

Research Article

Stability in a Simple Food Chain System with Michaelis-Menten Functional Response and Nonlocal Delays

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This paper is concerned with the asymptotical behavior of solutions to the reaction-diffusion system under homogeneous Neumann boundary condition. By taking food ingestion and species' moving into account, the model is further coupled with Michaelis-Menten type functional response and nonlocal delay. Sufficient conditions are derived for the global stability of the positive steady state and the semitrivial steady state of the proposed problem by using the Lyapunov functional. Our results show that intraspecific competition benefits the coexistence of prey and predator. Furthermore, the introduction of Michaelis-Menten type functional response positively affects the coexistence of prey and predator, and the nonlocal delay is harmless for stabilities of all nonnegative steady states of the system. Numerical simulations are carried out to illustrate the main results.

1. Introduction

The overall behavior of ecological systems continues to be of great interest to both applied mathematicians and ecologists. Two species predator-prey models have been extensively investigated in the literature. But recently more and more attention has been focused on systems with three or more trophic levels. For example, the predator-prey system for three species with Michaelis-Menten type functional response was studied by many authors [1–4]. However, the systems in [1–4] are either with discrete delay or without delay or without diffusion. In view of individuals taking time to move, spatial dispersal was dealt with by introducing diffusion term to corresponding delayed ODE model in previous literatures, namely, adding a Laplacian term to the ODE model. In recent years, it has been recognized that there are modelling difficulties with this approach. The difficulty is that diffusion and time delay are independent of each other, since individuals have not been at the same point at previous times. Britton [5] made a first comprehensive attempt to address this difficulty by introducing a nonlocal delay; that

is, the delay term involves a weighted-temporal average over the whole of the infinite domain and the whole of the previous times.

There are many results for reaction-diffusion equations with nonlocal delays [5–18]. The existence and stability of traveling wave fronts were studied in reaction-diffusion equations with nonlocal delay [5–9]. The stability of impulsive cellular neural networks with time varying was discussed in [10] by means of new Poincaré integral inequality. The asymptotic behavior of solutions of the reaction-diffusion equations with nonlocal delay was investigated in [11, 12] by using an iterative technique and in [13–15] by the Lyapunov functional. The stability and Hopf bifurcation were discussed in [16] for a diffusive logistic population model with nonlocal delay effect.

Motivated by the work above, we are concerned with the following food chain model with Michaelis-Menten type functional response:

$$\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 \left(a_1 - a_{11} u_1 - \frac{a_{12} u_2}{m_1 + u_1} \right),$$

$$\begin{aligned}
& \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 \\
&= u_2 \left(-a_2 + a_{21} \int_{\Omega} \int_{-\infty}^t \frac{u_1(s, y)}{m_1 + u_1(s, y)} \right. \\
&\quad \times K_1(x, y, t - s) ds dy \\
&\quad \left. - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_2} \right), \\
& \frac{\partial u_3}{\partial t} - d_3 \Delta u_3 \\
&= u_3 \left(-a_3 + a_{32} \int_{\Omega} \int_{-\infty}^t \frac{u_2(s, y)}{m_2 + u_2(s, y)} \right. \\
&\quad \times K_2(x, y, t - s) ds dy \\
&\quad \left. - a_{33} u_3 \right), \tag{1}
\end{aligned}$$

for $t > 0$, $x \in \Omega$ with homogeneous Neumann boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{2}$$

and initial conditions

$$\begin{aligned}
u_i(t, x) &= \phi_i(t, x) \geq 0 \quad (i = 1, 2), \quad (t, x) \in (-\infty, 0] \times \Omega, \\
u_3(0, x) &= \phi_3(x) \geq 0, \quad x \in \Omega, \tag{3}
\end{aligned}$$

where ϕ_i is bounded, Hölder continuous function and satisfies $\partial\phi_i/\partial\nu = 0$ ($i = 1, 2, 3$) on $(-\infty, 0] \times \partial\Omega$. Here, Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. $u_i(t, x)$ represents the density of the i th species (prey, predator, and top predator resp.) at time t and location x and thus only nonnegative $u_i(t, x)$ is of interest. The parameter a_1 is the intrinsic growth rate of the prey, and a_2 and a_3 are the death rates of the predator and top-predator. a_{ii} is the intraspecific competitive rate of the i th species. a_{12} is the maximum predation rate. a_{23} and a_{32} are the efficiencies of food utilization of the predator and top predator, respectively. We assume the predator and top predator show the Michaelis-Menten (or Holling type II) functional response with $u_1/(m_1 + u_1)$ and $u_2/(m_2 + u_2)$, respectively, where m_1 and m_2 are half-saturation constants. For a thorough biological background of similar models, see [18, 19]. As our most knowledge, the tritrophic food chain model has been found to have many interesting biological properties, such as the coexistence and the Hopf bifurcation. However, the effect of nonlocal time delays on the coexistence has not been reported. Our paper mainly concerns this perspective.

Additionally, $\int_{\Omega} \int_{-\infty}^t (u_i(s, y)/(m_i + u_i(s, y))) K_i(x, y, t - s) ds dy$ ($i = 1, 2$) represents the nonlocal delay due to the ingestion of predator; that is, mature adult predator can

only contribute to the production of predator biomass. The boundary condition in (2) implies that there is no migration across the boundary of Ω .

The main purpose of this paper is to study the global asymptotic behavior of the solution of system (1)–(3). The preliminary results are presented in Section 2. Section 3 contains sufficient conditions for the global asymptotic behaviors of the equilibria of system (1)–(3) by means of the Lyapunov functional. Numerical simulations are carried out to show the feasibility of the conditions in Theorems 8–10 in Section 4. Finally, a brief discussion is given to conclude this work.

2. Preliminary Results

In this section, we present several preliminary results that will be employed in the sequel.

Lemma 1 (see [3]). *Let a and b be positive constants. Assume that $\phi, \varphi \in C^1(a, +\infty)$, $\varphi(t) \geq 0$, and ϕ is bounded from below. If $\phi'(t) \leq -b\varphi(t)$ and $\varphi'(t) \leq K$ in $[a, +\infty)$ for some positive constant K , then $\lim_{t \rightarrow +\infty} \varphi(t) = 0$.*

The following lemma is the Positivity Lemma in [20].

Lemma 2. *Let $u_i \in C([0, T] \times \overline{\Omega}) \cap C^{1,2}((0, T] \times \Omega)$ ($i = 1, 2, 3$) and satisfy*

$$\begin{aligned}
u_{it} - d_i \Delta u_i &\geq \sum_{j=1}^3 b_{ij}(t, x) u_j(t, x) \\
&\quad + \sum_{j=1}^3 c_{ij} u_j(t - \tau_j, x) \quad \text{in } (0, T] \times \Omega, \tag{4}
\end{aligned}$$

$$\frac{\partial u_i(t, x)}{\partial \nu} \geq 0 \quad \text{on } (0, T] \times \partial\Omega,$$

$$u_i(t, x) \geq 0 \quad \text{in } [-\tau_i, 0] \times \Omega,$$

where $b_{ij}, c_{ij} \in C([0, T] \times \overline{\Omega})$, $u_{it} = (u_i)_t$. If $b_{ij} \geq 0$ for $j \neq i$ and $c_{ij} \geq 0$ for all $i, j = 1, 2, 3$. Then $u_i(t, x) \geq 0$ on $[0, T] \times \overline{\Omega}$. Moreover, if the initial function is nontrivial, then $u_i > 0$ in $(0, T] \times \overline{\Omega}$.

Lemma 3 (see [20]). *Let $\widehat{\mathbf{c}}$ and $\widetilde{\mathbf{c}}$ be a pair of constant vector satisfying $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}}$ and let the reaction functions satisfy local Lipschitz condition with $\Lambda = \langle \widehat{\mathbf{c}}, \widetilde{\mathbf{c}} \rangle$. Then system (1)–(3) admits a unique global solution $\mathbf{u}(t, x)$ such that*

$$\widehat{\mathbf{c}} \leq \mathbf{u}(t, x) \leq \widetilde{\mathbf{c}}, \quad \forall t > 0, \quad x \in \overline{\Omega}, \tag{5}$$

whenever $\widehat{\mathbf{c}} \leq \phi(t, x) \leq \widetilde{\mathbf{c}}$, $(t, x) \in (-\infty, 0] \times \Omega$.

The following result was obtained by the method of upper and lower solutions and the associated iterations in [21].

Lemma 4 (see [21]). Let $u(t, x) \in C([0, \infty) \times \overline{\Omega}) \cap C^{2,1}((0, \infty) \times \Omega)$ be the nontrivial positive solution of the system

$$\begin{aligned} u_t - d\Delta u &= Bu(t - \tau, x) \pm A_1 u(t, x) \\ &\quad - A_2 u^2(t, x), \quad (t, x) \in (0, \infty) \times \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad (t, x) \in (0, \infty) \times \partial\Omega, \\ u(t, x) &= \phi(t, x) \geq 0, \quad (t, x) \in (-\tau, 0) \times \Omega, \end{aligned} \quad (6)$$

where $A_1 \geq 0$, $B, A_2, \tau > 0$. The following results hold:

- (1) if $B \pm A_1 > 0$, then $u \rightarrow (B \pm A_1)/A_2$ uniformly on $\overline{\Omega}$ as $t \rightarrow \infty$;
- (2) if $B \pm A_1 \leq 0$, then $u \rightarrow 0$ uniformly on $\overline{\Omega}$ as $t \rightarrow \infty$.

Throughout this paper, we assume that

$$\begin{aligned} K_i(x, y, t) &= G_i(x, y, t) k_i(t), \quad x, y \in \Omega, \quad k_i(t) \geq 0; \\ \int_{\Omega} G_i(x, y, t) dx &= \int_{\Omega} G_i(x, y, t) dy = 1, \quad t \geq 0; \\ \int_0^{+\infty} k_i(t) dt &= 1, \quad tk_i(t) \in L^1((0, +\infty); R), \end{aligned} \quad (7)$$

where $G_i(x, y, t)$ is a nonnegative function, which is continuous in $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ for each $t \in [0, +\infty)$ and measurable in $t \in [0, +\infty)$ for each pair $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

Now we prove the following propositions which will be used in the sequel.

Proposition 5. For any nonnegative initial function, the corresponding solution of system (1)–(3) is nonnegative.

Proof. Suppose that (u_1, u_2, u_3) is a solution and satisfies $u_i \in C([0, T] \times \overline{\Omega}) \cap C^{1,2}((0, T) \times \Omega)$ with $T \leq +\infty$. Choose $0 < \tau < T$. Then

$$\begin{aligned} u_{1t} - d_1 \Delta u_1 &= A_1 u_1, \quad 0 < t < \tau, \quad x \in \Omega, \\ u_{2t} - d_2 \Delta u_2 &= A_2 u_2, \quad 0 < t < \tau, \quad x \in \Omega, \\ u_{3t} - d_3 \Delta u_3 &= A_3 u_3, \quad 0 < t < \tau, \quad x \in \Omega, \\ \frac{\partial u_i}{\partial \nu} &= 0, \quad 0 < t < \tau, \quad x \in \partial\Omega, \\ u_i(t, x) &\geq 0 \quad (i = 1, 2, 3), \quad t \leq 0, \quad x \in \overline{\Omega}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} A_1 &= a_1 - a_{11}u_1 - \frac{a_{12}u_2}{m_1 + u_1}, \\ A_2 &= -a_2 + a_{21} \int_{\Omega} \int_{-\infty}^t \frac{u_1(s, y)}{m_1 + u_1(s, y)} \\ &\quad \times K_1(x, y, t - s) ds dy \\ &\quad - a_{22}u_2 - \frac{a_{23}u_3}{m_2 + u_3}, \\ A_3 &= -a_3 + a_{32} \int_{\Omega} \int_{-\infty}^t \frac{u_2(s, y)}{m_2 + u_2(s, y)} \\ &\quad \times K_2(x, y, t - s) ds dy - a_{33}u_3. \end{aligned} \quad (9)$$

It follows from Lemma 2 that $u_i \geq 0$, $(t, x) \in [0, \tau] \times \overline{\Omega}$. Due to the arbitrariness of τ , we have $u_i \geq 0$ for $(t, x) \in [0, T] \times \overline{\Omega}$. \square

Proposition 6. System (1)–(3) with initial functions ϕ_i admits a unique global solution (u_1, u_2, u_3) satisfying $0 \leq u_i(t, x) \leq M_i$ for $i = 1, 2, 3$, where M_i is defined as

$$\begin{aligned} M_1 &= \max \left\{ \frac{a_1}{a_{11}}, \|\phi_1(t, x)\|_{L^\infty((-\infty, 0] \times \overline{\Omega})} \right\}, \\ M_2 &= \max \left\{ \frac{(a_{21} - a_2)M_1 - m_1 a_2}{a_{22}(m_1 + M_1)}, \|\phi_2(t, x)\|_{L^\infty((-\infty, 0] \times \overline{\Omega})} \right\}, \\ M_3 &= \max \left\{ \frac{(a_{32} - a_3)M_2 - m_2 a_3}{a_{22}(m_2 + M_2)}, \|\phi_3(x)\|_{L^\infty(\overline{\Omega})} \right\}. \end{aligned} \quad (10)$$

Proof. It follows from standard PDE theory that there exists a $T > 0$ such that problem (1)–(3) admits a unique solution in $[0, T] \times \overline{\Omega}$. From Lemma 2 we know that $u_i(t, x) \geq 0$ for $(t, x) \in [0, T] \times \overline{\Omega}$. Considering the first equation in (1), we have

$$\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 \leq u_1 (a_1 - a_{11}u_1). \quad (11)$$

Using the maximum principle gives that $u_1 \leq M_1$. In a similar way, we have $u_i \leq M_i$ for $i = 2, 3$. It is easy to see that $(0, 0, 0)$ and (M_1, M_2, M_3) are a pair of coupled upper and lower solutions to system (1)–(3) from the direct computation. In virtue of Lemma 3, system (1)–(3) admits a unique global solution (u_1, u_2, u_3) satisfying $0 \leq u_i(t, x) \leq M_i$ for $i = 1, 2, 3$. In addition, if $\phi_1(t, x), \phi_2(t, x), \phi_3(x), (t, x) \in (-\infty, 0] \times \Omega$ is nontrivial, it follows from the strong maximum principle that $u_i(t, x) > 0$ ($i = 1, 2, 3$) for all $t > 0, x \in \overline{\Omega}$. \square

3. Global Stability

In this section, we study the asymptotic behavior of the equilibrium of system (1)–(3). In the beginning, we show the existence and uniqueness of positive steady state.

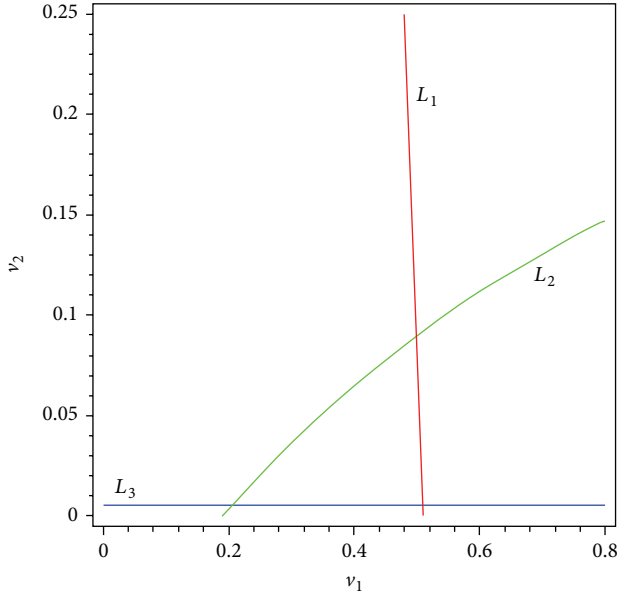


FIGURE 1: Graphs of the equations in (12). L_3 is the boundary condition ($a_3 = a_{32}v_2/(m_2 + v_2)$) and L_2 corresponds to $-a_2 + a_{21}(v_1/(m_1 + v_1)) - a_{22}v_2 = 0$, while L_1 corresponds to $a_1 - a_{11}v_1 - a_{12}v_2/(m_1 + v_1) = 0$. Intersection point between L_1 and L_2 gives the positive solution (v_1^*, v_2^*) to (12). Parameter values are listed in the example in Section 4.

Let us consider the following equations:

$$\begin{aligned} a_1 - a_{11}v_1 - \frac{a_{12}v_2}{m_1 + v_1} &= 0, \\ -a_2 + \frac{a_{21}v_1}{m_1 + v_1} - a_{22}v_2 &= 0. \end{aligned} \quad (12)$$

A direct computation shows that the above equations have only one positive solution (v_1^*, v_2^*) if and only if

$$H_1 : a_1(a_{21} - a_2) > m_1 a_2 a_{11}, \quad (13)$$

is satisfied. Taking H_1 into account, we consider the equation $a_3 = a_{32}v_2^*/(m_2 + v_2^*)$, which corresponds to L_3 in Figure 1.

It is clear that if $a_3 < a_{32}v_2^*/(m_2 + v_2^*)$, the intersection point (v_1^*, v_2^*) always lies in L_3 . Let

$$H_2 : a_3 m_2 < (a_{32} - a_3) v_2^*. \quad (14)$$

Suppose that H_1 and H_2 hold. We take u_3 as a parameter and consider the following system:

$$\begin{aligned} a_1 - a_{11}u_1 - \frac{a_{12}u_2}{m_1 + u_1} &= 0, \\ -a_2 + \frac{a_{21}u_1}{m_1 + u_1} - a_{22}u_2 - \frac{a_{23}u_3}{m_2 + u_2} &= 0, \\ -a_3 + \frac{a_{32}u_2}{m_2 + u_2} - a_{33}u_3 &= 0. \end{aligned} \quad (15)$$

When the parameter u_3 is sufficiently small, the first two equations in (15) can be approximated by (12). Moreover,

by continuously increasing the value of u_3 , L_3 goes up, and meanwhile the intersection point between L_1 and L_2 also goes up. However, L_3 goes up faster than L_2 , while L_1 keeps still. In other words, there exists a critical value u_3 such that the intersection point lies in L_3 , which implies that there is a unique positive solution $E^*(u_1^*, u_2^*, u_3^*)$ to (15), or equivalently (1)–(3).

Lemma 7. Assume that H_2 and G_2 hold, and then the positive steady state E^* of system (1)–(3) is locally asymptotically stable.

Proof. We get the local stability of E^* by performing a linearization and analyzing the corresponding characteristic equation. Similarly as in [22], let $0 < \mu_1 < \mu_2 < \mu_3 < \dots$ be the eigenvalues of $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\bar{\Omega})$, for $i = 1, 2, 3, \dots$. Let

$$\mathbf{X} = \left\{ \mathbf{u} = (u_1, u_2, u_3) \in [C^1(\bar{\Omega})]^3 \mid \partial_\eta \mathbf{u} = 0, x \in \partial\Omega \right\}, \quad (16)$$

$\{\varphi_{ij}, j = 1, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{c \cdot \varphi_{ij} \mid c \in \mathbb{R}^3\}$. Then

$$\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}, \quad \mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i. \quad (17)$$

Let $D = \text{diag}(D_1, D_2, D_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, $v_i = u_i - u_i^*$, ($i = 1, 2, 3$). Then the linearization of (1) is $\mathbf{v}_t = L\mathbf{v} = D\Delta\mathbf{v} + \mathbf{F}_v(E^*)\mathbf{v}$, $\mathbf{F}_v(E^*)\mathbf{v} = \{c_{ij}\}$, where $c_{ii} = b_{ii}v_i$, $i = 1, 2, 3$, $c_{12} = b_{12}v_2$, $c_{23} = b_{23}v_3$, $c_{21} = b_{21} \int_{\Omega} \int_{-\infty}^t v_1(s, y) K_1(x, y, t-s) ds dy$, and $c_{32} = b_{32} \int_{\Omega} \int_{-\infty}^t v_2(s, y) K_2(x, y, t-s) ds dy$. The coefficients b_{ij} are defined as follows:

$$\begin{aligned} b_{11} &= -a_{11}u_1^* + \frac{a_{12}u_1^*u_2^*}{(m_1 + u_1^*)^2}, \\ b_{12} &= -\frac{a_{12}u_1^*}{m_1 + u_1^*}, \quad b_{13} = 0, \\ b_{21} &= \frac{a_{21}m_1u_2^*}{(m_1 + u_1^*)^2}, \\ b_{22} &= -a_{22}u_2^* + \frac{a_{23}u_2^*u_3^*}{(m_2 + u_2^*)^2}, \quad b_{23} = -\frac{a_{23}u_2^*}{m_2 + u_2^*}, \\ b_{31} &= 0, \quad b_{32} = \frac{a_{32}m_2u_3^*}{(m_2 + u_2^*)^2}, \quad b_{33} = -a_{33}u_3^*. \end{aligned} \quad (18)$$

Since \mathbf{X}_i is invariant under the operator L for each $i \geq 1$, then the operator L on \mathbf{X}_i is $\mathbf{v}_t = L\mathbf{v} = D\mu_i\mathbf{v} + \mathbf{F}_v(E^*)\mathbf{v}$. Let $v_i = c_i e^{\lambda t}$ ($i = 1, 2, 3$) and we can get the characteristic equation

$\varphi_i(\lambda) = \lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$, where A_i , B_i , and C_i are defined as follows:

$$\begin{aligned} A_i &= (d_1 + d_2 + d_3)\mu_i - b_{11} - b_{22} - b_{33}, \\ B_i &= (d_1d_2 + d_2d_3 + d_3d_1)\mu_i^2 \\ &\quad + [d_3(-b_{11} - b_{22}) + d_1(-b_{33} - b_{22}) \\ &\quad + d_2(-b_{11} - b_{33})]\mu_i \\ &\quad + b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} \\ &\quad - b_{12}b_{21}\int_0^\infty k_1(s)e^{-\lambda s}ds - b_{23}b_{32}\int_0^\infty k_2(s)e^{-\lambda s}ds, \\ C_i &= d_1d_2d_3\mu_i^3 + [-b_{33}d_1d_2 - b_{11}d_2d_3 - b_{22}d_1d_3]\mu_i^2 \\ &\quad + [b_{11}b_{22}d_3 + b_{11}b_{33}d_2 + b_{22}b_{33}d_1 + b_{22}b_{33}d_1 \\ &\quad - d_3b_{12}b_{21}\int_0^\infty k_1(s)e^{-\lambda s}ds \\ &\quad - d_1b_{23}b_{32}\int_0^\infty k_2(s)e^{-\lambda s}ds]\mu_i \\ &\quad + [-b_{11}b_{22}b_{33} + b_{12}b_{21}b_{33}\int_0^\infty k_1(s)e^{-\lambda s}ds \\ &\quad + b_{11}b_{23}b_{32}\int_0^\infty k_2(s)e^{-\lambda s}ds]. \end{aligned} \quad (19)$$

It is easy to see that b_{12} , b_{23} , $b_{33} < 0$ and b_{21} , $b_{32} > 0$. It follows from assumption G_1 and G_2 that $a_{11} < 0$, $a_{22} < 0$. So $A_i > 0$, $B_i > 0$, $C_i > 0$ and $A_iB_i - C_i > 0$ for $i \geq 1$ from the direct calculation. According to the Routh-Hurwitz criterion, the three roots $\lambda_{i,1}$, $\lambda_{i,2}$, $\lambda_{i,3}$ of $\varphi_i(\lambda) = 0$ all have negative real parts.

By continuity of the roots with respect to μ_i and Routh-Hurwitz criterion, we can conclude that there exists a positive constant ε such that

$$\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\} \leq -\varepsilon, \quad i \geq 1. \quad (20)$$

Consequently, the spectrum of L , consisting only of eigenvalues, lies in $\{\operatorname{Re}\lambda \leq -\varepsilon\}$. It is easy to see that \mathbf{E}^* is locally asymptotically stable and follows from Theorem 5.1.1 of [23]. \square

Theorem 8. Assume that

$$H_3: \frac{a_1(a_{21} - a_2)}{m_1a_2} > a_{11} > \frac{a_{12}u_2^*}{m_1^2}, \quad (21)$$

H_2 and G_2 hold, and the positive steady state E^* of system (1)–(3) with nontrivial initial function is globally asymptotically stable.

Proof. It is easy to see that the equations in (1) can be rewritten as

$$\begin{aligned} \frac{\partial u_1}{\partial t} - d_1\Delta u_1 &= u_1 \left[-a_{11}(u_1 - u_1^*) \right. \\ &\quad \left. - \frac{a_{12}(u_2 - u_2^*)}{m_1 + u_1} + \frac{a_{12}u_2^*(u_1 - u_1^*)}{(m_1 + u_1)(m_1 + u_1^*)} \right], \\ \frac{\partial u_2}{\partial t} - d_2\Delta u_2 &= u_2 \left[\int_\Omega \int_{-\infty}^t K_1(x, y, t-s) \right. \\ &\quad \times \frac{a_{21}m_1(u_1(s, y) - u_1^*)}{(m_1 + u_1(s, y))(m_1 + u_1^*)} ds dy \\ &\quad - a_{22}(u_2 - u_2^*) - \frac{a_{23}(u_3 - u_3^*)}{m_2 + u_2} \\ &\quad \left. + \frac{a_{23}u_3^*(u_2 - u_2^*)}{(m_2 + u_2)(m_2 + u_2^*)} \right], \\ \frac{\partial u_3}{\partial t} - d_3\Delta u_3 &= u_3 \left[\int_\Omega \int_{-\infty}^t K_2(x, y, t-s) \right. \\ &\quad \times \frac{a_{32}m_2(u_2(s, y) - u_2^*)}{(m_2 + u_2(s, y))(m_2 + u_2^*)} ds dy \\ &\quad \left. - a_{33}(u_3 - u_3^*) \right]. \end{aligned} \quad (22)$$

Define

$$\begin{aligned} V(t) &= \alpha \int_\Omega \left(u_1 - u_1^* - u_1^* \ln \frac{u_1}{u_1^*} \right) dx \\ &\quad + \int_\Omega \left(u_2 - u_2^* - u_2^* \ln \frac{u_2}{u_2^*} \right) dx \\ &\quad + \beta \int_\Omega \left(u_3 - u_3^* - u_3^* \ln \frac{u_3}{u_3^*} \right) dx, \end{aligned} \quad (23)$$

where α and β are positive constants to be determined. Calculating the derivatives $V(t)$ along the positive solution to the system (1)–(3) yields

$$V'(t) = \Phi_1(t) + \Phi_2(t), \quad (24)$$

where

$$\begin{aligned} \Phi_1(t) &= - \int_\Omega \left(\frac{\alpha d_1 u_1^*}{u_1^2} |\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2} |\nabla u_2|^2 \right. \\ &\quad \left. + \frac{\beta d_3 u_3^*}{u_3^2} |\nabla u_3|^2 \right) dx, \end{aligned}$$

$$\begin{aligned}
\Phi_2(t) = & - \int_{\Omega} \alpha \left[a_{11} - \frac{a_{12}u_2^*}{(m_1 + u_1)(m_1 + u_1^*)} \right] (u_1 - u_1^*)^2 dx \\
& - \int_{\Omega} \left[a_{22} - \frac{a_{23}u_3^*}{(m_2 + u_2)(m_2 + u_2^*)} \right] (u_2 - u_2^*)^2 dx \\
& - \int_{\Omega} a_{33}\beta(u_3 - u_3^*)^2 dx \\
& - \int_{\Omega} \frac{a_{23}(u_3 - u_3^*)(u_2 - u_2^*)}{m_2 + u_2} dx \\
& - \int_{\Omega} \frac{\alpha a_{12}(u_2 - u_2^*)(u_1 - u_1^*)}{m_1 + u_1} dx \\
& + \iint_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \\
& \times \frac{a_{21}m_1(u_1(s, y) - u_1^*)(u_2(t, x) - u_2^*)}{(m_1 + u_1(s, y))(m_1 + u_1^*)} ds dy dx \\
& + \beta \iint_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \\
& \times \frac{a_{32}m_2(u_2(s, y) - u_2^*)(u_3(t, x) - u_3^*)}{(m_2 + u_2(s, y))(m_2 + u_2^*)} ds dy dx.
\end{aligned} \tag{25}$$

Applying the inequality $ab \leq \epsilon a^2 + (1/4\epsilon)b^2$, we derive from (25) that

$$\begin{aligned}
\Phi_2(t) \leq & - \int_{\Omega} \alpha \left[a_{11} - \frac{a_{12}u_2^*}{(m_1 + u_1)(m_1 + u_1^*)} \right] (u_1 - u_1^*)^2 dx \\
& - \int_{\Omega} a_{33}\beta(u_3 - u_3^*)^2 dx \\
& - \int_{\Omega} \left[a_{22} - \frac{a_{23}u_3^*}{(m_2 + u_2)(m_2 + u_2^*)} \right] (u_2 - u_2^*)^2 dx \\
& + \int_{\Omega} \frac{\alpha a_{12}}{m_1 + u_1} \left[\epsilon_1(u_1 - u_1^*)^2 + \frac{1}{4\epsilon_1}(u_2 - u_2^*)^2 \right] dx \\
& + \int_{\Omega} \frac{a_{23}}{m_2 + u_2} \left[\frac{1}{4\epsilon_2}(u_2 - u_2^*)^2 + \epsilon_2(u_3 - u_3^*)^2 \right] dx \\
& + \iint_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \\
& \times \frac{a_{21}m_1}{(m_1 + u_1(s, y))(m_1 + u_1^*)} \\
& \times \left[\epsilon_3(u_1(s, y) - u_1^*)^2 \right. \\
& \left. + \frac{1}{4\epsilon_3}(u_2(t, x) - u_2^*)^2 \right] ds dy dx
\end{aligned}$$

$$\begin{aligned}
& + \iint_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \\
& \times \frac{a_{32}\beta m_2}{(m_2 + u_2(s, y))(m_2 + u_2^*)} \\
& \times \left[\frac{1}{4\epsilon_4}(u_2(s, y) - u_2^*)^2 \right. \\
& \left. + \epsilon_4(u_3(t, x) - u_3^*)^2 \right] ds dy dx.
\end{aligned} \tag{26}$$

According to the property of the Kernel functions $K_i(x, y, t)$, ($i = 1, 2$), we know that

$$\begin{aligned}
\Phi_2(t) \leq & - \int_{\Omega} \left[\alpha a_{11} - \frac{\alpha a_{12}u_2^*}{m_1^2} - \frac{\alpha a_{12}\epsilon_1}{m_1} \right] (u_1 - u_1^*)^2 dx \\
& - \int_{\Omega} \left[a_{22} - \frac{a_{23}u_3^*}{m_2^2} - \frac{\alpha a_{12}}{4m_1\epsilon_1} - \frac{a_{23}}{4m_2\epsilon_2} \right. \\
& \left. - \frac{a_{21}}{4m_1\epsilon_3} \right] (u_2 - u_2^*)^2 dx \\
& - \int_{\Omega} \left[a_{33}\beta - \frac{a_{23}\epsilon_2}{m_2} - \frac{a_{32}\beta\epsilon_4}{m_2} \right] (u_3 - u_3^*)^2 dx \\
& + \frac{a_{21}\epsilon_3}{m_1} \iint_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \\
& \times (u_1(s, y) - u_1^*)^2 ds dy dx \\
& + \frac{a_{32}\beta}{4m_2\epsilon_4} \iint_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \\
& \times (u_2(s, y) - u_2^*)^2 ds dy dx.
\end{aligned} \tag{27}$$

Define a new Lyapunov functional as follows:

$$\begin{aligned}
E(t) = & V(t) \\
& + \frac{a_{21}\epsilon_3}{m_1} \iint_{\Omega} \int_0^{+\infty} \int_{t-r}^t K_1(x, y, r) \\
& \times (u_1(s, y) - u_1^*)^2 ds dr dy dx \\
& + \frac{a_{32}\beta}{4m_2\epsilon_4} \iint_{\Omega} \int_0^{+\infty} \int_{t-r}^t K_2(x, y, r) \\
& \times (u_2(s, y) - u_2^*)^2 ds dr dy dx.
\end{aligned} \tag{28}$$

It is derived from (27) and (28) that

$$\begin{aligned}
 E'(t) \leq & - \int_{\Omega} \left(\frac{\alpha d_1 u_1^*}{u_1^2} |\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2} |\nabla u_2|^2 \right. \\
 & \left. + \frac{\beta d_3 u_3^*}{u_3^2} |\nabla u_3|^2 \right) dx \\
 & - \left[\alpha a_{11} - \frac{\alpha a_{12} u_2^*}{m_1^2} - \frac{\alpha a_{12} \epsilon_1}{m_1} \right. \\
 & \left. - \frac{a_{21} \epsilon_3}{m_1} \right] \int_{\Omega} (u_1 - u_1^*)^2 dx \\
 & - \left[a_{22} - \frac{a_{23} u_3^*}{m_2^2} - \frac{\alpha a_{12}}{4m_1 \epsilon_1} - \frac{a_{23}}{4m_2 \epsilon_2} - \frac{a_{21}}{4m_1 \epsilon_3} \right. \\
 & \left. - \frac{a_{32} \beta}{4m_2 \epsilon_4} \right] \int_{\Omega} (u_2 - u_2^*)^2 dx \\
 & - \left[a_{33} \beta - \frac{a_{23} \epsilon_2}{m_2} - \frac{\beta a_{32} \epsilon_4}{m_2} \right] \int_{\Omega} (u_3 - u_3^*)^2 dx.
 \end{aligned} \quad (29)$$

Since (u_1, u_2, u_3) is the unique positive solution of system (1). Using Proposition 5, there exists a constant C which does not depend on $x \in \bar{\Omega}$ or $t \geq 0$ such that $\|u_i(\cdot, t)\|_{\infty} \leq C$ ($i = 1, 2, 3$) for $t \geq 0$. By Theorem A₂ in [24], we have

$$\|u_i(\cdot, t)\|_{C^{2,\alpha}(\bar{\Omega})} \leq C, \quad \forall t \geq 1. \quad (30)$$

Assume that

$$\begin{aligned}
 G_1 : a_{11} m_1^2 & > a_{12} u_2^*, \\
 G_2 : a_{22} m_2^2 & > a_{23} u_3^* + \frac{2a_{23} a_{32}}{a_{33}} + \frac{2a_{12} a_{21} m_2^2}{a_{11} m_1^2 - a_{12} u_2^*},
 \end{aligned} \quad (31)$$

and denote

$$\begin{aligned}
 l_1 &= \alpha a_{11} - \frac{\alpha a_{12} u_2^*}{m_1^2} - \frac{\alpha a_{12} \epsilon_1}{m_1} - \frac{a_{21} \epsilon_3}{m_1}, \\
 l_2 &= a_{22} - \frac{a_{23} u_3^*}{m_2^2} - \frac{\alpha a_{12}}{4m_1 \epsilon_1} - \frac{a_{23}}{4m_2 \epsilon_2} - \frac{a_{21}}{4m_1 \epsilon_3} - \frac{a_{32} \beta}{4m_2 \epsilon_4}, \\
 l_3 &= a_{33} \beta - \frac{a_{23} \epsilon_2}{m_2} - \frac{\beta a_{32} \epsilon_4}{m_2}.
 \end{aligned} \quad (32)$$

Then (29) is transformed into

$$\begin{aligned}
 E'(t) \leq & - \int_{\Omega} \left(\frac{\alpha d_1 u_1^*}{u_1^2} |\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2} |\nabla u_2|^2 \right. \\
 & \left. + \frac{\beta d_3 u_3^*}{u_3^2} |\nabla u_3|^2 \right) dx \\
 & - \sum_{i=1}^3 l_i \int_{\Omega} (u_i - u_i^*)^2 dx.
 \end{aligned} \quad (33)$$

If we choose

$$\begin{aligned}
 \alpha &= \frac{a_{21}}{a_{12}}, \quad \beta = \frac{a_{23}}{a_{32}}, \quad \epsilon_1 = \epsilon_3 = \frac{a_{11} m_1^2 - a_{12} u_2^*}{4m_1 a_{12}}, \\
 \epsilon_2 &= \epsilon_4 = \frac{m_2 a_{33}}{4a_{32}},
 \end{aligned} \quad (34)$$

then $l_i > 0$ ($i = 1, 2, 3$). Therefore, we have

$$E'(t) \leq - \sum_{i=1}^3 l_i \int_{\Omega} (u_i - u_i^*)^2 dx. \quad (35)$$

From Proposition 6 we can see that the solution of system (1) and (3) is bounded, and so are the derivatives of $(u_i - u_i^*)$ ($i = 1, 2, 3$) by the equations in (1). Applying Lemma 1, we obtain that

$$\lim_{t \rightarrow \infty} \int_{\Omega} (u_i - u_i^*) dx = 0, \quad (i = 1, 2, 3). \quad (36)$$

Recomputing $E'(t)$ gives

$$\begin{aligned}
 E'(t) \leq & - \int_{\Omega} \left(\frac{\alpha d_1 u_1^*}{u_1^2} |\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2} |\nabla u_2|^2 \right. \\
 & \left. + \frac{\beta d_3 u_3^*}{u_3^2} |\nabla u_3|^2 \right) dx
 \end{aligned} \quad (37)$$

$$\leq -c \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2) dx = -g(t),$$

where $c = \min\{\alpha d_1 u_1^*/M_1^2, d_2 u_2^*/M_2^2, \beta d_3 u_3^*/M_3^2\}$. Using (30) and (1), we obtain that the derivative of $g(t)$ is bounded in $[1, +\infty)$. From Lemma 1, we conclude that $g(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore

$$\lim_{t \rightarrow \infty} \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2) dx = 0. \quad (38)$$

Applying the Poincaré inequality

$$\int_{\Omega} \lambda |u - \bar{u}|^2 dx \leq \int_{\Omega} |\nabla u|^2 dx, \quad (39)$$

leads to

$$\lim_{t \rightarrow \infty} \int_{\Omega} (u_i - \bar{u}_i)^2 dx = 0, \quad (i = 1, 2, 3), \quad (40)$$

where $\bar{u}_i = (1/|\Omega|) \int_{\Omega} u_i dx$ and λ is the smallest positive eigenvalue of $-\Delta$ with the homogeneous Neumann condition. Therefore,

$$\begin{aligned}
 |\Omega| (\bar{u}_1(t) - u_1^*)^2 &= \int_{\Omega} (\bar{u}_1(t) - u_1^*)^2 dx \\
 &= \int_{\Omega} (\bar{u}_1(t) - u_1(t, x) + u_1(t, x) - u_1^*)^2 dx \\
 &\leq 2 \int_{\Omega} (\bar{u}_1(t) - u_1(t, x))^2 dx \\
 &\quad + 2 \int_{\Omega} (u_1(t, x) - u_1^*)^2 dx.
 \end{aligned} \quad (41)$$

So we have $\bar{u}_1(t) \rightarrow u_1^*$ as $t \rightarrow \infty$. Similarly, $\bar{u}_2(t) \rightarrow u_2^*$ and $\bar{u}_3(t) \rightarrow u_3^*$ as $t \rightarrow \infty$. According to (30), there exists a subsequence t_m , and non-negative functions $w_i \in C^2(\bar{\Omega})$, such that

$$\lim_{m \rightarrow \infty} \|u_i(\cdot, t_m) - w_i(\cdot)\|_{C^2(\bar{\Omega})} = 0 \quad (i = 1, 2, 3). \quad (42)$$

Applying (40) and noting that $\bar{u}_i(t) \rightarrow u_i^*$, we then have $w_i = u_i^*$, $(i = 1, 2, 3)$. That is,

$$\lim_{m \rightarrow \infty} \|u_i(\cdot, t_m) - u_i^*\|_{C^2(\bar{\Omega})} = 0 \quad (i = 1, 2, 3). \quad (43)$$

Furthermore, the local stability of E^* combining with (43) gives the following global stability. \square

Theorem 9. Assume that

$$H_5 : \frac{a_1(a_{21} - a_2)}{m_1 a_2} > a_{11} > \frac{a_{12} v_2^*}{m_1^2}, \quad (44)$$

H_4 and G_4 hold, and then the semi-trivial steady state E_2 of system (1)–(3) with non-trivial initial functions is globally asymptotically stable.

Proof. It is obvious that system (1)–(3) always has two non-negative equilibria (u_1, u_2, u_3) as follows: $E_0 = (0, 0, 0)$ and $E_1 = (a_1/a_{11}, 0, 0)$. If H_1 and H_4 are satisfied, system (1) has the other semitrivial solution denoted by $E_2(v_1^*, v_2^*, 0)$, where

$$H_4 : a_3 m_2 > (a_{32} - a_3) v_2^*. \quad (45)$$

We consider the stability of E_2 under condition H_1 and H_4 . Equation (1) can be rewritten as

$$\begin{aligned} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 &= u_1 \left[-a_{11}(u_1 - v_1^*) - \frac{a_{12}(u_2 - v_2^*)}{m_1 + u_1} \right. \\ &\quad \left. + \frac{a_{12} v_2^* (u_1 - v_1^*)}{(m_1 + u_1)(m_1 + v_1^*)} \right], \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 &= u_2 \left[\int_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \right. \\ &\quad \times \frac{a_{21} m_1 (u_1(s, y) - v_1^*)}{(m_1 + u_1(s, y))(m_1 + v_1^*)} ds dy \\ &\quad \left. - a_{22}(u_2 - v_2^*) - \frac{a_{23} u_3}{m_2 + u_2} \right], \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{\partial u_3}{\partial t} - d_3 \Delta u_3 &= u_3 \left[\int_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \right. \\ &\quad \times \frac{a_{32} m_2 (u_2(s, y) - v_2^*)}{(m_2 + u_2(s, y))(m_2 + v_2^*)} ds dy \\ &\quad \left. - a_{33} u_3 \right]. \end{aligned}$$

Similar to the argument of Theorem 8, we have $u_1(x, t) \rightarrow v_1^*$ and $u_2(x, t) \rightarrow v_2^*$ as $t \rightarrow \infty$ uniformly on $\bar{\Omega}$ provided that the following additional condition holds:

$$\begin{aligned} G_3 : \quad & a_{11} m_1^2 > a_{12} v_2^*, \\ G_4 : \quad & a_{22} m_2^2 > \frac{a_{23} a_{32}}{a_{33}} + \frac{2a_{12} a_{21} m_2^2}{a_{11} m_1^2 - a_{12} v_2^*}. \end{aligned} \quad (47)$$

Next, we consider the asymptotic behavior of $u_3(t, x)$. Let

$$\theta = \frac{a_3 - (a_{32} - a_3) v_2^*}{2\delta}, \quad \delta = |a_{32} - a_3|. \quad (48)$$

Then there exists $t_1 > 0$ such that

$$v_2^* - \theta < u_2(t, x) < v_2^* + \theta, \quad \forall t > t_1, \quad x \in \bar{\Omega}. \quad (49)$$

Consider the following two systems:

$$\begin{aligned} w_{3t} - d_3 \Delta w_3 &= w_3 \left[-a_3 + \frac{a_{32}(v_2^* + \theta)}{m_2 + v_2^* + \theta} - a_{33} w_3 \right], \\ (t, x) &\in [t_1, +\infty) \times \Omega, \\ \frac{\partial w_3}{\partial \nu} &= 0, \quad (t, x) \in [t_1, +\infty) \times \partial \Omega, \\ w_3 &= u_3, \quad (t, x) \in (-\infty, t_1] \times \Omega, \\ W_{3t} - d_3 \Delta W_3 &= W_3 \left[-a_3 + \frac{a_{32}(v_2^* - \theta)}{m_2 + v_2^* - \theta} - a_{33} W_3 \right], \\ (t, x) &\in [t_1, +\infty) \times \Omega, \\ \frac{\partial W_3}{\partial \nu} &= 0, \quad (t, x) \in [t_1, +\infty) \times \partial \Omega, \\ W_3 &= u_3, \quad (t, x) \in (-\infty, t_1] \times \Omega. \end{aligned} \quad (50)$$

Combining comparison principle with (50), we obtain that

$$W_3(t, x) \leq u_3(t, x) \leq w_3(t, x), \quad \forall t > t_1, \quad x \in \bar{\Omega}. \quad (51)$$

By Lemma 4, we obtain

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} W_3(t, x) \leq \inf u_3(t, x) \leq \sup u_3(t, x) \\ &\leq \lim_{t \rightarrow \infty} w_3(t, x) = 0, \quad \forall x \in \bar{\Omega}, \end{aligned} \quad (52)$$

which implies that $\lim_{t \rightarrow \infty} u_3(t, x) = 0$ uniformly on $\bar{\Omega}$. \square

Theorem 10. Suppose that G_5 and G_6 hold, and then the semi-trivial steady state E_1 of system (1)–(3) with non-trivial initial functions is globally asymptotically stable.

Proof. We study the stability of the semi-trivial solution $E_1 = (\tilde{u}_1, 0, 0)$. Similarly, the equations in (1) can be written as

$$\begin{aligned} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 &= u_1 \left[-a_{11} (u_1 - \tilde{u}_1) - \frac{a_{12} u_2}{m_1 + u_1} \right], \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 &= u_2 \left[-a_2 + \int_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \right. \\ &\quad \times \frac{a_{21} m_1 (u_1(s, y) - \tilde{u}_1)}{(m_1 + u_1(s, y)) (m_1 + \tilde{u}_1)} ds dy \\ &\quad \left. + \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_2} \right], \\ \frac{\partial u_3}{\partial t} - d_3 \Delta u_3 &= u_3 \left[-a_3 + \int_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \right. \\ &\quad \times \frac{a_{32} u_2(s, y)}{m_2 + u_2(s, y)} ds dy - a_{33} u_3 \left. \right]. \end{aligned} \quad (53)$$

Define

$$\begin{aligned} V(t) &= \alpha \int_{\Omega} \left[u_1 - \tilde{u}_1 - \tilde{u}_1 \log \frac{u_1}{\tilde{u}_1} \right] dx + \int_{\Omega} u_2 dx \\ &\quad + \beta \int_{\Omega} u_3 dx. \end{aligned} \quad (54)$$

Calculating the derivative of $V(t)$ along E_1 , we get from (54) that

$$\begin{aligned} V'(t) &\leq - \int_{\Omega} \left[\alpha \left(a_{11} - \frac{a_{12} \epsilon_1}{m_1} \right) \right] (u_1 - \tilde{u}_1)^2 dx \\ &\quad + \int_{\Omega} \left[a_{22} - \frac{\alpha a_{12}}{4m_1 \epsilon_1} - \frac{a_{23}}{4\epsilon_2 m_2} - \frac{a_{21}}{4m_1 \epsilon_3} \right] u_2^2 dx \\ &\quad + \int_{\Omega} \left[\beta a_{33} - \frac{a_{23} \epsilon_2}{m_2} \right] u_3^2 dx \\ &\quad - \int_{\Omega} \left[\left(a_2 - \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} \right) u_2 + a_3 u_3 \right] dx \\ &\quad + \frac{a_{21} \epsilon_3}{m_1} \int_{\Omega} \int_{-\infty}^t K_1(x, y, t-s) \\ &\quad \times (u_1(s, y) - \tilde{u}_1)^2 ds dy dx \\ &\quad + \frac{a_{32} \beta}{4\epsilon_4 m_2} \iint_{\Omega} \int_{-\infty}^t K_2(x, y, t-s) \\ &\quad \times u_2^2(s, y) ds dy dx. \end{aligned} \quad (55)$$

Define

$$\begin{aligned} E(t) &= V(t) \\ &\quad + \frac{a_{21} \epsilon_3}{m_1} \iint_{\Omega} \int_0^{+\infty} \int_{t-r}^t K_1(x, y, r) \\ &\quad \times (u_1(s, y) - u_1^*)^2 ds dr dy dx \\ &\quad + \frac{a_{32} \beta}{4\epsilon_4 m_2} \iint_{\Omega} \int_0^{+\infty} \int_{t-r}^t K_2(x, y, r) \\ &\quad \times (u_2(s, y) - u_2^*)^2 ds dr dy dx. \end{aligned} \quad (56)$$

It is easy to see that

$$\begin{aligned} E'(t) &\leq -\alpha d_1 \tilde{u}_1 \int_{\Omega} \frac{|\nabla u_1|^2}{u_1^2} dx \\ &\quad - \int_{\Omega} \left[\alpha \left(a_{11} - \frac{a_{12} \epsilon_1}{m_1} \right) - \frac{a_{21} \epsilon_3}{m_1} \right] (u_1 - \tilde{u}_1)^2 dx \\ &\quad + \int_{\Omega} \left[a_{22} - \frac{\alpha a_{12}}{4m_1 \epsilon_1} - \frac{a_{23}}{4m_2 \epsilon_2} - \frac{a_{21}}{4m_1 \epsilon_3} \right. \\ &\quad \left. - \frac{a_{32}}{4m_2 \epsilon_4} \right] u_2^2 dx \\ &\quad + \int_{\Omega} \left[\beta a_{33} - \frac{a_{23} \epsilon_2}{m_2} - \frac{a_{32} \beta \epsilon_4}{m_2} \right] u_3^2 dx \\ &\quad - \int_{\Omega} \left[\left(a_2 - \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} \right) u_2 - a_3 \beta u_3 \right] dx. \end{aligned} \quad (57)$$

Assume that

$$\begin{aligned} G_5 : \quad &a_1 (a_{21} - a_2) < m_1 a_2 a_{11}, \\ G_6 : \quad &a_{22} m_2^2 > \frac{a_{23} a_{32}}{a_{33}} + \frac{2a_{12} a_{21} m_2^2}{a_{11} m_1^2}. \end{aligned} \quad (58)$$

Let

$$\begin{aligned} l_1 &= \alpha a_{11} - \frac{\alpha a_{12} \epsilon_1}{m_1} - \frac{a_{21} \epsilon_3}{m_1}, \\ l_2 &= a_{22} - \frac{\alpha a_{12}}{4m_1 \epsilon_1} - \frac{a_{23}}{4m_2 \epsilon_2} - \frac{a_{21}}{4m_1 \epsilon_3} - \frac{a_{32} \beta}{4m_2 \epsilon_4}, \\ l_3 &= a_{33} \beta - \frac{a_{23} \epsilon_2}{m_2} - \frac{\beta a_{32} \epsilon_4}{m_2}. \end{aligned} \quad (59)$$

Choose

$$\begin{aligned} \alpha &= \frac{a_{21}}{a_{12}}, \quad \beta = \frac{a_{23}}{a_{32}}, \quad \epsilon_1 = \epsilon_3 = \frac{a_{11} m_1}{4a_{12}}, \\ \epsilon_2 &= \epsilon_4 = \frac{m_2 a_{33}}{4a_{32}}. \end{aligned} \quad (60)$$

Then we get $l_i > 0$ ($i = 1, 2, 3$). Therefore we have

$$\lim_{t \rightarrow \infty} u_i(t, x) = \tilde{u}_i, \quad (61)$$

uniformly on $\bar{\Omega}$. \square

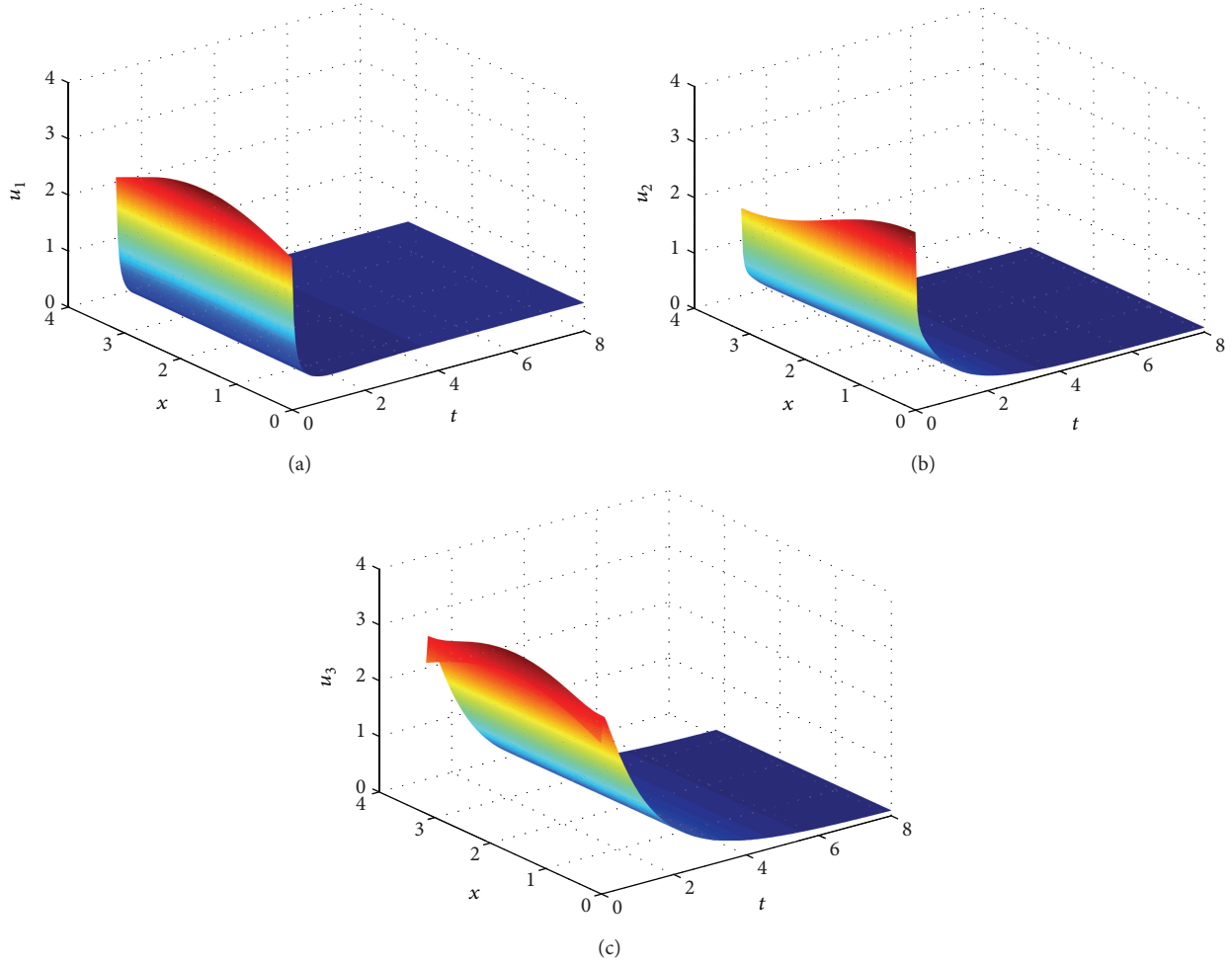


FIGURE 2: Global stability of the positive equilibrium E^* with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5 \sin x, 2.7 + 0.5 \cos x, 2.7 + 0.5 \sin x)$.

Next, we consider the asymptotic behavior of $u_2(t, x)$ and $u_3(t, x)$. For any $T > 0$, integrating (57) over $[0, T]$ yields

$$E(T) + \alpha d_1 \left\| \frac{\nabla u_1}{u_1} \right\|_2^2 + l_1 \|u_1 - \tilde{u}_1\|_2^2 + l_2 \|u_2\|_2^2 + l_3 \|u_3\|_2^2 \leq E(0), \quad (62)$$

where $\|u_i\|_2^2 = \int_0^T \int_\Omega u_i^2 dx dt$. It implies that $\|u_i\|_2 \leq C_i$ ($i = 2, 3$) for the constant C_i which is independent of T . Now we consider the boundedness of $\|\nabla u_2\|_2$ and $\|\nabla u_3\|_2$. From the Green's identity, we obtain

$$\begin{aligned} & d_2 \int_0^T \int_\Omega |\nabla u_2(t, x)|^2 dx dt \\ &= -d_2 \int_0^T \int_\Omega u_2(t, x) \Delta u_2(t, x) dx dt \\ &= - \int_0^T \int_\Omega u_2 \frac{\partial u_2}{\partial t} dx dt - a_2 \int_0^T \int_\Omega u_2^2 dx dt \end{aligned}$$

$$\begin{aligned} & -a_{22} \int_0^T \int_\Omega u_2^3 dx dt - a_{23} \int_0^T \int_\Omega \frac{u_2^2 u_3}{m_2 + u_3} dx dt \\ &+ a_{21} \int_0^T \iint_\Omega \int_{-\infty}^t \frac{u_1(s, y) u_2^2(t, x)}{m_1 + u_1(s, y)} \\ &\quad \times K_1(x, y, t-s) ds dy dx dt. \end{aligned} \quad (63)$$

Note that

$$\begin{aligned} & \int_0^T \int_\Omega u_2(t, x) \frac{\partial u_2(t, x)}{\partial t} dx dt \\ &= \int_0^T \int_\Omega \frac{1}{2} \frac{\partial u_2^2}{\partial t} dt dx \\ &= \frac{1}{2} \int_\Omega u_2^2(T, x) dx - \frac{1}{2} \int_\Omega u_2^2(0, x) dx \leq M_2^2 |\Omega|, \\ & \int_0^T \int_\Omega u_2^2 u_3 dx dt \leq M_3 \int_0^T \int_\Omega u_2^2 dx dt, \end{aligned}$$

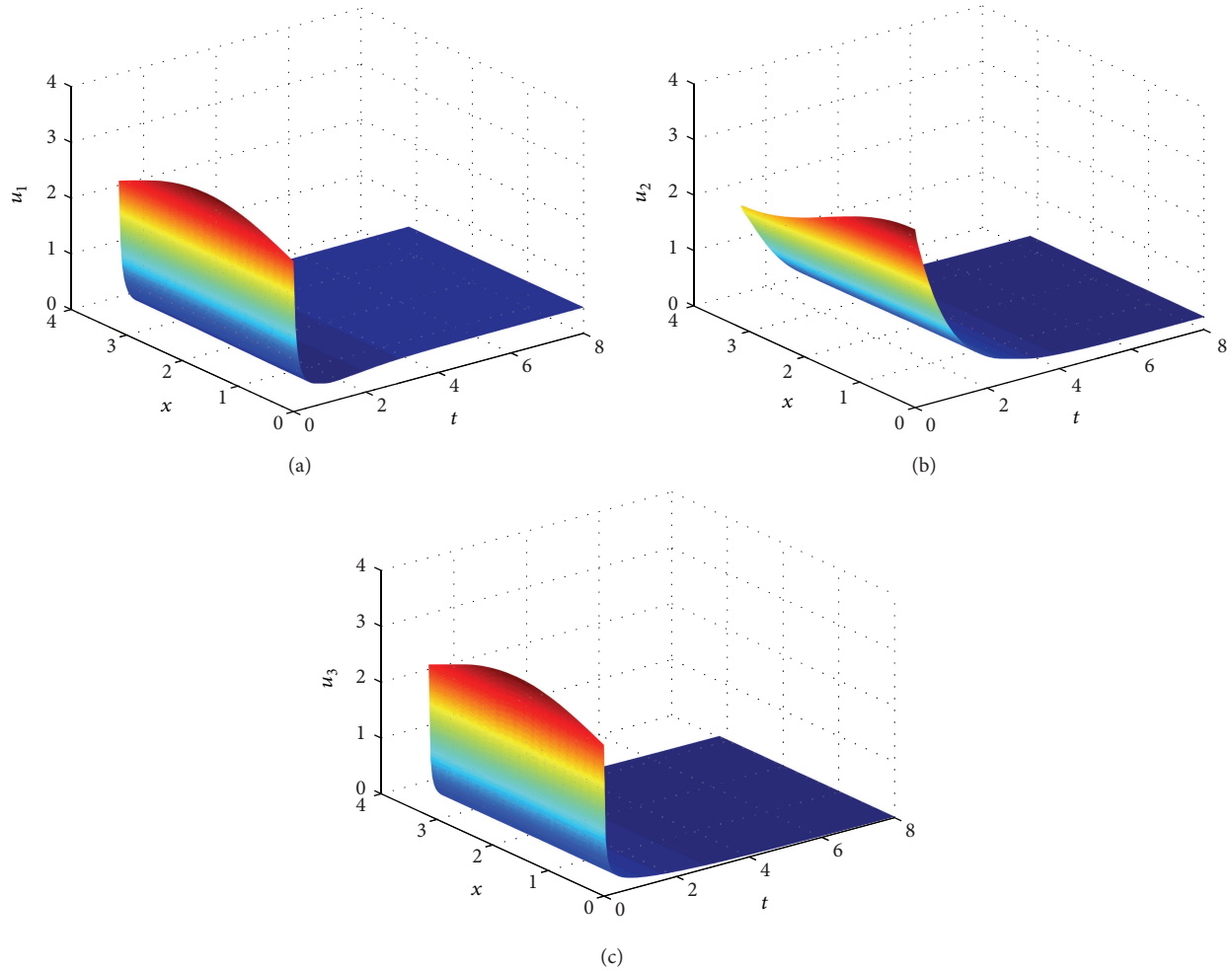


FIGURE 3: Global stability of the positive equilibrium E_2 with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5 \sin x, 2.7 + 0.5 \cos x, 2.7 + 0.5 \sin x)$.

$$\begin{aligned} \int_0^T \int_{\Omega} u_2^3 dx dt &\leq M_2 \int_0^T \int_{\Omega} u_2^2 dx dt, \\ \int_0^T \iint_{\Omega} \int_{-\infty}^t \frac{u_1(s, y) u_2^2(t, x)}{m_1 + u_1(s, y)} K_1(x, y, t - s) ds dy dx dt \\ &\leq \int_0^T \int_{\Omega} u_2^2(t, x) dx dt. \end{aligned} \quad (64)$$

Thus, we get $\|\nabla u_2\|_2 \leq C_4$. In a similar way, we have $\|\nabla u_3\|_2 \leq C_5$. Here C_4 and C_5 are independent of T .

It is easy to see that $u_2(t, x), u_3(t, x) \in L^2((0, \infty); W^{1,2}(\Omega))$. These imply that

$$\lim_{t \rightarrow \infty} \|u_i(\cdot, t)\|_{W^{1,2}(\Omega)} = 0, \quad i = 2, 3. \quad (65)$$

From the Sobolev compact embedding theorem, we know

$$\lim_{t \rightarrow \infty} \|u_i(\cdot, t)\|_{C(\bar{\Omega})} = 0, \quad i = 2, 3. \quad (66)$$

In the end, we show that the trivial solution E_0 is an unstable equilibrium. Similarly to the local stability to E^* , we can get the characteristic equation of E_0 as

$$(\lambda + \mu_1 D_1 - a_1)(\lambda + \mu_1 D_2 + a_2)(\lambda + \mu_1 D_3 + a_3) = 0. \quad (67)$$

If $i = 1$, then $\mu_1 = 0$. It is easy to see that this equation admits a positive solution $\lambda = a_1$. According to Theorem 5.1 in [23], we have the following result.

Theorem 11. *The trivial equilibrium E_0 is an unstable equilibrium of system (1)–(3).*

4. Numerical Illustrations

In this section, we perform numerical simulations to illustrate the theoretical results given in Section 3.

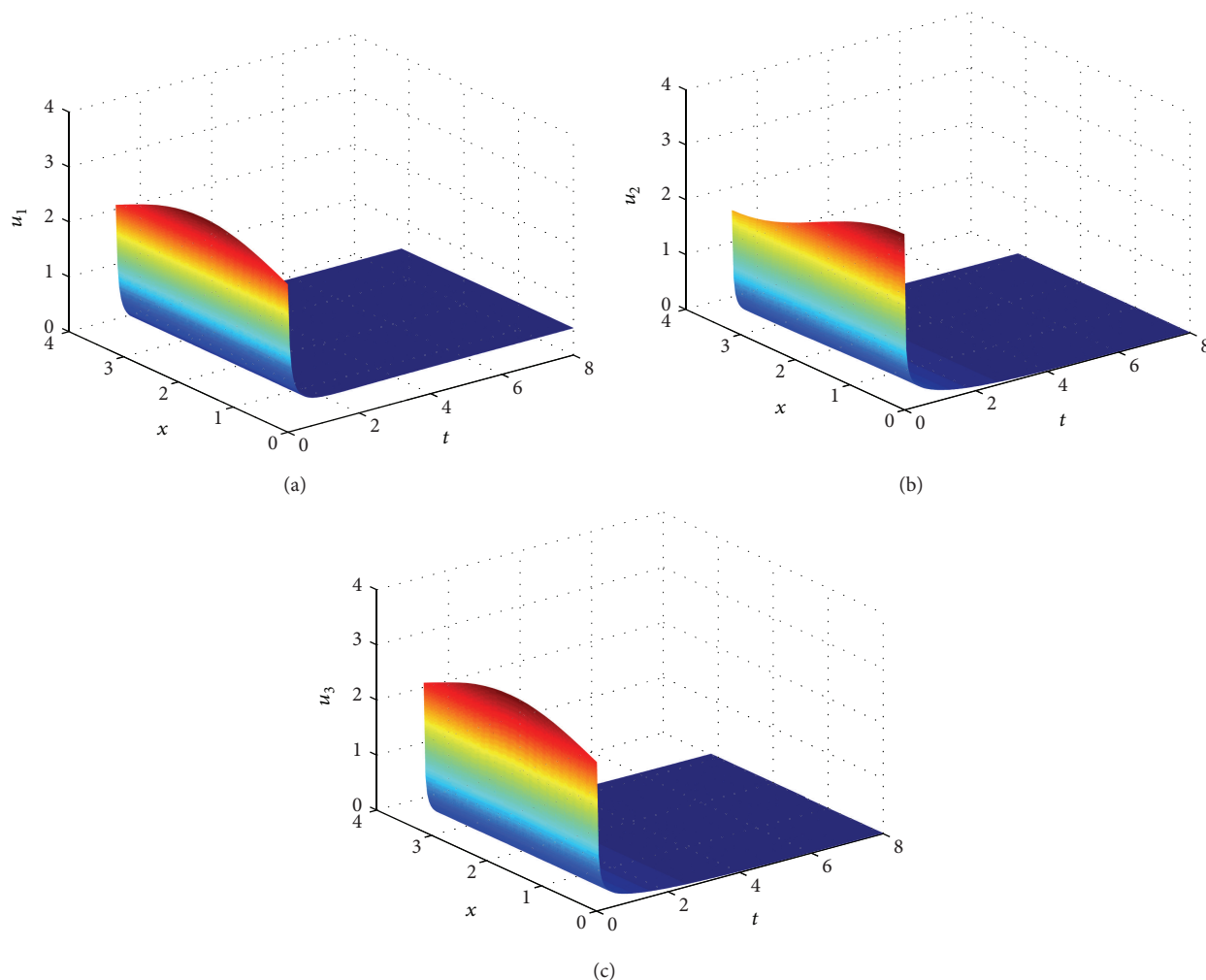


FIGURE 4: Global stability of the positive equilibrium E_1 with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5 \sin x, 2.7 + 0.5 \cos x, 2.7 + 0.5 \sin x)$.

In the following, we always take $\Omega = [0, \pi]$, $K_i(x, y, t) = G_i(x, y, t)k_i(t)$, where $k_i(t) = (1/\tau^*)e^{-t/\tau^*}$ ($i = 1, 2$) and

$$G_1(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-d_2 n^2 t} \cos nx \cos ny, \quad (68)$$

$$G_2(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-d_3 n^2 t} \cos nx \cos ny.$$

However, it is difficult for us to simulate our results directly because of the nonlocal term. Similar to [25], the equations in (1) can be rewritten as follows:

$$\begin{aligned} \frac{\partial u_1}{\partial t} - d_1 \Delta u_1 &= u_1 \left(a_1 - a_{11} u_1 - \frac{a_{12} u_2}{m_1 + u_1} \right), \\ \frac{\partial u_2}{\partial t} - d_2 \Delta u_2 &= u_2 \left(-a_2 + a_{21} v_1 - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_3} \right), \\ \frac{\partial u_3}{\partial t} - d_3 \Delta u_3 &= u_3 (-a_3 + a_{32} v_2 - a_{33} u_3), \end{aligned}$$

$$\begin{aligned} \frac{\partial v_1}{\partial t} - d_2 \Delta v_1 &= \frac{1}{\tau^*} \left(\frac{u_1}{m_1 + u_1} - v_1 \right), \\ \frac{\partial v_2}{\partial t} - d_3 \Delta v_2 &= \frac{1}{\tau^*} \left(\frac{u_2}{m_2 + u_2} - v_2 \right), \end{aligned} \quad (69)$$

where

$$v_i = \int_0^\pi \int_{-\infty}^t G_i(x, y, t-s) \frac{1}{\tau^*} e^{-(t-s)/\tau^*} \frac{u_i(s, y)}{m_i + u_i(s, y)} dy ds, \quad i = 1, 2. \quad (70)$$

Each component is considered with homogeneous Neumann boundary conditions, and the initial condition of v_i is

$$v_i(0, x) = \int_0^\pi \int_{-\infty}^0 G_i(x, y, -s) \frac{1}{\tau^*} e^{s/\tau^*} \frac{u_i(s, y)}{m_i + u_i(s, y)} dy ds, \quad i = 1, 2. \quad (71)$$

In the following examples, we fix some coefficients and assume that $d_1 = d_2 = d_3 = 1$, $a_1 = 3$, $a_{12} = 1$, $m_1 = m_2 = 1$, $a_2 = 1$, $a_{22} = 12$, $a_{23} = 1$, $a_{33} = 12$, and $\tau^* = 1$. The asymptotic behaviors of system (1)–(3) are shown by choosing different coefficients a_{11} , a_{21} , a_3 , and a_{32} .

Example 12. Let $a_{11} = 53/9$, $a_{21} = 81/13$, $a_3 = 1/13$ and $a_{32} = 14$. Then it is easy to see that the system admits a unique positive equilibrium $E^*(1/2, 1/12, 1/12)$. By Theorem 8, we see that the positive solution $(u_1(t, x), u_2(t, x), u_3(t, x))$ of system (1)–(3) converges to E^* as $t \rightarrow \infty$. See Figure 2.

Example 13. Let $a_{11} = 6$, $a_{21} = 12$, $a_3 = 1/4$ and $a_{32} = 1$. Clearly, H_2 does not hold. Hence, the positive steady state is not feasible. System (1) admits two semi-trivial steady state $E_2(0.4732, 0.2372, 0)$ and $E_1(1/2, 0, 0)$. According to Theorem 9, we know that the positive solution $(u_1(t, x), u_2(t, x), u_3(t, x))$ of system (1)–(3) converges to E_2 as $t \rightarrow \infty$. See Figure 3.

Example 14. Let $a_{11} = 6$, $a_{21} = 2$, $a_3 = 1/4$ and $a_{32} = 1$. Clearly, H_1 does not hold. Hence, E^* and E_2 are not feasible. System (1) has a unique semi-trivial steady state $E_1(1/2, 0, 0)$. According to Theorem 10 we know that the positive solution $(u_1(t, x), u_2(t, x), u_3(t, x))$ of system (1)–(3) converges to E_1 as $t \rightarrow \infty$. See Figure 4.

5. Discussion

In this paper, we incorporate nonlocal delay into a three-species food chain model with Michaelis-Menten functional response to represent a delay due to the gestation of the predator. The conditions, under which the spatial homogeneous equilibria are asymptotically stable, are given by using the Lyapunov functional.

We now summarize the ecological meanings of our theoretical results. Firstly, the positive equilibrium E^* of system (1)–(3) exists under the high birth rate of the prey (a_1) and low death rates (a_2 and a_3) of predator and top predator. E^* is globally stable if the intraspecific competition a_{11} is neither too big nor too small and the maximum harvest rates a_{12} , a_{23} are small enough. Secondly, the semi-trivial equilibrium E_2 of system (1)–(3) exists if the birth rate of the prey (a_1) is high, death rate of predator a_2 is low, and the death rate (a_3) exceeds the conversion rate from predator to top predator (a_{32}). E_2 is globally stable if the maximum harvest rates a_{12} , a_{23} are small and the intra-specific competition a_{11} is neither too big nor too small. Thirdly, system (1)–(3) has only one semi-trivial equilibrium E_1 when the death rate (a_2) exceeds the conversion rate from prey to predator (a_{21}). E_1 is globally stable if intra-specific competitions (a_{11} , a_{22} , and a_{33}) are strong. Finally, E_0 is unstable and the non-stability of trivial equilibrium tells us that not all of the populations go to extinction. Furthermore, our main results imply that the nonlocal delay is harmless for stabilities of all non-negative steady states of system (1)–(3).

There are still many interesting and challenging problems with respect to system (1)–(3), for example, the permanence

and stability of periodic solution or almost periodic solution. These problems are clearly worthy for further investigations.

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