

Research Article

The Local Time of the Fractional Ornstein-Uhlenbeck Process

Guangjun Shen, Dongjin Zhu, Yong Ren, and Xueping Ding

Department of Mathematics, Anhui Normal University, 1 East Beijing Road, Wuhu 241000, China

Correspondence should be addressed to Guangjun Shen; gjshen@163.com

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We investigate the Hölder regularity of the local time of the fractional Ornstein-Uhlenbeck process $X^H = \{X_t^H, t \geq 0\}$. As a related problem, we study the collision local time of two independent fractional Ornstein-Uhlenbeck $X^{H_i} = \{X_t^{H_i}, t \geq 0\}$, $i = 1, 2$ with respective indices $H_1, H_2 \in (0, 1)$.

1. Introduction

The Brownian motion and the Ornstein-Uhlenbeck process are the two most well-studied and widely applied stochastic processes. The Einstein-Smoluchowski theory may be seen as an idealized Ornstein-Uhlenbeck theory, and predictions of either cannot be distinguished by the experiment. However, if the Brownian particle is under the influence of an external force, the Einstein-Smoluchowski theory breaks down, while the Ornstein-Uhlenbeck theory remains successful. It is well known that a diffusion process $X = (X_t)_{t \geq 0}$ starting from $x \in \mathbb{R}$ is called Ornstein-Uhlenbeck process with coefficients $\nu > 0$ if its infinitesimal generator is

$$L = \frac{1}{2} \nu^2 \frac{d^2}{dx^2} - x \frac{d}{dx}. \quad (1)$$

The Ornstein-Uhlenbeck process (see, e.g., Revuz and Yor [1]) has a remarkable history in physics. It is introduced to model the velocity of the particle diffusion process, and later it has been heavily used in finance, and thus in econophysics. It can be constructed as the unique strong solution of Itô stochastic differential equation

$$dX_t = -X_t dt + \nu dB_t, \quad X_0 = x, \quad (2)$$

where B is a standard Brownian motion starting at 0.

Recently, as an extension of Brownian motion, fractional Brownian motion has become an object of intense study, due to its interesting properties and its applications in various scientific areas including condensed matter physics, biological physics, telecommunications, turbulence, image processing, finance, and econophysics (see, e.g., Gouyet [2], Nualart [3],

Biagini et al. [4], Mishura [5], Willinger et al. [6], and references therein). Recall that fractional Brownian motion B^H with Hurst index $H \in (0, 1)$ is a central Gaussian process with $B_0^H = 0$ and the covariance function

$$E[B_t^H B_s^H] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}] \quad (3)$$

for all $t, s \geq 0$. This process was first introduced by Kolmogorov and studied by Mandelbrot and van Ness [7], where a stochastic integral representation in terms of a standard Brownian motion was established. For $H = 1/2$, B^H coincides with the standard Brownian motion B . B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, and so many of the powerful techniques from stochastic analysis are not available when dealing with B^H . It has self-similar, long-range dependence, Hölder paths, and it has stationary increments. These properties make B^H an interesting tool for many applications.

On the other hand, extensions of the classical Ornstein-Uhlenbeck process have been suggested mainly on demand of applications. The fractional Ornstein-Uhlenbeck process is an extension of the Ornstein-Uhlenbeck process, where fractional Brownian motion is used as integrator

$$dX_t = -X_t dt + \nu dB_t^H, \quad X_0 = x. \quad (4)$$

Then (4) has a unique solution $X_t^H = \{X_t^H, 0 \leq t \leq T\}$, which can be expressed as

$$X_t^H = e^{-t} \left(x + \nu \int_0^t e^s dB_s^H \right), \quad (5)$$

and the solution is called the fractional Ornstein-Uhlenbeck process. More work for the process can be found in Cheridito et al. [8], Lim and Muniandy [9], Metzler and Klafter [10], and Yan et al. [11, 12]. Clearly, when $H = 1/2$, the fractional Ornstein-Uhlenbeck process is the classical Ornstein-Uhlenbeck process X with parameter ν starting at $x \in \mathbb{R}$. An advantage of using fractional Ornstein-Uhlenbeck process is to realize stationary long range dependent processes.

The intuitive idea of local time $L(t, x)$ for a stochastic process X is that $L(t, x)$ measures the amount of time X spends at the level x during the interval $[0, t]$. Moreover, since the work of Varadhan [13], the local time of stochastic processes has become an important subject. Therefore, it seems interesting to study the local time of fractional Ornstein-Uhlenbeck process, a rather special class of Gaussian processes.

In this paper, we focus our attention on the Hölder regularity of the local time of fractional Ornstein-Uhlenbeck process.

The rest of this paper is organized as follows. Section 2 contains a brief review on the local times of Gaussian processes and the approach of chaos expansion of the Gaussian process. In Section 3, we give Hölder regularity of the local time. In Section 4, as a related problem, we study the so-called collision local time of two independent fractional Ornstein-Uhlenbeck $X^{H_i} = \{X_t^{H_i}, t \geq 0\}$, $i = 1, 2$ with respective indices $H_1, H_2 \in (0, 1)$.

2. Preliminaries

2.1. Local Times and Local Nondeterminism. We recall briefly the definition of local time. For a comprehensive survey on local times of both random and nonrandom vector fields, we refer to Alder [14], Geman and Horowitz [15], and Xiao [16–18]. Let $X(t)$ be any Borel function on \mathbb{R} with values in \mathbb{R} . For any Borel set $B \subset \mathbb{R}$, the occupation measure of X is defined by

$$\mu_B(A) = \lambda_1 \{t \in B, X(t) \in A\}, \quad (6)$$

for all Borel set $A \subseteq \mathbb{R}$, where λ_1 is the one-dimensional Lebesgue measure. If μ_B is absolutely continuous with respect to the Lebesgue measure λ_1 on \mathbb{R} , we say that $X(t)$ has a local time on B and define its local time $L(B, x)$ to be the Radon-Nikodym derivative of μ_B . If $B = [0, t]$, we simply write $L(B, x)$ as $L(t, x)$. If $I = [0, T]$ and $L(t, x)$ is continuous as a function of $(t, x) \in I \times \mathbb{R}$, then we say that X has a jointly continuous local time on I . In this latter case, the set function $L(\cdot, x)$ can be extended to be a finite Borel measure on the level set (see Adler [14, Theorem 8.6.1])

$$X_I^{-1}(x) = \{t \in I : X(t) = x\}. \quad (7)$$

This fact has been used by many authors to study fractal properties of level sets, inverse image, and multiple times of stochastic processes. For example, Xiao [16] and Hu [19] have studied the Hausdorff dimension, and exact Hausdorff and packing measure of the level sets of iterated Brownian motion, respectively.

For a fixed sample function at fixed t , the Fourier transform on x of $L(t, x)$ is the function

$$f(t, u) = \int_{\mathbb{R}} e^{iux} L(t, x) dx. \quad (8)$$

Using the density of occupation formula we have

$$f(t, u) = \int_0^t e^{iuX(s)} ds. \quad (9)$$

We can express the local times $L(t, x)$ as the inverse Fourier transform of $f(t, u)$, namely,

$$L(t, x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_0^t e^{iu(X(s)-x)} ds \right) du. \quad (10)$$

It follows from (10) that for any $x, y \in \mathbb{R}$, $t, t + \omega \in [0, T]$ and any integer $n \geq 2$, we have (see, e.g., Boufoussi et al. [20, 21])

$$\begin{aligned} & E(L(t + h, x) - L(t, x))^n \\ &= \frac{1}{(2\pi)^n} \int_{[t, t+\omega]^n} \int_{\mathbb{R}^n} e^{-ix \sum_{j=1}^n u_j} E \left(e^{i \sum_{j=1}^n u_j X(s_j)} \right) \\ & \quad \times \prod_{j=1}^n du_j \prod_{j=1}^n ds_j, \end{aligned} \quad (11)$$

and for every even integer $n \geq 2$,

$$\begin{aligned} & E(L(t + h, x) - L(t, x) - L(t + h, y) + L(t, y))^n \\ &= \frac{1}{(2\pi)^n} \int_{[t, t+\omega]^n} \int_{\mathbb{R}^n} \prod_{j=1}^n [e^{-ixu_j} - e^{-iyu_j}] \\ & \quad \times E \left(e^{i \sum_{j=1}^n u_j X(s_j)} \right) \prod_{j=1}^n du_j \prod_{j=1}^n ds_j. \end{aligned} \quad (12)$$

The concept of local nondeterminism was first introduced by Berman [22] to unify and extend his methods for studying local times of real-valued Gaussian processes. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a real-valued, separable Gaussian process with mean 0 and let $T \subset \mathbb{R}_+$ be an open interval. Assume that $E[X(t)^2] > 0$ for all $t \in T$ and there exists $\delta > 0$ such that

$$E[(X(s) - X(t))^2] > 0, \quad (13)$$

for $s, t \in T$ with $0 < |s - t| < \delta$.

Recall from Berman [22] that X is called locally nondeterministic on T if for every integer $n \geq 2$,

$$\lim_{\epsilon \rightarrow 0} \inf_{t_n - t_1 \leq \epsilon} V_n > 0, \quad (14)$$

where V_n is the relative prediction error as follows:

$$V_n = \frac{\text{Var}(X(t_n) - X(t_{n-1}) | X(t_1), \dots, X(t_{n-1}))}{\text{Var}(X(t_n) - X(t_{n-1}))}, \quad (15)$$

and the infimum in (14) is taken over all ordered points $t_1 < t_2 < \dots < t_n$ in T with $t_n - t_1 \leq \epsilon$. Roughly speaking, (14)

means that a small increment of the process X is not almost relatively predictable based on a finite number of observations from the immediate past.

It follows from Berman [22, Lemma 2.3] that (14) is equivalent to the following property which says that X has locally approximately independent increments: for any positive integer $n \geq 2$, there exist positive constants C_n and δ (both may depend on n) such that

$$\begin{aligned} & \text{Var} \left(\sum_{j=1}^n u_j [X(t_j) - X(t_{j-1})] \right) \\ & \geq C_n \sum_{j=1}^n u_j^2 \text{Var} [X(t_j) - X(t_{j-1})], \end{aligned} \quad (16)$$

for all ordered points $0 = t_0 < t_1 < t_2 < \dots < t_n$ in T with $t_n - t_1 < \delta$ and all $u_j \in \mathbb{R}$ ($1 \leq j \leq n$). We refer to Nolan [23, Theorem 2.6] for a proof of the above equivalence in a much more general setting.

For simplicity throughout this paper we let C_n stand for a positive constant depending only on the subscripts and its value may be different in different appearances, and this assumption is also adaptable to C, C_H .

2.2. Chaos Expansion. Let Ω be the space of continuous \mathbb{R}^1 -valued functions ω on $[0, T]$. Then Ω is a Banach space with respect to the supreme norm. Let \mathcal{F} be the σ -algebra on Ω . Let P be the probability measure on the measurable space (Ω, \mathcal{F}) . Let \mathbb{E} denote the expectation on this probability space. The set of all square integrable functionals is denoted by $L^2(\Omega, P)$, that is,

$$E(F^2) = \int_{\Omega} |F(\omega)|^2 P(d\omega) < \infty. \quad (17)$$

We can introduce the chaos expansion, which is an orthogonal decomposition of $L^2(\Omega, P)$. We refer to Hu [24], Nualart [3], and the references therein for more details. Let $X := \{X_t, t \in [0, T]\}$ be a Gaussian process defined on the probability space (Ω, \mathcal{F}, P) . If $p_n(x)$ is a polynomial of degree n in x , then we call $p_n(X_t)$ a polynomial function of X with $t \in [0, T]$. Let \mathcal{P}_n be the completion with respect to the $L^2(\Omega, P)$ norm of the set $\{p_m(X_t) : 0 \leq m \leq n, t \in [0, T]\}$. Clearly, \mathcal{P}_n is a subspace of $L^2(\Omega, P)$. If \mathcal{C}_n denotes the orthogonal complement of \mathcal{P}_{n-1} in \mathcal{P}_n , then $L^2(\Omega, P)$ is actually the direct sum of \mathcal{C}_n , that is,

$$L^2(\Omega, P) = \bigoplus_{n=0}^{\infty} \mathcal{C}_n. \quad (18)$$

Namely, for any functional $F \in L^2(\Omega, P)$, there are F_n in \mathcal{C}_n , $n = 0, 1, 2, \dots$, such that

$$F = \sum_{n=0}^{\infty} F_n. \quad (19)$$

The decomposition equation (19) is called the *chaos expansion* of F , and F_n is called the n th chaos of F . Clearly, we have

$$E(|F|^2) = \sum_{n=0}^{\infty} E(|F_n|^2). \quad (20)$$

Recall that Meyer-Watanabe test function space \mathcal{U} (see Watanabe [25]) is defined as

$$\mathcal{U} := \left\{ F \in L^2(\Omega, P) : F = \sum_{n=0}^{\infty} F_n, \sum_{n=0}^{\infty} n E(|F_n|^2) < \infty \right\}, \quad (21)$$

and $F \in L^2(\Omega, P)$ is said to be smooth if $F \in \mathcal{U}$.

Now, for $F \in L^2(\Omega, P)$, we define an operator Γ_u with $u \in [0, 1]$ by

$$\Gamma_u F := \sum_{n=0}^{\infty} u^n F_n. \quad (22)$$

Set $\Theta(u) := \Gamma_{\sqrt{u}} F$. Then $\Theta(1) = F$. Define $\Phi_{\Theta}(u) := (d/du)(\|\Theta(u)\|^2)$, where $\|F\|^2 := E(|F|^2)$ for $F \in L^2(\Omega, P)$. We have

$$\Phi_{\Theta}(u) = \sum_{n=1}^{\infty} n u^{n-1} E(|F_n|^2). \quad (23)$$

Note that $\|\Theta(u)\|^2 = E(|\Theta(u)|^2) = \sum_{n=0}^{\infty} E(u^n |F_n|^2)$.

Proposition 1. Let $F \in L^2(\Omega, P)$. Then $F \in \mathcal{U}$ if and only if $\Phi_{\Theta}(1) < \infty$.

Consider two independent fractional Ornstein-Uhlenbeck $X^{H_i} = \{X_t^{H_i}, t \geq 0\}$, $i = 1, 2$, with respective indices $H_i \in (0, 1)$. Let $H_n(x)$ and $x \in \mathbb{R}$ be the Hermite polynomials of degree n . That is,

$$H_n(x) = (-1)^n \frac{1}{n!} e^{x^2/2} \frac{\partial^n}{\partial x^n} e^{-x^2/2}. \quad (24)$$

Then,

$$e^{tx-t^2/2} = \sum_{n=0}^{\infty} t^n H_n(x), \quad (25)$$

for all $t \in \mathcal{C}$ and $x \in \mathbb{R}$, this implies that

$$\begin{aligned} & \exp \left(iu \xi (X_t^{H_1} - X_t^{H_2}) + \frac{1}{2} u^2 \xi^2 \text{Var} (X_t^{H_1} - X_t^{H_2}) \right) \\ & = \sum_{n=0}^{\infty} (iu)^n \sigma^n(t, \xi) H_n \left(\frac{\xi (X_t^{H_1} - X_t^{H_2})}{\sigma(t, \xi)} \right), \end{aligned} \quad (26)$$

where $i = \sqrt{-1}$ and $\sigma(t, \xi) = \sqrt{\text{Var}(X_t^{H_1} - X_t^{H_2}) \xi^2}$ for $\xi \in \mathbb{R}$. Because of the orthogonality of $\{H_n(x), x \in \mathbb{R}\}_{n \in \mathbb{Z}_+}$, we will get from (19) that

$$(iu)^n \sigma^n(t, \xi) H_n \left(\frac{\xi (X_t^{H_1} - X_t^{H_2})}{\sigma(t, \xi)} \right) \quad (27)$$

is the n th chaos of $\exp(iu \xi (X_t^{H_1} - X_t^{H_2}) + (1/2) u^2 \xi^2 \text{Var}(X_t^{H_1} - X_t^{H_2}))$ for all $t \geq 0$.

3. Local Time of Fractional Ornstein-Uhlenbeck Process

In this section, we offer the Hölder regularity of the local time of fractional Ornstein-Uhlenbeck process.

Theorem 2. Let $\{X_t^H, t \geq 0\}$ be the fractional Ornstein-Uhlenbeck process. Then, for every $t \in \mathbb{R}^+$ and any $x \in \mathbb{R}$, there exist positive and finite constants C_1 and C_2 such that

$$\limsup_{h \rightarrow 0} \sup_x \frac{L(t+h, x) - L(t, x)}{h^{1-H}(\log \log(h^{-1}))^H} \leq C_1 \quad a.s. \quad (28)$$

$$\limsup_{h \rightarrow 0} \sup_{x,t} \frac{L(t+h, x) - L(t, x)}{h^{1-H}(\log(h^{-1}))^H} \leq C_2 \quad a.s. \quad (29)$$

Proof. Let $t \geq 0$ be a fixed point. Following the Fourier analytic approach of Berman [26], we have

$$\begin{aligned} & E[L(t+h, x) - L(t, x)]^n \\ &= \frac{1}{(2\pi)^n} \\ & \times \int_{[t, t+h]^n} \int_{\mathbb{R}^n} E \left(\exp \left(i \sum_{j=1}^n u_j (X_{s_j}^H - X_t^H) \right) \right) \\ & \times \prod_{j=1}^n du_j \prod_{j=1}^n ds_j. \end{aligned} \quad (30)$$

Let $\Delta X_s^H = X_s^H - X_t^H, s \geq 0$, and denote by $R(s_1, s_2, \dots, s_n)$ the covariance matrix of $(\Delta X_{s_1}^H, \dots, \Delta X_{s_n}^H)$ for different s_1, \dots, s_n , then we have

$$\begin{aligned} & \det R(s_1, s_2, \dots, s_n) \\ &= \text{Var}(\Delta X_{s_1}^H) \text{Var}(\Delta X_{s_2}^H | \Delta X_{s_1}^H) \cdots \\ & \text{Var}(\Delta X_{s_n}^H | \Delta X_{s_1}^H, \dots, \Delta X_{s_{n-1}}^H). \end{aligned} \quad (31)$$

By Yan et al. [11], one can write the fractional Ornstein-Uhlenbeck process starting from zero as

$$X_t^H = \nu \int_0^t F(t, u) dB_u, \quad 0 \leq t \leq T, \quad (32)$$

where B is a standard Brownian motion with $B_0 = 0$, and for $0 < u < t$

$$\begin{aligned} F(t, u) &= \left(H - \frac{1}{2} \right) \kappa_H e^{-t} u^{1/2-H} \\ & \times \int_u^t s^{H-1/2} (s-u)^{H-3/2} e^s ds, \end{aligned} \quad (33)$$

with $1/2 < H < 1$, $\kappa_H = (2H\Gamma((3/2) - H)/\Gamma(H + (1/2))\Gamma(2-2H))^{1/2}$, and

$$\begin{aligned} F(t, u) &= \kappa_H u^{1/2-H} \\ & \times \left(-e^{-t} \int_u^t (s-u)^{H-1/2} s^{H-1/2} e^s ds \right. \\ & \quad \left. + t^{H-1/2} (t-u)^{H-1/2} + \frac{2}{1-2H} e^{-t} \right. \\ & \quad \left. \times \int_u^t (s-u)^{H-1/2} s^{H-3/2} e^s ds \right), \end{aligned} \quad (34)$$

with $0 < H < 1/2$.

For any $r, s \in [t, t+h]$ such that $r < s$, we have

$$\begin{aligned} \text{Var}(\Delta X_s^H | \Delta X_u^H, u \leq r) &\geq \text{Var}(\Delta X_s^H - \Delta X_r^H | B_u, u \leq r) \\ &= \text{Var}(X_s^H - X_r^H | B_u, u \leq r) \\ &= \text{Var}(X_s^H | B_u, u \leq r), \end{aligned} \quad (35)$$

where the last equality follows from the fact that X_r^H is measurable with respect to $\sigma(B_u, u \leq r)$. Moreover, we can write

$$X_s^H = \nu \int_0^s F(s, u) dB_u = \nu \int_0^r F(s, u) dB_u + \nu \int_r^s F(s, u) dB_u. \quad (36)$$

Hence, by using the measurability of $\int_0^r F(s, u) dB_u$ with respect to $\sigma(B_u, u \leq s)$, we have

$$\begin{aligned} \text{Var}(X_s^H | B_u, u \leq r) &= \text{Var}\left(\nu \int_r^s F(s, u) dB_u | B_u, u \leq r\right) \\ &= \text{Var}\left(\nu \int_r^s F(s, u) dB_u\right) \\ &\geq C_H (s-r)^{2H}, \end{aligned} \quad (37)$$

where, to obtain the second equality, we have used the fact that $\int_r^s F(s, u) dB_u$ is independent of $\sigma(B_u, u \leq s)$ (by the independence of the increments of the Brownian motion). Combining (31), inequation (35), and inequation (37), we have

$$\det R(s_1, s_2, \dots, s_n) \geq C_H \prod_{j=1}^n (s_j - s_{j-1})^{2H} > 0, \quad (38)$$

where $s_0 = 0$. Hence, the change of variable $V = R^{1/2}U$, $U = (u_1, \dots, u_n)$ implies that

$$\begin{aligned} & \int_{\mathbb{R}^n} E \left(\exp \left(i \sum_{j=1}^n u_j (X_{s_j}^H - X_t^H) \right) \right) \prod_{j=1}^n du_j \\ &= \int_{\mathbb{R}^n} E \left(\exp \left(i \sum_{j=1}^n u_j \Delta X_{s_j}^H \right) \right) \prod_{j=1}^n du_j \quad (39) \\ &= \frac{(2\pi)^{n/2}}{(\det R(s_1, s_2, \dots, s_n))^{1/2}}. \end{aligned}$$

Hence,

$$\begin{aligned} & E[L(t+h, x) - L(t, x)]^n \\ &= \frac{n!}{(2\pi)^{n/2}} \\ & \times \int_{t < s_1 < \dots < s_n < t+h} \frac{1}{(\det R(s_1, s_2, \dots, s_n))^{1/2}} ds_1 \cdots ds_n \\ &\leq C_H \frac{n!}{(2\pi)^{n/2}} \int_{t < s_1 < \dots < s_n < t+h} \prod_{j=1}^n \frac{1}{(s_j - s_{j-1})^H} ds_1 \cdots ds_n \\ &\leq C_H \frac{n!}{(2\pi)^{n/2}} h^{n(1-H)} \frac{(\Gamma(1-H))^n}{\Gamma(1+n(1-H))}. \quad (40) \end{aligned}$$

Following from Stirling's formula, we have $n!/\Gamma(1+n(1-H)) \leq A^n n^H$, $n \geq 2$, for a suitable finite number A . So

$$E \left(\frac{L(t+h, x) - L(t, x)}{h^{1-H}} \right)^n \leq C^n n^H. \quad (41)$$

Following, we first prove that for any $K > 0$, there exists a positive and finite constant $B > 0$, depending on t , such that for sufficiently small u

$$P \left(L(t+h, x) - L(t, x) \geq \frac{Bh^{1-H}}{u^H} \right) \leq e^{-(K/u)}. \quad (42)$$

First consider u of the form $u = 1/n$. By Chebyshev's inequality and inequation (41), we have

$$\begin{aligned} & P(L(t+h, x) - L(t, x) \geq Bh^{1-H}n^H) \\ &\leq E \left(\frac{L(t+h, x) - L(t, x)}{Bh^{1-H}n^H} \right)^n \\ &\leq \frac{C^n}{B^n} \left(\frac{1}{n} \right)^{nH} n!^H \text{ (by Stirling's formula)} \quad (43) \\ &\leq \frac{C^n}{B^n} (2\pi n)^{H/2} e^{-Hn} \\ &= \exp \left(n \left(\log \left(\frac{C}{B} \right) - H \right) + \frac{H}{2} (\log n + \log 2\pi) \right). \end{aligned}$$

Choose $B > C$ and n_0 large such that for any $n \geq n_0$, to dominate (43) by e^{-2Kn} . Moreover, for u sufficiently small, there exists $n \geq n_0$ such that $u_{n+1} < u < u_n$ and since $n \geq 1$, $n/(m+1) \geq 1/2$. This proves inequation (42).

On the other hand, if we take $u(h) = 1/\log \log(1/h)$ and consider h_n of the form 2^{-n} , then inequation (42) implies

$$\begin{aligned} & P \left(L(t+h_n, x) - L(t, x) \geq Bh_n^{1-H} \left(\log \log \left(\frac{1}{h_n} \right) \right)^H \right) \\ &\leq n^{-2}, \quad (44) \end{aligned}$$

for large n . So, following that Borel-Cantelli lemma and monotonicity arguments, we have

$$\frac{L(t+h, x) - L(t, x)}{h^{1-H}} \leq B \left(\log \log \left(\frac{1}{h} \right) \right)^H \text{ a.s.} \quad (45)$$

This completes the proof of inequation (28). we can obtain inequation (29) in the similar manner. \square

4. Existence and Smoothness of Collision Local Time

In this section we will study the so-called collision local time of two independent fractional Ornstein-Uhlenbeck $X^{H_i} = \{X_t^{H_i}, t \geq 0\}$, $i = 1, 2$. It is defined formally by the following expression:

$$\ell_T = \int_0^T \delta_0(X_t^{H_1} - X_t^{H_2}) dt, \quad (46)$$

where δ_0 is the Dirac delta function. It is a measure of the amount of time for which the trajectories of the two processes, $X_t^{H_1}$ and $X_t^{H_2}$, collide on the time interval $[0, T]$. The collision local time for fractional Brownian motion has been studied by Jiang and Wang [27]. We shall show that the random variable ℓ_T exists in L^2 . We approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/2\varepsilon} \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\varepsilon(\xi^2/2)} d\xi. \quad (47)$$

For $\varepsilon > 0$ we define

$$\begin{aligned} \ell_{\varepsilon, T} &= \int_0^T p_\varepsilon(X_t^{H_1} - X_t^{H_2}) dt \\ &= \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} e^{i\xi(X_t^{H_1} - X_t^{H_2})} \cdot e^{-\varepsilon(\xi^2/2)} d\xi dt, \quad (48) \end{aligned}$$

and a natural question to study is that of the behavior of $\ell_{\varepsilon, T}$ as ε tends to zero.

Theorem 3. For $H_i \in (0, 1)$, $i = 1, 2$. Then $\ell_{\varepsilon, T}$ converges in $L^2(\Omega, \mathcal{F}, P)$, as $\varepsilon \downarrow 0$. Moreover, the limit is denoted by ℓ_T , then $\ell_T \in L^2(\Omega, \mathcal{F}, P)$.

Proof. First we claim that $\ell_{\varepsilon,T} \in L^2(\Omega, \mathcal{F}, P)$ for every $\varepsilon > 0$. By (48) we have

$$\begin{aligned} E(\ell_{\varepsilon,T}^2) &= \frac{1}{4\pi^2} \iint_0^T \int_{\mathbb{R}^2} E e^{i\xi(X_t^{H_1} - X_t^{H_2}) + i\eta(X_s^{H_1} - X_s^{H_2})} \\ &\quad \times e^{-((\varepsilon(\xi^2 + \eta^2))/2)} d\xi d\eta ds dt \\ &= \frac{1}{4\pi^2} \iint_0^T \int_{\mathbb{R}^2} e^{-(1/2)\sigma^2} e^{-((\varepsilon(\xi^2 + \eta^2))/2)} d\xi d\eta ds dt, \end{aligned} \quad (49)$$

where σ^2 denotes the variance of random variable $\xi(X_t^{H_1} - X_t^{H_2}) + \eta(X_s^{H_1} - X_s^{H_2})$, that is,

$$\sigma^2 := \text{Var}(\xi(X_t^{H_1} - X_t^{H_2}) + \eta(X_s^{H_1} - X_s^{H_2})). \quad (50)$$

According to the property of local nondeterminism (see Theorem 3.1 in [11]), we have

$$\begin{aligned} \sigma^2 &= \text{Var}(\xi(X_t^{H_1} - X_s^{H_1}) - \xi(X_t^{H_2} - X_s^{H_2}) \\ &\quad + (\xi + \eta)(X_s^{H_1} - X_s^{H_2})) \\ &\geq C[\xi^2((t-s)^{2H_1} + (t-s)^{2H_2}) \\ &\quad + (\xi + \eta)^2(s^{2H_1} + s^{2H_2})]. \end{aligned} \quad (51)$$

Thus, we have

$$\begin{aligned} &\frac{1}{4\pi^2} \iint_0^T \int_{\mathbb{R}^2} e^{-(1/2)\sigma^2} e^{-((\varepsilon(\xi^2 + \eta^2))/2)} d\xi d\eta ds dt \\ &\leq \int_0^T \int_0^t \int_{\mathbb{R}^2} e^{-(C/2)[\xi^2((t-s)^{2H_1} + (t-s)^{2H_2}) + (\xi + \eta)^2(s^{2H_1} + s^{2H_2})]} d\xi d\eta ds dt \\ &= C \int_0^T \int_0^t [(t-s)^{2H_1} + (t-s)^{2H_2}] \\ &\quad \times (s^{2H_1} + s^{2H_2})^{-1/2} ds dt \\ &\leq C \int_0^T \int_0^t (t-s)^{-(1/2)(H_1+H_2)} s^{-(1/2)(H_1+H_2)} ds dt < \infty, \end{aligned} \quad (52)$$

because of $H_i \in (0, 1)$, which yields

$$E(\ell_{\varepsilon,T}^2) < \infty, \quad (53)$$

for all $\varepsilon \in (0, 1]$.

Second, we claim that the sequence $\{\ell_{\varepsilon,T}, \varepsilon > 0\}$ is of Cauchy in $L^2(\Omega, \mathcal{F}, P)$. For any $\theta, \varepsilon > 0$ we have

$$\begin{aligned} &E(|\ell_{\varepsilon,T} - \ell_{\theta,T}|^2) \\ &= \frac{1}{4\pi^2} \iint_0^T \int_{\mathbb{R}^2} E e^{i\xi(X_t^{H_1} - X_t^{H_2}) + i\eta(X_s^{H_1} - X_s^{H_2})} \\ &\quad \cdot (e^{-(\varepsilon/2)\xi^2} - e^{-(\theta/2)\xi^2}) \\ &\quad \times (e^{-(\varepsilon/2)\eta^2} - e^{-(\theta/2)\eta^2}) d\xi d\eta ds dt \\ &\leq \frac{1}{4\pi^2} \sup_{\xi \in \mathbb{R}} (1 - e^{-((\varepsilon - \theta)|\xi|^2)/2})^2 \\ &\quad \times \iint_0^T \int_{\mathbb{R}^2} e^{-(1/2)\sigma^2} d\xi d\eta ds dt. \end{aligned} \quad (54)$$

Thus, dominated convergence theorem yields

$$E(|\ell_{\varepsilon,T} - \ell_{\theta,T}|^2) \longrightarrow 0, \quad (55)$$

as $\varepsilon \rightarrow 0$ and $\theta \rightarrow 0$, which leads to $\ell_{\varepsilon,T}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, P)$. Consequently, $\lim_{\varepsilon \rightarrow 0} \ell_{\varepsilon,T}$ exists in $L^2(\Omega, \mathcal{F}, P)$. This completes the proof. \square

For the increments of collision local time we have the following.

Theorem 4. Let $H_1, H_2 \in (0, 1)$ and $\beta = \min\{H_1, H_2\}$. Then the collision local time ℓ_T satisfies the following estimate:

$$E(|\ell_t - \ell_s|^2) \leq C_{H_1, H_2} (t-s)^{2-2\beta}, \quad (56)$$

for all $s, t, s < t$.

Proof. For any $0 \leq r, l \leq T$ we denote

$$\sigma_{r,l}^2 := \text{Var}(\xi(X_r^{H_1} - X_r^{H_2}) + \eta(X_l^{H_1} - X_l^{H_2})). \quad (57)$$

Then the property of local nondeterminism (see Theorem 3.1 in [11]) yields

$$\begin{aligned} \sigma_{r,l}^2 &\geq C[\xi^2((r-l)^{2H_1} + (r-l)^{2H_2}) \\ &\quad + (\xi + \eta)^2(l^{2H_1} + l^{2H_2})] \end{aligned} \quad (58)$$

for a constant $C > 0$. It follows from (48) that for $0 \leq s \leq t \leq T$

$$\begin{aligned} &E(|\ell_{\varepsilon,t} - \ell_{\varepsilon,s}|^2) \\ &= \frac{2}{(2\pi)^2} \int_s^t \int_s^r dr dl \int_{\mathbb{R}^2} e^{-(1/2)\sigma_{r,l}^2} e^{-(\varepsilon/2)(\xi^2 + \eta^2)} d\xi d\eta \\ &\leq C \int_s^t dr \int_s^r (r-l)^{-\beta} l^{-\beta} dl \\ &\leq C(t-s)^{2-2\beta}. \end{aligned} \quad (59)$$

Thus, Theorem 3 and Fatou's lemma yield

$$\begin{aligned} E(|\ell_t - \ell_s|^2) &= E\left(\lim_{\varepsilon \rightarrow 0} |\ell_{\varepsilon,t} - \ell_{\varepsilon,s}|^2\right) \\ &\leq \liminf_{\varepsilon \rightarrow 0} E(|\ell_{\varepsilon,t} - \ell_{\varepsilon,s}|^2) \leq C(t-s)^{2-2\beta}. \end{aligned} \quad (60)$$

This completes the proof. \square

Let $\lambda_t = \text{Var}(X_t^{H_1} - X_t^{H_2})$ for $t \geq 0$ and

$$\rho_{s,t} = E\left[(X_t^{H_1} - X_t^{H_2})(X_s^{H_1} - X_s^{H_2})\right], \quad (61)$$

for $s, t \geq 0$.

Lemma 5 (An and Yan [28]). *For any $x \in [-1, 1]$ we have*

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n-2)!!} x^n = x(1-x)^{-3/2}, \quad (62)$$

where $(2n-2)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ and $(2n-1)!! = 2 \cdot 4 \cdot 6 \cdots (2n-2)$.

By Cauchy-Schwartz's inequality, we have $\rho_{s,t}^2 \leq \lambda_s \lambda_t$. Hence,

$$\begin{aligned} \frac{\rho_{s,t}^2}{(\lambda_s \lambda_t - \rho_{s,t}^2)^{3/2}} &= \frac{\rho_{s,t}^2}{\lambda_s \lambda_t} \left(1 - \frac{\rho_{s,t}^2}{\lambda_s \lambda_t}\right)^{-3/2} \left(\frac{1}{\lambda_s \lambda_t}\right)^{1/2} \\ &= \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n-2)!!} \left(\frac{\rho_{s,t}^2}{\lambda_s \lambda_t}\right)^n \left(\frac{1}{\lambda_s \lambda_t}\right)^{1/2}, \end{aligned} \quad (63)$$

for all $t, s \geq 0$ and $s \neq t$.

Below, we consider the smoothness of the collision local time. Our main object is to explain and prove the following theorem.

Theorem 6. *Let $\ell_T, T \geq 0$ be the collision local time process of two independent fractional Ornstein-Uhlenbeck $X^{H_i} = \{X_t^{H_i}, t \geq 0\}$, $i = 1, 2$, with respective indices $H_i \in (0, 1)$. Then ℓ_T is smooth in the sense of the Meyer-Watanabe if and only if*

$$\min\{H_1, H_2\} < \frac{1}{3}. \quad (64)$$

Proof. By Yan et al. [11], we have

$$\begin{aligned} \lambda_t \lambda_s - \rho_{s,t}^2 &\asymp (s^{2H_1} + s^{2H_2})(t^{2H_1} + t^{2H_2}) \\ &\quad - \frac{1}{2}(t^{2H_1} + s^{2H_1} - |t-s|^{2H_1} \\ &\quad + t^{2H_2} + s^{2H_2} - |t-s|^{2H_2}) \\ &\asymp (s^{2H_1} + s^{2H_2})[(t-s)^{2H_1} + (t-s)^{2H_2}], \end{aligned} \quad (65)$$

where the notation $F \asymp G$ means that there are positive constants c_1 and c_2 so that

$$c_1 G(x) \leq F(x) \leq c_2 G(x), \quad (66)$$

in the common domain of definition for F and G . Hence, following Theorem 2 in An and Yan [28], we have $\iint_0^T (\rho_{s,t}^2 / (\lambda_s \lambda_t - \rho_{s,t}^2)^{3/2}) ds dt < \infty$ if and only if $\min\{H_1, H_2\} < 1/3$. Therefore, in order to prove Theorem 6, it only needs to prove: for $T \geq 0$, ℓ_T is smooth in the sense of the Meyer-Watanabe if and only if

$$\iint_0^T \rho_{s,t}^2 (\lambda_t \lambda_s - \rho_{s,t}^2)^{-3/2} ds dt < \infty. \quad (67)$$

In fact, for $\varepsilon > 0, T \geq 0$ we denote

$$\Theta_\varepsilon(u, T, \ell_{\varepsilon,T}) := E(|\Gamma_{\sqrt{u}} \ell_{\varepsilon,T}|^2), \quad (68)$$

and $\Theta(u, T, \ell_T) := E(|\Gamma_{\sqrt{u}} \ell_T|^2)$. Thus, by Proposition 1 to prove that (67) holds if and only if $\Phi_\Theta(1) < \infty$. Clearly, we have

$$\begin{aligned} \ell_{\varepsilon,T} &= \int_0^T p_\varepsilon(X_t^{H_1} - X_t^{H_2}) dt \\ &= \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} e^{i\xi(X_t^{H_1} - X_t^{H_2})} \cdot e^{-(1/2)\varepsilon\xi^2} d\xi dt \\ &= \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} e^{-(1/2)\xi^2 \text{Var}(X_t^{H_1} - X_t^{H_2})} \\ &\quad \times e^{-(1/2)\varepsilon\xi^2} \sum_{n=0}^{\infty} i^n \sigma^n(t, \xi) H_n \\ &\quad \times \left(\frac{\xi(X_t^{H_1} - X_t^{H_2})}{\sigma(t, \xi)} \right) d\xi dt \\ &\equiv \sum_{n=0}^{\infty} F_n. \end{aligned} \quad (69)$$

Notice that

$$\begin{aligned} \Phi_{\Theta_\varepsilon}(1) &= \sum_{n=0}^{\infty} n E(|F_n|^2) \\ &= \sum_{n=0}^{\infty} \frac{n}{4\pi^2} \\ &\quad \times E\left[\left(\iint_0^T \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\varepsilon(|\xi|^2 + |\eta|^2)\right) \right. \right. \\ &\quad \times \sigma^n(t, \xi) \sigma^n(s, \eta) \\ &\quad \cdot \exp\left(-\frac{1}{2}(\xi^2 \text{Var}(X_t^{H_1} - X_t^{H_2}) \right. \\ &\quad \left. \left. + \eta^2 \text{Var}(X_s^{H_1} - X_s^{H_2}))\right) \right) \end{aligned}$$

$$\begin{aligned}
& \cdot H_n \left(\frac{\xi (X_t^{H_1} - X_t^{H_2})}{\sigma(t, \xi)} \right) \\
& \times H_n \left(\frac{\eta (X_s^{H_1} - X_s^{H_2})}{\sigma(s, \eta)} \right) d\xi d\eta ds dt \Big] \\
& = \sum_{n=1}^{\infty} \frac{1}{4\pi^2 (2n-1)!} \\
& \times \left[\int_0^T \int_{\mathbb{R}^2} (\xi \eta)^{2n} \right. \\
& \times \left[E \left((X_t^{H_1} - X_t^{H_2}) (X_s^{H_1} - X_s^{H_2}) \right) \right]^{2n} \\
& \cdot \exp \left(- \left((\xi^2 \text{Var}((X_t^{H_1} - X_t^{H_2}) + \varepsilon) \right. \right. \\
& \quad \left. \left. + \eta^2 \text{Var}((X_s^{H_1} - X_s^{H_2}) \right. \right. \\
& \quad \left. \left. + \varepsilon) \right) \times \frac{1}{2} \right) \Big] d\xi d\eta ds dt \Big], \quad (70)
\end{aligned}$$

for all $T \geq 0$, where we have used the following fact: For two random variables X, Y with joint Gaussian distribution such that $E(X) = E(Y) = 0$ and $E(X^2) = E(Y^2) = 1$ we have (see, for example, Nualart [3])

$$E(H_n(X) H_m(Y)) = \begin{cases} 0, & m \neq n, \\ \frac{1}{n!} [E(XY)]^n, & m = n. \end{cases} \quad (71)$$

We obtain

$$\begin{aligned}
\Phi_{\Theta_\varepsilon}(1) &= \sum_{n=1}^{\infty} \frac{(\Gamma(n+1/2))^2 2^{2n+1}}{4\pi^2 (2n-1)!} \\
&\times \int_0^T \int_0^T \frac{\rho_{s,t}^{2n}}{((\lambda_s + \varepsilon)(\lambda_t + \varepsilon))^{n+(1/2)}} ds dt \\
&= \sum_{n=1}^{\infty} \frac{1}{2\pi} \frac{(2n-1)!!}{(2n-2)!!} \\
&\times \int_0^T \int_0^T \frac{\rho_{s,t}^{2n}}{((\lambda_s + \varepsilon)(\lambda_t + \varepsilon))^{n+(1/2)}} ds dt \\
&= \frac{1}{2\pi} \int_0^T \int_0^T \frac{\rho_{s,t}^2}{((\lambda_s + \varepsilon)(\lambda_t + \varepsilon) - \rho_{s,t}^2)^{3/2}} ds dt, \quad (72)
\end{aligned}$$

where we have used the following equality:

$$\begin{aligned}
& \int_{\mathbb{R}} \xi^{2n} \exp \left(- \frac{\xi^2 (\text{Var}(X_t^{H_1} - X_t^{H_2}) + \varepsilon)}{2} \right) d\xi \\
&= 2^{n+(1/2)} \Gamma \left(n + \frac{1}{2} \right) (\text{Var}(X_t^{H_1} - X_t^{H_2}) + \varepsilon)^{-(n+(1/2))}. \quad (73)
\end{aligned}$$

Hence, we have

$$\lim_{\varepsilon \rightarrow 0} \Phi_{\Theta_\varepsilon}(1) = \frac{1}{2\pi} \int_0^T \int_0^T \frac{\rho_{s,t}^2}{(\lambda_s \lambda_t - \rho_{s,t}^2)^{3/2}} ds dt, \quad (74)$$

for all $T \geq 0$. This completes the proof. \square

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