Thus, (2) implies $G = c_1$, which combining (41) yields that

$$\phi_1^{-1}(-\infty, r] \subseteq \left\{u \in E_{(k)} : \max_{t\in[-kT,kT]} |u(t)| \le c_1\right\}, \quad k \in N. \quad (42)$$

Therefore,

$$\begin{aligned}\sup_{u\in\phi_1^{-1}(-\infty,r]} (-\phi_2(u)) &= \sup_{u\in\phi_1^{-1}(-\infty,r]} \int_{-kT}^{kT} F_{(k)}(t, u(t))\,dt \\ &\le \sup_{|u(t)|\le c_1} \int_{-kT}^{kT} F_{(k)}(t, u(t))\,dt \\ &\le 2kT \max_{(t,x)\in[-kT,kT]\times[-c_1,c_1]} F_{(k)}(t, x), \\ &\qquad k \in N.\end{aligned} \quad (43)$$

Since $F_{(k)}(t, 0) = 0$, we obtain

$$2kT \max_{(t,x)\in[-kT,kT]\times[-c_1,c_1]} F_{(k)}(t, x) \ge 0, \quad k \in N. \quad (44)$$

It follows from $\delta_1^2 + \delta_2^2 \ne 0$ that $K_2 + K_3^p \int_{-kT}^{kT} s(t)dt > 0$, which combining $c_1, M > 0$ yields that $E > 0$. Therefore, in view of (V1) and (44), we get $\Omega > 0$. Thus, it follows from (38) and (40) that

$$\begin{aligned}\frac{r\int_{-kT}^{kT} F_{(k)}(t, u_1(t))\,dt}{\phi_1(u_1)} &\ge \frac{r\Omega}{(1/p)\|u_1\|^p} \\ &\ge \frac{\Omega(c_1/M)^p}{K_2 + K_3^p \int_{-kT}^{kT} s(t)\,dt}, \\ &\qquad k \in N.\end{aligned} \quad (45)$$

From (43), (45), and (V1), we have

$$\sup_{u\in\phi_1^{-1}(-\infty,r]} (-\phi_2(u)) < r\frac{-\phi_2(u_1)}{\phi_1(u_1)}. \quad (46)$$

It is obvious that $\phi_1(0) = -\phi_2(0) = 0$. Owing to Lemma 6, choosing $h(\lambda) = \rho\lambda$, we obtain

$$\begin{aligned}&\sup_{\lambda\ge 0} \inf_{u\in E_{(k)}} (\phi_1(u) + \lambda\phi_2(u) + h(\lambda)) \\ &\quad < \inf_{u\in E_{(k)}} \sup_{\lambda\ge 0} (\phi_1(u) + \lambda\phi_2(u) + h(\lambda)),\end{aligned} \quad (47)$$

which combining $J_{(k)}(u, \lambda) = \phi_1(u) + \lambda\phi_2(u)$ implies the conclusion. □

Lemma 11. *If (V2) holds, then for each $k \in N$, $\lim_{\|u\|\to\infty} J_{(k)}(u, \lambda) = +\infty$ and $J_{(k)}(\cdot, \lambda)$ satisfies the PS condition.*

Proof. Let $\{u_n^{(k)}\}$ be a sequence in $E_{(k)}$ such that $\lim_{n\to+\infty} J'_{(k)u}(u_n^{(k)}, \lambda) = 0$ and $J'_{(k)}(u_n^{(k)}, \lambda)$ is bounded, for each $k \in N$.

Lemma 5 implies that

$$|u(t)| \le \|u\|_{C_{[-kT,kT]}}^{\infty} \le M\|u\|_{E_{(k)}} \quad \forall t \in [-kT, kT]. \quad (48)$$

It follows from (V2) and (48) that

$$\int_{-kT}^{kT} F_{(k)}(t,u(t))\,dt \le \int_{-kT}^{kT} \tau_1(t)\,|u(t)|^{\mu} dt + \int_{-kT}^{kT} \tau_2(t)\,dt \le M^{\mu}\|u(t)\|^{\mu}\int_{-kT}^{kT}\tau_1(t)\,dt + \int_{-kT}^{kT}\tau_2(t)\,dt, \tag{49}$$

which yields that

$$J_{(k)}(u,\lambda) \ge \frac{1}{p}\|u\|^{p} - \lambda M^{\mu}\|u(t)\|^{\mu}\int_{-kT}^{kT}\tau_1(t)\,dt - \lambda\int_{-kT}^{kT}\tau_2(t)\,dt, \tag{50}$$

for each $k \in N$. Noting that $\mu \in [0,p)$, the above inequality implies that $\lim_{\|u\|\to\infty} J_{(k)}(u,\lambda) = +\infty$ and $\{u_n^{(k)}\}$ is bounded in $E_{(k)}$. Next, we will prove that $\{u_n^{(k)}\}$ converges strongly to some $u^{(k)}$ in $E_{(k)}$. The proof is similar to [22]. Since $\{u_n^{(k)}\}$ is bounded in $E_{(k)}$, there exists a subsequence of $\{u_n^{(k)}\}$ (for simplicity denoted again by $\{u_n^{(k)}\}$) such that $\{u_n^{(k)}\}$ converges weakly to some $u^{(k)}$ in $E_{(k)}$. Then $\{u_n^{(k)}\}$ converges uniformly to $u^{(k)}$ on $[-kT,kT]$ (see [28]). Therefore,

$$\int_{-kT}^{kT}\left(f_{(k)}\left(t,u_n^{(k)}(t)\right) - f_{(k)}\left(t,u^{(k)}(t)\right)\right)\left(u_n^{(k)}(t)-u^{(k)}(t)\right)dt \longrightarrow 0 \tag{51}$$

as $n \to +\infty$, for each $k \in N$. In view that $\lim_{n\to+\infty} J'_{(k)u}(u_n^{(k)},\lambda) = 0$ and $\{u_n^{(k)}\}$ converges weakly to some $u^{(k)}$, we get

$$\left\langle J'_{(k)u}\left(u_n^{(k)},\lambda\right) - J'_{(k)u}\left(u^{(k)},\lambda\right), u_n^{(k)} - u^{(k)}\right\rangle \longrightarrow 0 \tag{52}$$

as $n \to +\infty$, for each $k \in N$. Then, from (15), one has

$$\begin{aligned}
&\left\langle J'_{(k)u}\left(u_n^{(k)},\lambda\right) - J'_{(k)u}\left(u^{(k)},\lambda\right), u_n^{(k)} - u^{(k)}\right\rangle \\
&= \int_{-kT}^{kT}\rho(t)\left(\Phi_p\left(u_n'^{(k)}(t)\right) - \Phi_p\left(u'^{(k)}(t)\right)\right)\times\left(u_n'^{(k)}(t) - u'^{(k)}(t)\right)dt \\
&+ \int_{-kT}^{kT} s(t)\left(\Phi_p\left(u_n^{(k)}(t)\right) - \Phi_p\left(u^{(k)}(t)\right)\right)\times\left(u_n^{(k)}(t)-u^{(k)}(t)\right)dt \\
&- \lambda\int_{-kT}^{kT}\left(f_{(k)}\left(t,u_n^{(k)}(t)\right) - f_{(k)}\left(t,u^{(k)}(t)\right)\right)\times\left(u_n^{(k)}(t)-u^{(k)}(t)\right)dt
\end{aligned} \tag{53}$$

for each $k \in N$. By [29], for each $k \in N$, there exist $c_p, d_p > 0$ such that

$$\begin{aligned}
&\int_{-kT}^{kT}\rho(t)\left(\Phi_p\left(u_n'^{(k)}(t)\right) - \Phi_p\left(u'^{(k)}(t)\right)\right)\times\left(u_n'^{(k)}(t)-u'^{(k)}(t)\right)dt \\
&+ \int_{-kT}^{kT} s(t)\left(\Phi_p\left(u_n^{(k)}(t)\right) - \Phi_p\left(u^{(k)}(t)\right)\right)\times\left(u_n^{(k)}(t)-u^{(k)}(t)\right)dt \\
&\ge \begin{cases}
c_p\displaystyle\int_{-kT}^{kT}\left(\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{p} + s(t)\left|u_n^{(k)}(t)-u^{(k)}(t)\right|^{p}\right)dt, & \text{if } p \ge 2,\\[2ex]
d_p\displaystyle\int_{-kT}^{kT}\left(\frac{\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{2}}{\left(\left|u_n'^{(k)}(t)\right| + \left|u'^{(k)}(t)\right|\right)^{2-p}} + \frac{s(t)\left|u_n^{(k)}(t)-u^{(k)}(t)\right|^{2}}{\left(\left|u_n^{(k)}(t)\right|+\left|u^{(k)}(t)\right|\right)^{2-p}}\right)dt, & \text{if } 1<p<2.
\end{cases}
\end{aligned} \tag{54}$$

If $p \ge 2$, it follows from (51)–(54) that $\|u_n^{(k)} - u^{(k)}\| \to 0$ as $n \to +\infty$.

If $1 < p < 2$, by Holder's inequality, we obtain

$$\begin{aligned}
&\int_{-kT}^{kT}\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{p}dt \\
&\le \left(\int_{-kT}^{kT}\frac{\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{2}}{\left(\left|u_n'^{(k)}(t)\right|+\left|u'^{(k)}(t)\right|\right)^{2-p}}dt\right)^{p/2} \times\left(\int_{-kT}^{kT}\rho(t)\left|u_n'^{(k)}(t)\right| + \left|u'^{(k)}(t)\right|^{p}dt\right)^{(2-p)/2} \\
&\le 2^{(p-1)(2-p)/2}\times\left(\int_{-kT}^{kT}\frac{\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{2}}{\left(\left|u_n'^{(k)}(t)\right|+\left|u'^{(k)}(t)\right|\right)^{2-p}}dt\right)^{p/2} \times\left(\int_{-kT}^{kT}\rho(t)\left|u_n'^{(k)}(t)\right| + \left|u'^{(k)}(t)\right|^{p}dt\right)^{(2-p)/2} \\
&\le 2^{(p-1)(2-p)/2}\times\left(\int_{-kT}^{kT}\frac{\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^{2}}{\left(\left|u_n'^{(k)}(t)\right|+\left|u'^{(k)}(t)\right|\right)^{2-p}}dt\right)^{p/2} \times\left(\left\|u_n^{(k)}\right\| + \left\|u^{(k)}\right\|\right)^{(2-p)p/2}
\end{aligned} \tag{55}$$

for each $k \in N$. Similarly,

$$\begin{aligned}&\int_{-kT}^{kT} s(t)\left|u_n^{(k)}(t)-u^{(k)}(t)\right|^p dt\\&\le 2^{(p-1)(2-p)/2}\left(\int_{-kT}^{kT}\frac{s(t)\left|u_n^{(k)}(t)-u^{(k)}(t)\right|^2}{\left(\left|u_n^{(k)}(t)\right|+\left|u^{(k)}(t)\right|\right)^{2-p}}dt\right)^{p/2}\\&\quad\times\left(\left\|u_n^{(k)}\right\|+\left\|u^{(k)}\right\|\right)^{(2-p)p/2}.\end{aligned} \tag{56}$$

It follows from $1<p<2$ and (54)–(56) that

$$\begin{aligned}&\int_{-kT}^{kT}\rho(t)\left(\Phi_p\left({u'}_n^{(k)}(t)\right)-\Phi_p\left(u'^{(k)}(t)\right)\right)\\&\quad\times\left({u'}_n^{(k)}(t)-u'^{(k)}(t)\right)dt\\&\quad+\int_{-kT}^{kT}s(t)\left(\Phi_p\left(u_n^{(k)}(t)\right)-\Phi_p\left(u^{(k)}(t)\right)\right)\\&\quad\quad\times\left(u_n^{(k)}(t)-u^{(k)}(t)\right)dt\\&\ge\frac{2^{(p-1)(2-p)/2}d_p}{\left(\left\|u_n^{(k)}\right\|+\left\|u^{(k)}\right\|\right)^{2-p}}\\&\quad\times\left[\left(\int_{-kT}^{kT}\rho(t)\left|u_n'^{(k)}(t)-u'^{(k)}(t)\right|^p dt\right)^{2/p}\right.\\&\quad\left.+\left(\int_{-kT}^{kT}s(t)\left|u_n^{(k)}(t)-u^{(k)}(t)\right|^p dt\right)^{2/p}\right]\\&\ge\frac{d_p 2^{p-2}\left\|u_n^{(k)}-u^{(k)}\right\|^2}{\left(\left\|u_n^{(k)}\right\|+\left\|u^{(k)}\right\|\right)^{2-p}}.\end{aligned} \tag{57}$$

In view of (51)–(53) and (57), we have $\|u_n^{(k)}-u^{(k)}\|\to 0$ as $n\to+\infty$, for each $k\in N$.

Therefore, $\{u_n^{(k)}\}$ converges strongly to $u^{(k)}$ in $E_{(k)}$, for each $k\in N$. Thus, for each $k\in N$, $J_{(k)}(\cdot,\lambda)$ satisfies the *PS* condition. □

Lemma 12. *Assume that (V1) and (V2) hold; then there exist an open interval $\Lambda\subseteq[0,+\infty)$ and a positive real number σ, such that, for each $\lambda\in\Lambda$ and $k\in N$, (P_k) has at least three $2kT$-periodic solutions in $E_{(k)}$ whose norms are less than σ.*

Proof. Let $\{u_n^{(k)}\}$ be a weakly convergent sequence to $u^{(k)}$ in $E_{(k)}$; then $\{u_n^{(k)}\}$ converges uniformly sequence to $u^{(k)}$ on $[-kT,kT]$. The continuity and convexity of $(1/p)\|u^{(k)}\|^p$ imply that $(1/p)\|u^{(k)}\|^p$ is sequentially weakly lower continuous [28, Lemma 1.2], for each $k\in N$, which combining the continuity of $f_{(k)}$ yields that

$$\begin{aligned}&\liminf_{n\to+\infty}\left[\frac{1}{p}\left\|u_n^{(k)}\right\|^p-\lambda\int_{-kT}^{kT}F_{(k)}\left(t,u_n^{(k)}\right)dt\right]\\&\ge\frac{1}{p}\left\|u^{(k)}\right\|^p-\lambda\int_{-kT}^{kT}F_{(k)}\left(t,u^{(k)}\right)dt.\end{aligned} \tag{58}$$

Hence, $J_{(k)}(\cdot,\lambda)$ is sequentially weakly lower semi-continuous, for each $k\in N$.

It is obvious that $J_{(k)}(u,\cdot)$ is continuous and concave for each $u\in E_{(k)}$. In view of Lemmas 10 and 11, it follows from Lemma 7 that there exist an open interval $\Lambda\subseteq[0,+\infty)$ and a positive real number σ, such that, for each $\lambda\in\Lambda$ and $k\in N$, $J_{(k)}(\cdot,\lambda)$ has at least three critical points in $E_{(k)}$ whose norms are less than σ. Therefore, we can reach our conclusion by using Lemma 2. □

Lemma 13. *Assume that (V3) holds. Let $\tilde{u}_{(k)}\in E_{(k)}$ be one of the three $2kT$-periodic solutions of system (P_k) obtained by Lemma 12 for each $k\in N$. Then there exists a subsequence $\{\tilde{u}_{(k_j)}\}$ of $\{\tilde{u}_{(k)}\}_{k\in N}$ convergent to a certain $\tilde{u}_0\in C^1(R,R^n)$ in $C^1_{loc}(R,R^n)$.*

Proof. From Lemma 12, we have

$$\left\|\tilde{u}_{(k)}\right\|_{E_{(k)}}<\sigma, \tag{59}$$

which combining Lemma 5 yields that there exists a positive constant M_1 independent of k such that

$$\left\|\tilde{u}_{(k)}\right\|_{L^\infty_{2kT}}\le M_1. \tag{60}$$

Thus, we obtain that $\{\tilde{u}_{(k)}\}_{k\in N}$ is a uniformly bounded sequence. Next, we will show that $\{\tilde{u}'_{(k)}\}_{k\in N}$ and $\{\rho\Phi_p(\tilde{u}'_{(k)})'\}_{k\in N}$ are also uniformly bounded sequences. Since $\{\tilde{u}(t)_{(k)}\}$ is a $2kT$-periodic solutions of system (P_k) for every $t\in[-kT,kT)$, we have

$$\begin{aligned}&\left(\rho(t)\Phi_p\left(\tilde{u}'_{(k)}(t)\right)\right)'\\&\quad=s(t)\Phi_p\left(\tilde{u}_{(k)}(t)\right)-\lambda f_{(k)}\left(t,\tilde{u}_{(k)}(t)\right).\end{aligned} \tag{61}$$

By (60), (61), and (V3), we get

$$\begin{aligned}\left|\left(\rho(t)\Phi_p\left(\tilde{u}'_{(k)}(t)\right)\right)'\right|&\le\left|s(t)\Phi_p\left(\tilde{u}_{(k)}(t)\right)\right|\\&\quad+\lambda\left|f_{(k)}\left(t,\tilde{u}_{(k)}(t)\right)\right|\\&\le\sup_{0\le t<kT,|x|\le M_1}\left|s(t)\Phi_p(x)\right|\\&\quad+\lambda\sup_{0\le t<kT,|x|\le M_1}\left|f(t,x)\right|\\&\equiv M_2,\quad t\in[-kT,kT),\end{aligned} \tag{62}$$

which yields that

$$\left\|\left(\rho(t)\Phi_p\left(\tilde{u}_{(k)}'(t)\right)\right)'\right\|_{L^\infty_{2kT}} \le M_0, \quad k \in N. \tag{63}$$

Then, from (63), (V3), and the definition of $\Phi_p(x)$, we obtain

$$\left\|\tilde{u}_{(k)}''(t)\right\|_{L^\infty_{2kT}} \le M_3, \quad k \in N. \tag{64}$$

For $i = -k, -k+1, \ldots, k-1$, by the continuity of $\tilde{u}_{(k)}'(t)$, we can choose $t_i^{(k)} \in [iT, (i+1)T]$ such that

$$\begin{aligned}\tilde{u}_{(k)}'\left(t_i^{(k)}\right) &= \frac{1}{T}\int_{iT}^{(i+1)T} \tilde{u}_{(k)}'(s)\,ds \\ &= T^{-1}\left[u_{(k)}((i+1)T) - u_{(k)}(iT)\right];\end{aligned} \tag{65}$$

it follows that for $t \in [iT, (i+1)T]$, $i = -k, -k+1, \ldots, k-1$

$$\begin{aligned}\left|\tilde{u}_{(k)}'(t)\right| &= \left|\int_{t_i^{(k)}}^{t} \tilde{u}_{(k)}''(s)\,ds + \tilde{u}_{(k)}'\left(t_i^{(k)}\right)\right| \\ &\le \int_{iT}^{(i+1)T}\left|\tilde{u}_{(k)}''(s)\right|ds + \left|\tilde{u}_{(k)}'\left(t_i^{(k)}\right)\right| \\ &\le M_3T + T^{-1}\left|u_{(k)}((i+1)T) - u_{(k)}(iT)\right| \\ &\le M_3T + 2M_1T^{-1} \equiv M_4.\end{aligned} \tag{66}$$

Consequently,

$$\left\|\tilde{u}_{(k)}'(t)\right\|_{L^\infty_{2kT}} \le M_4, \quad k \in N. \tag{67}$$

Now we prove that the sequences $\{\tilde{u}_{(k)}\}_{k\in N}$ and $\{\tilde{u}_{(k)}'\}_{k\in N}$ are uniformly bounded and equicontinuous. In fact, for every $k \in N$ and $t_1, t_2 \in R$, we have by (67)

$$\begin{aligned}\left|\tilde{u}_{(k)}(t_1) - \tilde{u}_{(k)}(t_2)\right| &= \left|\int_{t_1}^{t_2}\tilde{u}_{(k)}'(s)\,ds\right| \\ &\le \int_{t_1}^{t_2}\left|\tilde{u}_{(k)}'(s)\right|ds \le M_4\left|t_1 - t_2\right|.\end{aligned} \tag{68}$$

Similarly, from (64), we have

$$\left|\tilde{u}_{(k)}'(t_1) - \tilde{u}_{(k)}'(t_2)\right| \le M_3\left|t_1 - t_2\right|. \tag{69}$$

Then, by application of the Arzelà-Ascoli Theorem, we obtain the existence of a subsequence $\{\tilde{u}_{(k_j)}\}$ of $\{\tilde{u}_{(k)}\}_{k\in N}$ and a function $\tilde{u}_0$ such that

$$\tilde{u}_{(k_j)} \longrightarrow \tilde{u}_0, \quad \text{as } j \longrightarrow \infty \text{ in } C^1_{\text{loc}}(R, R^n). \tag{70}$$

Thus, Lemma 13 is proved. □

Lemma 14. *Let $\tilde{u}_0 \in C^1(R, R^n)$ be determined by Lemma 13. Then $\tilde{u}_0$ is a nontrivial homoclinic solution of system (P).*

Proof. The first step is to show that $\tilde{u}_0$ is a solution of system (P). By Lemma 13, one has

$$\begin{aligned}\left(\rho(t)\Phi_p\left(\tilde{u}_{(k_j)}'(t)\right)\right)' &= s(t)\Phi_p\left(\tilde{u}_{(k_j)}(t)\right) \\ &\quad - \lambda f_{(k_j)}\left(t, \tilde{u}_{(k_j)}(t)\right),\end{aligned} \tag{71}$$

for $t \in [-k_jT, k_jT)$, $j \in N$. Take $a, b \in R$ with $a < b$. There exists $j_0 \in N$ such that for all $j > j_0$ one has

$$\begin{aligned}\left(\rho(t)\Phi_p\left(\tilde{u}_{(k_j)}'(t)\right)\right)' &= s(t)\Phi_p\left(\tilde{u}_{(k_j)}(t)\right) \\ &\quad - \lambda f\left(t, \tilde{u}_{(k_j)}(t)\right), \quad \text{for } t \in [a, b].\end{aligned} \tag{72}$$

Integrating (72) from a to $t \in [a, b]$, we obtain

$$\begin{aligned}&\rho(t)\Phi_p\left(\tilde{u}_{(k_j)}'(t)\right) - \rho(a)\Phi_p\left(\tilde{u}_{(k_j)}'(a)\right) \\ &\quad = \int_a^t\left[s(v)\Phi_p\left(\tilde{u}_{(k_j)}(v)\right) - \lambda f\left(v, \tilde{u}_{(k_j)}(v)\right)\right]dv,\end{aligned} \tag{73}$$

for $t \in [a, b]$. Since (70) shows that $\tilde{u}_{(k_j)} \to \tilde{u}_0$ uniformly on $[a, b]$ and $\tilde{u}_{(k_j)}' \to \tilde{u}_0'$ uniformly on $[a, b]$ as $j \to \infty$. Let $j \to \infty$ in (73), we get

$$\begin{aligned}&\rho(t)\Phi_p\left(\tilde{u}_0'(t)\right) - \rho(a)\Phi_p\left(\tilde{u}_0'(a)\right) \\ &\quad = \int_a^t\left[s(v)\Phi_p\left(\tilde{u}_0(v)\right) - \lambda f\left(v, \tilde{u}_0(v)\right)\right]dv,\end{aligned} \tag{74}$$

for $t \in [a, b]$. Since a and b are arbitrary, (74) yields that $\tilde{u}_0$ is a solution of system (P). It is easy to see that $u = 0$ is not a solution of system (P) for $f(t, 0) \neq 0$ and so $\tilde{u}_0 \neq 0$.

Secondly, we will prove that $\tilde{u}_0(t) \to 0$ as $t \to \pm\infty$. By (59), we have

$$\int_{-kT}^{kT}\left(\rho(t)\left|\tilde{u}_{(k)}'(t)\right|^p + s(t)\left|\tilde{u}_{(k)}(t)\right|^p\right)dt \le \sigma^p, \quad k \in N. \tag{75}$$

For every $l \in N$, there exists $j_1 \in N$ such that for $j > j_1$

$$\int_{-lT}^{lT}\left(\rho(t)\left|\tilde{u}_{(k_j)}'(t)\right|^p + s(t)\left|\tilde{u}_{(k_j)}(t)\right|^p\right)dt \le \sigma^p. \tag{76}$$

Let $j \to \infty$ in the above and use (70), and it follows that for each $l \in N$,

$$\int_{-lT}^{lT}\left(\rho(t)\left|\tilde{u}_0'(t)\right|^p + s(t)\left|\tilde{u}_0(t)\right|^p\right)dt \le \sigma^p. \tag{77}$$

Let $l \to \infty$ in the above, and we get

$$\int_{-\infty}^{\infty}\left(\rho(t)\left|\tilde{u}_0'(t)\right|^p + s(t)\left|\tilde{u}_0(t)\right|^p\right)dt \le \sigma^p. \tag{78}$$

Thus

$$\int_{|t|\ge r}\left(\rho(t)\left|\tilde{u}_0'(t)\right|^p + s(t)\left|\tilde{u}_0(t)\right|^p\right)dt \longrightarrow 0, \quad \text{as } r \longrightarrow \infty. \tag{79}$$

Combining the above with (V3) we have

$$\int_{|t|\geq r}\left(\left|\tilde{u}_0'(t)\right|^p+\left|\tilde{u}_0(t)\right|^p\right)dt\longrightarrow 0,\quad \text{as } r\longrightarrow\infty. \tag{80}$$

By (26), we obtain

$$\left|\tilde{u}_0(t)\right|\leq p^{1/p}\left(\int_{t-1/2}^{t+1/2}\left(\left|\tilde{u}_0(s)\right|^p+\left|\tilde{u}_0'(s)\right|^p\right)ds\right)^{1/p}. \tag{81}$$

Combining (80) with (81), we get $\tilde{u}_0(t)\rightarrow 0$ as $t\rightarrow\pm\infty$.

Finally, we show that

$$\tilde{u}_0'(t)\longrightarrow 0\quad \text{as } t\longrightarrow\pm\infty. \tag{82}$$

From (60) and (70), one has

$$\left|\tilde{u}_0(t)\right|\leq M_1,\quad \text{for } t\in R. \tag{83}$$

From this and (64), we have

$$\left\|\tilde{u}_0''(t)\right\|\leq M_3,\quad \text{for } t\in R. \tag{84}$$

If (82) does not hold, then there exist $\varepsilon_0\in(0,1/2)$ and a sequence $\{t_k\}$ such that

$$\begin{gathered}\left|t_1\right|<\left|t_2\right|<\left|t_3\right|<\cdots<\left|t_k\right|<\left|t_{k+1}\right|,\quad k=1,2,\ldots,\\ \left|\tilde{u}_0'(t_k)\right|\geq 2\varepsilon_0,\quad k=1,2,\ldots,\end{gathered} \tag{85}$$

which yield that for $t\in[t_k,t_k+\varepsilon_0/(1+M_3)]$

$$\begin{aligned}\left|\tilde{u}_0'(t)\right|&=\left|\tilde{u}_0'(t_k)+\int_{t_k}^{t}\tilde{u}_0''(s)\,ds\right|\\&\geq\left|\tilde{u}_0'(t_k)\right|-\int_{t_k}^{t}\left|\tilde{u}_0''(s)\right|ds\geq\varepsilon_0.\end{aligned} \tag{86}$$

It follows that

$$\int_{-\infty}^{\infty}\left|\tilde{u}_0'(t)\right|^p dt\geq\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\varepsilon_0/(1+M_3)}\left|\tilde{u}_0'(t)\right|^p dt=\infty, \tag{87}$$

which contradicts to (78) and so (82) holds. The proof is completed. □

Lemmas 13 and 14 imply that the limit of the $2kT$-periodic solutions of system (P_k) is a nontrivial homoclinic solution of system (P). Combining this with Lemma 10–Lemma 12, we can get the following.

Theorem 15. *Assume that (V1), (V2), and (V3) hold. Then system (P) possesses three nontrivial homoclinic solutions.*

4. Example

Example 1. Consider the following p-Laplacian problem:

$$\begin{gathered}\left((t+3)\,\Phi_3\left(u'(t)\right)\right)'-(2t+2)\,\Phi_3(u(t))\\+\lambda f(t,u(t))=0,\end{gathered} \tag{88}$$

where $\lambda\in[0,+\infty)$, $kT=2$, and

$$f_{(k)}(t,x)=tx+1,\quad \forall(t,x)\in[-2,2]\times(-\infty,+\infty). \tag{89}$$

It is obvious that (V3) holds and for every $t\in[-2,2]$,

$$F_{(k)}(t,x)=\frac{t}{2}x^2+x-2,\quad \forall(t,x)\in[-2,2]\times(-\infty,+\infty). \tag{90}$$

Then,

$$\lim_{x\rightarrow+\infty}\frac{F_{(k)}(t,x)}{x^2}=\frac{t}{2} \tag{91}$$

for each $t\in[-2,2]$. Thus, there exists $c_4>0$ such that

$$F_{(k)}(t,x)\leq 2|x|^2\quad \forall t\in[-2,2],\ |x|\geq c_4, \tag{92}$$

which combining the continuity of $F_{(k)}(t,x)-2|x|^2$ on $[-2,2]\times[-c_4,c_4]$ yields that there exists constant $c_5>0$ such that

$$\begin{gathered}F_{(k)}(t,x)\leq 2|x|^2+c_5\\ \text{for each }(t,x)\in[-2,2]\times[-\infty,+\infty).\end{gathered} \tag{93}$$

Therefore, (V2) is satisfied. Furthermore, in view of Lemma 5, $M=4$. Let $\eta_1=\eta_2=4$, $\delta_1=1$, $\delta_2=1$, and $c_1=(\sqrt{2}-1)/2$; then $K_1=0$, $K_2=6$, $K_3=1$, $E=1.112\times10^{-3}$, $\Omega=1/2$, and $[2-(-2)]\max_{(t,x)\in[-2,2]\times[-(\sqrt{2}-1)/2,(\sqrt{2}-1)/2]}F_{(k)}(t,x)\leq 0$. Thus (V1) is satisfied. Moreover, $f_{(k)}(t,0)=1\neq 0$. In view of Theorem 15, we have that Example 1 possesses three nontrivial homoclinic solutions.

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References

[1] Z.-Q. Ou and C.-L. Tang, "Existence of homoclinic solution for the second order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 1, pp. 203–213, 2004.

[2] L. Wan and C.-L. Tang, "Homoclinic orbits for a class of the second order Hamiltonian systems," *Acta Mathematica Scientia. Series B*, vol. 30, no. 1, pp. 312–318, 2010.

[3] Z.-H. Zhang and R. Yuan, "Homoclinic solutions for some second order non-autonomous Hamiltonian systems with the globally superquadratic condition," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 3-4, pp. 1809–1819, 2010.

[4] A. Daouas, "Homoclinic orbits for superquadratic Hamiltonian systems without a periodicity assumption," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 11, pp. 3407–3418, 2011.

[5] L. Yang, H.-B. Chen, and J.-T. Sun, "Infinitely many homoclinic solutions for some second order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 17, pp. 6459–6468, 2011.

[6] Q.-Y. Zhang and C.-G. Liu, "Infinitely many homoclinic solutions for second order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 2, pp. 894–903, 2010.

[7] X.-H. Tang and X.-Y. Lin, "Infinitely many homoclinic orbits for Hamiltonian systems with indefinite sign subquadratic potentials," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 17, pp. 6314–6325, 2011.

[8] J. Yang and F.-B. Zhang, "Infinitely many homoclinic orbits for the second-order Hamiltonian systems with super-quadratic potentials," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1417–1423, 2009.

[9] M.-H. Yang and Z.-Q. Han, "Infinitely many homoclinic solutions for second-order Hamiltonian systems with odd nonlinearities," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 74, no. 7, pp. 2635–2646, 2011.

[10] X.-H. Tang and L. Xiao, "Homoclinic solutions for a class of second-order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 3-4, pp. 1140–1152, 2009.

[11] J.-C. Wei and J. Wang, "Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials," *Journal of Mathematical Analysis and Applications*, vol. 366, no. 2, pp. 694–699, 2010.

[12] M. Izydorek and J. Janczewska, "Homoclinic solutions for a class of the second order Hamiltonian systems," *Journal of Differential Equations*, vol. 219, no. 2, pp. 375–389, 2005.

[13] X. Lv, S. Lu, and P. Yan, "Existence of homoclinic solutions for a class of second-order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 1, pp. 390–398, 2010.

[14] D. Wu, X. Wu, and C. Tang, "Homoclinic solutions for a class of nonperiodic and noneven second-order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 1, pp. 154–166, 2010.

[15] A. Daouas, "Homoclinic orbits for superquadratic Hamiltonian systems with small forcing terms," *Nonlinear Dynamics and Systems Theory*, vol. 10, no. 4, pp. 339–348, 2010.

[16] Y. Tian and W. Ge, "Applications of variational methods to boundary-value problem for impulsive differential equations," *Proceedings of the Edinburgh Mathematical Society*, vol. 51, no. 2, pp. 509–527, 2008.

[17] B. Ricceri, "On a three critical points theorem," *Archiv der Mathematik*, vol. 75, no. 3, pp. 220–226, 2000.

[18] Z. Du, X. Lin, and C.-C. Tisdell, "A multiplicity result for p-Lapacian boundary value problems via critical points theorem," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 231–237, 2008.

[19] T. He and Y. Su, "On discrete fourth-order boundary value problems with three parameters," *Journal of Computational and Applied Mathematics*, vol. 233, no. 10, pp. 2506–2520, 2010.

[20] G. Bonanno and R. Livrea, "Multiplicity theorems for the Dirichlet problem involving the p-Laplacian," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 54, no. 1, pp. 1–7, 2003.

[21] G.-A. Afrouzi and S. Heidarkhani, "Three solutions for a quasilinear boundary value problem," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 69, no. 10, pp. 3330–3336, 2008.

[22] L. Bai and B.-X. Dai, "Three solutions for a p-Laplacian boundary value problem with impulsive effects," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 9895–9904, 2011.

[23] Y. Tian and W. Ge, "Multiple positive solutions for a second order Sturm-Liouville boundary value problem with a p-Laplacian via variational methods," *The Rocky Mountain Journal of Mathematics*, vol. 39, no. 1, pp. 325–342, 2009.

[24] G. Bonanno and G. Riccobono, "Multiplicity results for Sturm-Liouville boundary value problems," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 294–297, 2009.

[25] Y. Tian and W. Ge, "Second-order Sturm-Liouville boundary value problem involving the one-dimensional p-Laplacian," *The Rocky Mountain Journal of Mathematics*, vol. 38, no. 1, pp. 309–327, 2008.

[26] B. Ricceri, "Existence of three solutions for a class of elliptic eigenvalue problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1485–1494, 2000.

[27] J. Yang, Z. Wei, and K. Liu, "Existence of symmetric positive solutions for a class of Sturm-Liouville-like boundary value problems," *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 424–432, 2009.

[28] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[29] J. Simon, "Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^n$," in *Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977)*, P. Benilan and J. Robert, Eds., vol. 665 of *Lecture Notes in Mathematics*, pp. 205–227, Springer, Berlin, Germany, 1978.