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Research Article

On the Behaviour of Singular Semigroups in Intermediate and Interpolation Spaces and Its Applications to Maximal Regularity for Degenerate Integro-Differential Evolution Equations

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For those semigroups, which may have power type singularities and whose generators are abstract multivalued linear operators, we characterize the behaviour with respect to a certain set of intermediate and interpolation spaces. The obtained results are then applied to provide maximal time regularity for the solutions to a wide class of degenerate integro- and non-integro-differential evolution equations in Banach spaces.

1. Introduction

Let X be a complex Banach space and let $\{\mathcal{T}_A(t)\}_{t\geq 0}$ be a semigroup of operators on X, which is generated by a multivalued linear operator $A : \mathcal{D}(A) \subseteq X \to X$ and which may have a power type singularity at the origin $t = 0$, that is,

$$\begin{gathered} \|\mathcal{T}_A(t)\|_{\mathcal{L}(X)} \leq C_0 t^{\nu}, \quad \forall t > 0, \\ \mathcal{T}_A(0)\, x = x, \quad \forall x \in X, \end{gathered} \tag{1}$$

for some nonnegative constant C_0 and nonpositive exponent ν, where $\mathcal{L}(X)$ denotes the Banach algebra of all endomorphisms of X endowed with the uniform operator norm. In this context our aim here is twofold. The first is to characterize the behaviour of $\{\mathcal{T}_A(t)\}_{t\geq 0}$ with respect to some intermediate and interpolation spaces between X and the domain $\mathcal{D}(A)$ of A. The second is to investigate how this behaviour reflects on the question of maximal time regularity for the solutions to a class of degenerate integro- and non-integrodifferential initial value problems in X.

The class of operators we will deal with consists precisely of those multivalued linear operators A whose single-valued resolvents satisfy the following estimate:

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq C(|\lambda| + 1)^{-\beta}, \quad \forall \lambda \in \Sigma_\alpha. \tag{2}$$

Here, I is the identity operator, C is a positive constant, $\beta \in (0, 1]$, and Σ_α is the complex region $\{z \in \mathbf{C} : \mathfrak{Re}\, z \geq -c(|\mathfrak{Im}\, z| + 1)^\alpha, \mathfrak{Im}\, z \in \mathbf{R}\}, c > 0, \alpha \in [\beta, 1]$. It thus happens (cf. [1–3]) that A is the infinitesimal generator of a semigroup of linear bounded operators in X satisfying (1) with $\nu = \nu_{\alpha,\beta}$, where $\nu_{\alpha,\beta} = (\beta - 1)/\alpha$.

To outline the motivations of our research, let us assume for a moment that A is a single-valued linear operator satisfying (2). It is well known that if $\beta = 1$, then A is the infinitesimal generator of a bounded analytic semigroup. For this case, an extensive literature exists concerning the behaviour of $\{\mathcal{T}_A(t)\}_{t\geq 0}$ with respect to the real interpolation spaces $(X, \mathcal{D}(A))_{\gamma,p}$, $\gamma \in (0, 1)$, $p \in [1, \infty]$, and its application to questions of maximal regularity for the solutions to nondegenerate (possibly nonautonomous) integro- and non-integrodifferential abstract Cauchy problems. See, for instance, [4–11]. Due to (1) with $\nu = \nu_{1,\beta}$, the case of $\alpha = 1$ and $\beta \in (0, 1)$ is definitely worsened and the literature for it is considerably less conspicuous, although estimate of type (2), with $(\mathfrak{Re}\, \lambda + |\mathfrak{Im}\, \lambda|^\beta)^{-1}$ in place of $(|\lambda| + 1)^{-\beta}$, goes back even to [12, Remark p. 383] in the ambit of Abel summable semigroups admitting uniform derivatives of all orders. One of the main problems with the case $\beta \in (0, 1)$ is that some equivalent characterizations of $(X, \mathcal{D}(A))_{\gamma,p}$ begin to fail (cf. [13]), so that some spaces which were just real interpolation spaces between X and $\mathscr{D}(A)$ in the case $\beta = 1$ become only intermediate spaces in the case $\beta \in (0,1)$. However, avoiding questions of interpolation theory and of maximal regularity, a quite satisfactorily semigroup theory for the single-valued case with $\beta \in (0,1)$ and its application to the unique solvability of some concrete partial (non-integro-) differential equations have been developed in [14–18]. Since the multivalued case embraces the single-valued one, our contribution in this field is to fill this gap, supplying a theory for the behaviour of singular semig n intermediate and interpolation spaces which, in the case $\beta = 1$, reduces to that in [9, 11]. As an effect of this theory, there is the possibility of investigating questions of maximal time regularity for an entire class of nondegenerate evolution equations which does not fall within the case $\beta = 1$.

The case when A is really a multivalued linear operator arises naturally when we shift our attention to degenerate evolution equations of the type considered in [1–3]. There, a semigroup theory for multivalued linear operators was introduced as a tool to handle degenerate equations by means of analogous techniques of the nondegenerate ones. Such a theory has been then successfully applied to questions of maximal regularity for the solutions to a wide class of degenerate integro- and non-integrodifferential equations. We quote [2, 19–23] where, in general and unless $\beta = 1$, it is shown that the time regularity of the solutions decreases with respect to that of the data. In this respect, we mention the recent results in [20] where, under an additional condition of space regularity on the data and provided that α and β are large enough, the loss of time regularity is restored. Regrettably (cf. the appendix below), we have found some inaccuracies in [20, Section 4], and for this reason we must indicate some changes to that paper. On the other side, fortunately, the basic idea in [20] is correct and remedy can be applied to all the inappropriate items. Furthermore, unexpectedly, we will see that the more delicate approach followed in this paper not only corrects the mistakes in [20], but also gives rise to an effective improvement of the achievable results. In fact, here, we will straighten out, refine, and extend [20], enlarging the class of the admissible spaces to which the data may belong, weakening the assumption for the pair (α, β), and complicating the structure of the underlying equations. This is why we will first analyze the behaviour of the semigroup generated by A with respect to some intermediate and interpolation spaces which turn out to be equivalent only in the case $\beta = 1$. Indeed, the phenomena exhibited in [13] for the single-valued case extend to the multivalued one (cf. [24]), and, until now, for the mentioned behaviour there exist no more than some partial results obtained in [2, 19, 24].

We now give the detailed plan of the paper. In Section 2, for a multivalued linear operator A having domain $\mathscr{D}(A)$ and satisfying (2), we introduce the corresponding generated semigroup $\{e^{tA}\}_{t\ge 0}$. This leads us to define also the linear bounded operators $[(-A)^{\theta}]^{\circ} e^{tA}$, $\Re e\,\theta \ge 0$, $t > 0$, ($[(-A)^{0}]^{\circ} e^{tA} = e^{tA}$) and to recall the fundamental estimates for their $\mathscr{L}(X)$-norm. For the operators $[(-A)^{\theta}]^{\circ} e^{tA}$ a semigroup type property is proven in Proposition 1. We then introduce the spaces we will deal with in this paper, that is, the interpolation spaces $(X, \mathscr{D}(A))_{\gamma,p}$ and the spaces $X_A^{\gamma,p}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. Special attention is given to the embeddings linking these two classes of spaces which, in general, are equivalent only in the case $\beta = 1$. Some relations existing between the spaces $X_A^{\gamma,p}$ for different values of γ and p are proven in Proposition 2 and discussed in Remarks 3–5. We conclude the section recalling the estimates proven in [19, 24] for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})}$, $\Re e\,\theta \ge 0$, and $\|[(-A)^{1}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. In Remarks 7 and 8 we explain why, unless we renounce to optimality, in the case $\beta < 1$ these estimates can not be directly extended to the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;X_A^{\gamma,p})}$ and $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re e\,\theta \ge 1$, respectively.

In Section 3, we investigate the behaviour of the operators $[(-A)^{\theta}]^{\circ} e^{tA}$ with respect to both of the spaces $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. First, in Proposition 9, we deal with the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;X_A^{\gamma,p})}$, $\Re e\,\theta \ge 0$, and we show that, except for replacing $(X,\mathscr{D}(A))_{\gamma,p}$ with $X_A^{\gamma,\infty}$ if $p = \infty$ and with $X_A^{\beta\gamma,p}$ if $p \in [1,\infty)$, the *same* estimates of [19] for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})}$ continue to hold. The second significant result is Proposition 12 where, extending those in [24] to values of θ other than one, we establish estimates for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re e\,\theta \ge 1$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. As a byproduct we deduce the basic Corollary 14, which in Section 5 will be a key tool in proving the equivalence between the following problem (3) and the fixed-point equation (179). The estimates in Proposition 12 are then merged together with those in [19] to achieve estimates for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p},(X,\mathscr{D}(A))_{\delta,p})}$, $\Re e\,\theta \ge 1$. In particular, two different estimates are obtained, if $\gamma + \delta < 1$ or not. For if $\gamma + \delta < 1$, then (cf. the proof of Proposition 16) we can take advantage of the reiteration theorem for interpolation spaces and obtain estimates that, unless $\beta = 1$, are better than those rougher estimates derived in the general case $\gamma, \delta \in (0,1)$ (see Remarks 17 and 18). We stress that if $\beta = 1$, $\theta \in \mathbf{N}$ and A is single-valued, then we restore the estimates in [9]. Finally, in Proposition 20, a combination of Propositions 9 and 12 yields the estimate for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X_A^{\gamma,p},X_A^{\delta,p})}$, $\Re e\,\theta \ge 1$. Since $\beta < 1$, the spaces $X_A^{\sigma,q}$ are, in general, only intermediate spaces between X and $\mathscr{D}(A)$ for $\sigma \in (0,\beta)$; here the reiteration theorem does not apply and a weaker result is obtained (cf. (101)–(103)).

The estimates of Section 3 are applied in Section 4 to study the time regularity of those operator functions Q_j, $j = 1,\ldots,6$, that we will need in Section 5. In particular (cf. formula (106)), we modify the definition of Q_2 in [20, Section 4] in order that it is well defined, at least when acting on functions $g \in C^{\delta}([0,T];X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$ (cf. Corollary 26). Consequently, operators Q_3 and Q_4 in [20] change too, and the new Q_5 and Q_6 should be introduced (cf. formulae (107)–(110)). The Hölder in time regularity of the Q_j's is characterized in Lemmas 22, 24, 30, and 32 and Propositions 29 and 36. The main feature of these results is to show that the loss of regularity produced by Q_2 and

Q_5 can be restored, in Q_3 and Q_6 respectively, employing the regularization property established in [20, Section 3] for a wide range of general convolution operators.

In Section 5 we analyze the maximal time regularity of the strict solutions v to the following class of degenerate integrodifferential equations in a complex Banach space X:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mv(t)) = [\lambda_0 M + L]\, v(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2} + f(t), \quad t \in I_T, \qquad Mv(0) = Mv_0. \tag{3}$$

Here, $I_T = [0,T]$, $\lambda_0 \in \mathbf{C}$, $n_1, n_2 \in \mathbf{N}$, $h_{i_2} : I_T \to \mathbf{C}$, $y_{i_2} \in X$, $i_2 = 1,\dots,n_2$, whereas, Z being another complex Banach space and $\mathscr{P} : Z\times X \to X$ being a bilinear bounded operator, $k_{i_1} : I_T \to Z$, and $\mathscr{K}(k_{i_1}, L_{i_1} v)(t) = \int_0^t \mathscr{P}(k_{i_1}(t-s), L_{i_1} v(s))\mathrm{d}s$, $i_1 = 1,\dots,n_1$. Of course, if $Z = \mathbf{C}$, then $\mathscr{P}$ may be the scalar multiplication in X. As M, L, and L_{i_1}, $i_1 = 1,\dots,n_1$, we take closed single-valued linear operators from X to itself, whose domains fulfill the relation $\mathscr{D}(L) \subseteq \bigcap_{i_1=1}^{n_1}[\mathscr{D}(M) \cap \mathscr{D}(L_{i_1})]$, and we require L to have a bounded inverse, allowing M to be *not* invertible. Hence, in general, $A = LM^{-1}$ is only a multivalued linear operator in X having domain $\mathscr{D}(A) = M(\mathscr{D}(L))$. Assuming that A satisfies (2) and that the data k_{i_1}, h_{i_2}, y_{i_2} and f, $i_l = 1,\dots,n_l$, $l = 1,2$, are suitably chosen, problem (3) is then reduced to an equivalent fixed point-equation for the new unknown $w = L(v - v_0)$, $v_0 \in \mathscr{D}(L)$. It is here that the results of Sections 3 and 4 play their role, leading us to Theorem 48. In that theorem, provided that $5\alpha + 2\beta > 6$, we will prove that if $k_{i_1} \in C^{\eta_{i_1}}(I_T;Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T;\mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, $Y^r_{\gamma_{i_2}} \in \{(X,\mathscr{D}(A))_{\gamma_{i_2},r}, X_A^{\gamma_{i_2},r}\}$, and $f \in C^{\mu}(I_T;X)$ for opportunely chosen η_{i_1}, σ_{i_2}, γ_{i_2}, and μ, $i_l = 1,\dots,n_l$, $l = 1,2$, then problem (3) has a unique strict solution $v \in C^{\tau}(I_T;\mathscr{D}(L))$ satisfying $v(0) = v_0$ and Lv, $\mathrm{d}Mv/\mathrm{d}t \in C^{\tau}(I_T;X)$, where $\tau = \min_{i_l=1,\dots,n_l,l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$ (cf. Remark 51). Section 5 concludes with applications of Theorem 48 to integral and nonintegral subcases of (3), (cf. Theorems 52–54 and 56). We stress that Theorem 48 repairs, generalizes, and improves [20, Theorems 5.6 and 5.7], where similar results were proven only for the case $(n_1, n_2, Y_\psi^p) = (1,1,X_A^{\psi,p})$ and under the stronger condition $3\alpha + 8\beta > 10$.

In Section 6, we give an application of Theorem 48 to a concrete case of problem (3) arising in the theory of heat conduction for materials with memory. In particular, we show how Theorem 48 characterizes the appropriate functional framework where to search for the solution of the inverse problem of recovering both v and the vector $(k_1,\dots,k_{r_1})$, $r_1 \le n_1$, in (3) with $(i_2, n_2) = (i_1, n_1)$ and $h_{i_1} = k_{i_1}$, $i_1 = 1,\dots,n_1$.

Finally, in the Appendix we explain how to amend [20, Theorems 5.6 and 5.7] in accordance to Theorem 48.

2. Multivalued Linear Operators, Singular Semigroups, and the Spaces $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$

Let X be a complex Banach space endowed with norm $\|\cdot\|_X$ and let $\mathscr{P}(X)$ be the collection of all the subsets of X. For a number $\lambda \in \mathbf{C}$ and elements $\mathscr{U},\mathscr{V},\mathscr{W} \in \mathscr{P}(X)\setminus\emptyset$, $\lambda\mathscr{U}$, and $\mathscr{V}+\mathscr{W}$ denote the subsets of X defined by $\{\lambda u : u \in \mathscr{U}\}$ and $\{v + w : v \in \mathscr{V}, w \in \mathscr{W}\}$, respectively. Then, a mapping A from X into $\mathscr{P}(X)$ is called a *multivalued linear operator* in X if its domain $\mathscr{D}(A) = \{x \in X : Ax \ne \emptyset\}$ is a linear subspace of X and A satisfies the following: (i) $Ax + Ay \subset A(x+y)$, for all $x, y \in \mathscr{D}(A)$; (ii) $\lambda Ax \subset A(\lambda x)$, for all $\lambda \in \mathbf{C}$, for all $x \in \mathscr{D}(A)$. From now on, the shortening m. l. will be always used for multivalued linear.

The set $\mathscr{R}(A) = \bigcup_{x\in\mathscr{D}(A)} Ax$ is called the range of A. If $\mathscr{R}(A) = X$, then A is said to be surjective. The following properties of a m. l. operator A are immediate consequences of its definition (cf. [1, Theorems 2.1 and 2.2]): (iii) $Ax+Ay = A(x+y)$, for all $x, y \in \mathscr{D}(A)$; (iv) $\lambda Ax = A(\lambda x)$, for all $\lambda \in \mathbf{C}\setminus\{0\}$, for all $x \in \mathscr{D}(A)$; (v) $A0$ is a linear subspace of X and $Ax = y + A0$ for any $y \in Ax$, $x \in \mathscr{D}(A)$. In particular, A is single-valued if and only if $A0 = \{0\}$.

If A is an m. l. operator in X, then its inverse A^{-1} is defined to be the operator having domain $\mathscr{D}(A^{-1}) = \mathscr{R}(A)$ such that $A^{-1}y = \{x \in \mathscr{D}(A) : y \in Ax\}$, $y \in \mathscr{D}(A^{-1})$. A^{-1} is an m. l. operator in X too, and $(A^{-1})^{-1} = A$. The set $A^{-1}0 = \{x \in \mathscr{D}(A) : 0 \in Ax\}$ is called the kernel of A and denoted by $\mathscr{N}(A)$. If $\mathscr{N}(A) = \{0\}$; that is, if A^{-1} is single-valued, then A is said to be injective. Observe that (v) yields $Ax = A0$ if and only if $x \in \mathscr{N}(A)$.

Given $\mathscr{U} \in \mathscr{P}(X)\setminus\emptyset$, we write $A(\mathscr{U}) = \bigcup_{u\in\mathscr{U}\cap\mathscr{D}(A)} Au$, so that, in particular, $A(X) = A(\mathscr{D}(A)) = \mathscr{R}(A)$. If A_j, $j = 1,2$ are m. l. operators in X and $\lambda \in \mathbf{C}$, then the scalar multiplication λA_1, the sum $A_1 + A_2$, and the product $A_1 A_2$ are defined by

$$\begin{gathered}
\mathscr{D}(\lambda A_1) = \mathscr{D}(A_1), \\
(\lambda A_1)x = \lambda A_1 x, \quad x \in \mathscr{D}(\lambda A_1), \\
\mathscr{D}(A_1 + A_2) = \mathscr{D}(A_1) \cap \mathscr{D}(A_2), \\
(A_1 + A_2)x = A_1 x + A_2 x, \quad x \in \mathscr{D}(A_1 + A_2), \\
\mathscr{D}(A_1 A_2) = \{x \in \mathscr{D}(A_2) : A_1(A_2 x) \ne \emptyset\}, \\
(A_1 A_2)x = A_1(A_2 x), \quad x \in \mathscr{D}(A_1 A_2),
\end{gathered} \tag{4}$$

where λA_1, $A_1 + A_2$ and $A_1 A_2$ are m. l. operators in X and $(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$.

Let A and B be m. l. operators in X. We write $A \subset B$ if $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $Ax \subseteq Bx$ for every $x \in \mathscr{D}(A)$. Clearly, $A \subset B \subset A$ if and only if $A = B$. If $A \subset B$ and $Ax = Bx$ for every $x \in \mathscr{D}(A)$, then B is called an extension of A. If a linear single-valued operator S has domain $\mathscr{D}(S) = \mathscr{D}(A)$ and $S \subset A$, that is, $Sx \in Ax$ for every $x \in \mathscr{D}(A)$, then S is called a section of A. With an arbitrary section S, it holds $Ax = Sx + A0$, $x \in \mathscr{D}(A)$, and $\mathscr{R}(A) = \mathscr{R}(S) + A0$, but this latter sum may or may not be direct (cf. [25, p. 14]). A method for constructing sections is provided in [25, Proposition I.5.2].

If X_j, $j = 1, 2$, are two complex Banach spaces, then the linear space of all bounded *single-valued* linear operators L from $X_1 = \mathcal{D}(L)$ to X_2 is denoted by $\mathcal{L}(X_1; X_2)$ ($\mathcal{L}(X_1)$ if $X_1 = X_2$) and it is equipped with the uniform operator norm $\|L\|_{\mathcal{L}(X_1;X_2)} = \sup_{\|x\|_{X_1}\le 1}\|Lx\|_{X_2} = \inf_{K\ge 0}\{\|Lx\|_{X_2} \le K\|x\|_{X_1} : x \in X_1\}$. Then the resolvent set $\rho(A)$ of a m. l. operator A is defined to be the set $\{z \in \mathbf{C} : (zI - A)^{-1} \in \mathcal{L}(X)\}$, with I being the identity operator in X. The basic properties of the resolvent set of single-valued linear operators hold the same for m. l. operators. First, if $\rho(A) \ne \emptyset$, then A is closed; that is, its graph $\{(x, y) \in X \times X : x \in \mathcal{D}(A), y \in Ax\}$ is closed (cf. [25, p. 43]). Further (cf. [1, Theorem 2.6]), $\rho(A)$ is an open set and the operator function $z \in \rho(A) \to (zI - A)^{-1} \in \mathcal{L}(X)$ is holomorphic. Finally (cf. [1, formula (2.1)]), the resolvent equation $(\lambda_2-\lambda_1)(\lambda_1 I-A)^{-1}(\lambda_2 I-A)^{-1} = (\lambda_1 I-A)^{-1}-(\lambda_2 I-A)^{-1}$, $\lambda_1, \lambda_2 \in \rho(A)$, is satisfied, too. Unlike the single-valued case, instead, for $z \in \rho(A)$ the following inclusions hold (cf. [1, Theorem 2.7]):

$$(zI - A)^{-1}A \subset z(zI - A)^{-1} - I \subset A(zI - A)^{-1}. \tag{5}$$

Then, in general, $z(zI - A)^{-1} - I$, $z \in \rho(A)$, is only a bounded section of the m. l. operator $A(zI - A)^{-1}$. Throughout this paper, we denote this bounded section by $A^\circ(zI - A)^{-1}$, but we warn the reader that here A° does not necessarily denote a section of A itself. Of course, if A is single-valued, then $A^\circ(zI - A)^{-1}$ reduces to $A(zI - A)^{-1}$. Notice that (5) implies that $(zI-A)^{-1}A$, $z \in \rho(A)$, is single-valued on $\mathcal{D}(A)$ and $(zI-A)^{-1}Ax = (zI - A)^{-1}y$ with any $y \in Ax$, $x \in \mathcal{D}(A)$. Another difference with the single-valued case is that for every $z \in \rho(A)$ it holds $\mathcal{N}((zI - A)^{-1}) = A0$. Indeed, $((zI - A)^{-1})^{-1}0 = (zI - A)0 = A0$. Therefore, in the m. l. case, $\{0\} \subsetneq \mathcal{N}((zI - A)^{-1})$, $z \in \rho(A)$. However (cf. [24, Lemma 2.1]), if $0 \in \rho(A)$, then $\mathcal{N}(A^\circ(zI - A)^{-1}) = \{0\}$, and, in addition, $x \notin A0$ if and only if $A^\circ(zI - A)^{-1}x \notin A0$, $z \in \rho(A)$. We also recall that for every $\lambda_1, \lambda_2 \in \rho(A)$ the following slight variants of the resolvent equation hold (cf. [24, Lemma 2.2]):

$$\begin{aligned}
&(\lambda_2 - \lambda_1)(\lambda_1 I - A)^{-1}A^\circ(\lambda_2 I - A)^{-1} \\
&\quad = A^\circ(\lambda_1 I - A)^{-1} - A^\circ(\lambda_2 I - A)^{-1}, \\
&(\lambda_2 - \lambda_1)A^\circ(\lambda_1 I - A)^{-1}(\lambda_2 I - A)^{-1} \\
&\quad = A^\circ(\lambda_1 I - A)^{-1} - A^\circ(\lambda_2 I - A)^{-1}.
\end{aligned} \tag{6}$$

In particular, if $0 \in \rho(A)$, then, since $A^\circ(0I - A)^{-1} = -I$, the first in (6) with $(\lambda_1, \lambda_2) = (0, \lambda)$ yields $\lambda(-A)^{-1}A^\circ(\lambda I-A)^{-1} = -I - A^\circ(\lambda I - A)^{-1} = -\lambda(\lambda I - A)^{-1}$; that is,

$$A^{-1}A^\circ(\lambda I - A)^{-1} = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A). \tag{7}$$

Let $(A, \mathcal{D}(A))$ be a m. l. operator in X satisfying the following resolvent condition:

(H1) $\rho(A)$ contains a region $\Sigma_\alpha = \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge -c(|\mathfrak{Im}\, z| + 1)^\alpha, \mathfrak{Im}\, z \in \mathbf{R}\}$, $\alpha \in (0, 1]$, $c > 0$, and for some exponent $\beta \in (0, \alpha]$ and constant $C > 0$ the following estimate holds:

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \le C(|\lambda| + 1)^{-\beta}, \quad \forall \lambda \in \Sigma_\alpha. \tag{8}$$

Introduce the family $\{e^{tA}\}_{t\ge 0} \in \mathcal{L}(X)$ defined by $e^{0A} = I$ and

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda I - A)^{-1}d\lambda, \quad t > 0, \tag{9}$$

where $\Gamma \subsetneq \Sigma_\alpha \setminus \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge 0\}$ is the contour parametrized by $\lambda = -c(|\eta| + 1)^\alpha + i\eta$, $\eta \in (-\infty, \infty)$. Then (cf. [1, pp. 360, 361]), $\{e^{tA}\}_{t\ge 0}$ is a semigroup on X, infinitely many times strongly differentiable for $t > 0$ with

$$D_t^k e^{tA} = \frac{1}{2\pi i}\int_\Gamma \lambda^k e^{t\lambda}(\lambda I - A)^{-1}d\lambda, \qquad t > 0, \quad k \in \mathbf{N} = \{1, 2, \ldots\}, \tag{10}$$

where $D_t^k = d^k/dt^k$. In general, no analyticity should be expected for e^{tA}. For if $\alpha < 1$ in (H1), then Σ_α does not contain any sector $\Lambda_{\omega+\pi/2} = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \omega + \pi/2\}$, $\omega \in (0, \pi/2)$, and [15, Theorem 5.3], which extends e^{tA} analytically to the sector Λ_ω containing the positive real axis, is not applicable. We stress that (9) and $\mathcal{N}((zI - A)^{-1}) = A0$, $z \in \rho(A)$, imply $A0 \subseteq \mathcal{N}(e^{tA})$ for every $t > 0$, whereas $\mathcal{N}(e^{0A}) = \mathcal{N}(I) = \{0\}$. Hence, if A is really an m. l. operator, then $\{0\} \subsetneq A0 \subseteq \bigcap_{t>0}\mathcal{N}(e^{tA})$. From the semigroup property it also follows that $\mathcal{N}(e^{t_0 A}) \subseteq \mathcal{N}(e^{t_1 A})$ for $t_1 \ge t_0 \ge 0$.

Now, for every $\theta \in \mathbf{C}$ such that $\mathfrak{Re}\,\theta \ge 0$ we set

$$\left[(-A)^\theta\right]^\circ e^{tA} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^\theta e^{t\lambda}(\lambda I - A)^{-1}d\lambda, \quad t > 0. \tag{11}$$

Here, for the multivalued function $(-\lambda)^\theta = e^{\theta \mathrm{Ln}(-\lambda)}$ we choose the principal branch holomorphic in the region $\mathbf{C} \setminus \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge 0\}$, where for principal branch we mean the principal determination $\ln|z| + i\arg(z)$ of $\mathrm{Ln}(z)$. We briefly recall the main properties of operators $[(-A)^\theta]^\circ e^{tA}$. Of course, $[(-A)^0]^\circ e^{tA} = e^{tA}$, $t > 0$. As shown in [26, p. 426], $[(-A)^k]^\circ e^{tA}$, $k \in \mathbf{N}$, $t > 0$, is a section of $(-A)^k e^{tA}$, so that from (10) we get

$$(-1)^k D_t^k e^{tA} = \left[(-A)^k\right]^\circ e^{tA} \subset (-A)^k e^{tA}, \quad t > 0, \ k \in \mathbf{N}. \tag{12}$$

Moreover (cf. [19, formula (22)] with $\theta \ge 0$ being replaced by $\mathfrak{Re}\,\theta \ge 0$), we get

$$\left[(-A)^\theta\right]^\circ e^{tA} - \left[(-A)^\theta\right]^\circ e^{sA} = -\int_s^t \left[(-A)^{\theta+1}\right]^\circ e^{\xi A}d\xi, \qquad \mathfrak{Re}\,\theta \ge 0, \quad 0 < s < t. \tag{13}$$

Finally, (H1) implies the following estimates (cf. [1, 24, Section 3]):

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathcal{L}(X)} \le \tilde{c}_{\alpha,\beta,\theta}t^{(\beta-\mathfrak{Re}\,\theta-1)/\alpha}, \quad \mathfrak{Re}\,\theta \ge 0, \ t > 0, \tag{14}$$

where the $\tilde{c}_{\alpha,\beta,\theta}$'s are positive constants depending on α, β, and θ. Thus, letting $\theta = 0$ in (14), we see that if $\beta \in (0,1)$, then the operator function $t \in (0,\infty) \to \mathrm{e}^{tA} \in \mathscr{L}(X)$ may be singular at the origin and the semigroup is not necessarily strongly continuous in the X-norm on the closure $\overline{\mathscr{D}(A)}$ of $\mathscr{D}(A)$ in X. Notice that if $\alpha + \beta > 1$, then the singularity is a *weak* one, in the sense that $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is integrable in norm in any interval $[0,\tau]$, $\tau > 0$. Further (cf. [24, Lemma 3.9]), if $\alpha + \beta > 1$, then $A0 = \bigcap_{t>0} \mathscr{N}(\mathrm{e}^{tA})$, and if $\alpha = 1$, then $A0 = \mathscr{N}(\mathrm{e}^{tA})$ for every $t > 0$.

Observe that $A0 \subseteq \mathscr{N}([(-A)^{\theta}]^{\circ}\mathrm{e}^{tA})$, $\Re\mathrm{e}\,\theta \geq 0$, $t > 0$, so that $A0 \subseteq \bigcap_{t>0} \mathscr{N}([(-A)^{\theta}]^{\circ}\mathrm{e}^{tA})$, $\Re\mathrm{e}\,\theta \geq 0$. The operators $[(-A)^{\theta}]^{\circ}\mathrm{e}^{tA}$ satisfy the following semigroup type property.

Proposition 1. *Let $\theta_j \in \mathbf{C}$, $\Re\mathrm{e}\theta_j \geq 0$, and let $t_j > 0$, $j = 1,2$. Then*

$$\left[(-A)^{\theta_1}\right]^{\circ} e^{t_1 A}\left[(-A)^{\theta_2}\right]^{\circ} e^{t_2 A} = \left[(-A)^{\theta_1+\theta_2}\right]^{\circ} e^{(t_1+t_2)A}. \tag{15}$$

Proof. First, the function $\lambda \in \rho(A) \to (-\lambda)^{\theta}\mathrm{e}^{t\lambda}(\lambda I - A)^{-1} \in \mathscr{L}(X)$ being holomorphic for every $\Re\mathrm{e}\,\theta \geq 0$ and $t > 0$, and the contour Γ in (11) with $(\theta, t) = (\theta_2, t_2)$ can be replaced with the contour $\Gamma' \subsetneq \Sigma_\alpha \setminus \{z \in \mathbf{C} : \Re\mathrm{e}\,z \geq 0\}$ parametrized by $\mu = -c'(|\eta|+1)^{\alpha} + i\eta$, $\eta \in (-\infty,\infty)$, $c' \in (0,c)$, and lies to the right of Γ. Then, for every $x \in X$, from the resolvent equation we obtain

$$\begin{aligned}
&\left[(-A)^{\theta_1}\right]^{\circ}\mathrm{e}^{t_1 A}\left[(-A)^{\theta_2}\right]^{\circ}\mathrm{e}^{t_2 A}x \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma} (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda} \\
&\qquad \times \left[\int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\lambda I - A)^{-1}(\mu I - A)^{-1}x\,\mathrm{d}\mu\right]\mathrm{d}\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma} (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda I - A)^{-1} \\
&\qquad \times \left[\left(\int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu - \lambda)^{-1}\mathrm{d}\mu\right)x\right]\mathrm{d}\lambda \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu I - A)^{-1} \\
&\qquad \times \left[\left(\int_{\Gamma} (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda - \mu)^{-1}\mathrm{d}\lambda\right)x\right]\mathrm{d}\mu.
\end{aligned} \tag{16}$$

Now, after having enclosed Γ and Γ' on the left with an arc Δ_R of the circle $\{z \in \mathbf{C} : |z + c'| = R\}$, $R > c - c'$, we apply the residue theorem and let R go to infinity. To this purpose, we observe that since the contours Γ and Γ' both lie in the half-plane $\{z \in \mathbf{C} : \Re\mathrm{e}\,z \leq -c'\}$, the arc Δ_R may be parametrized in polar coordinates by $\Re\mathrm{e}\,z = -c' + R\cos\varphi$, $\Im\mathrm{m}\,z = R\sin\varphi$, $\varphi \in (\pi/2, 3\pi/2)$. Then, for every $z \in \Delta_R$ we have

$$\begin{aligned}
\left|(-z)^{\theta}\mathrm{e}^{tz}\right| &= |z|^{\Re\mathrm{e}\,\theta}\mathrm{e}^{-\Im\mathrm{m}\,\theta\arg(-z)}\mathrm{e}^{t\Re\mathrm{e}\,z} \\
&\leq \left(R + c'\right)^{\Re\mathrm{e}\,\theta}\mathrm{e}^{(\pi/2)|\Im\mathrm{m}\,\theta|}\mathrm{e}^{-tc'}\mathrm{e}^{tR\cos\varphi}.
\end{aligned} \tag{17}$$

Since $t > 0$ and $\varphi \in (\pi/2, 3\pi/2)$, the right-hand side of the latter inequality goes to zero as R goes to infinity, so that $\lim_{R\to\infty, z\in\Delta_R}(-z)^{\theta}\mathrm{e}^{tz} = 0$ for every $\Re\mathrm{e}\,\theta \geq 0$ and $t > 0$. The residue theorem together with the fact that Γ' lies to the right of Γ thus yields $\int_{\Gamma'}(-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu-\lambda)^{-1}\mathrm{d}\mu = 2\pi i(-\lambda)^{\theta_2}\mathrm{e}^{t_2\lambda}$ and $\int_{\Gamma}(-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda-\mu)^{-1}\mathrm{d}\lambda = 0$. Replacing these identities in (16) and using the equality $(-\lambda)^{\theta_1}(-\lambda)^{\theta_2} = (-\lambda)^{\theta_1+\theta_2}$ which is satisfied for the principal branch of the function $(-\lambda)^{\theta} = \mathrm{e}^{\theta\mathrm{Ln}(-\lambda)}$, we finally find

$$\begin{aligned}
&\left[(-A)^{\theta_1}\right]^{\circ}\mathrm{e}^{t_1 A}\left[(-A)^{\theta_2}\right]^{\circ}\mathrm{e}^{t_2 A}x \\
&\quad = \frac{1}{2\pi i}\int_{\Gamma} (-\lambda)^{\theta_1+\theta_2}\mathrm{e}^{(t_1+t_2)\lambda}(\lambda I - A)^{-1}x\,\mathrm{d}\lambda.
\end{aligned} \tag{18}$$

The right-hand side being precisely $[(-A)^{\theta_1+\theta_2}]^{\circ}\mathrm{e}^{(t_1+t_2)A}x$, the proof is complete. □

For an m. l. operator A satisfying (H1) we introduce now the spaces $(X, \mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. We first specify a topology on $\mathscr{D}(A)$ equipping it with the norm $\|x\|_{\mathscr{D}(A)} = \inf_{y\in Ax}\|y\|_X$, $x \in \mathscr{D}(A)$. Since $A^{-1} \in \mathscr{L}(X)$, this norm is equivalent to the graph norm and makes $\mathscr{D}(A)$ a complex Banach space (cf. [2, Proposition 1.11]). As X_1 and X_2 being given normed complex linear spaces, we will write $X_1 \hookrightarrow X_2$ if $X_1 \subseteq X_2$ and there exists a positive constant C_0 such that $\|x\|_{X_2} \leq C_0\|x\|_{X_1}$ for every $x \in X_1$. If $X_1 \hookrightarrow X_2 \hookrightarrow X_1$, that is, if $X_1 = X_2$ and the norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ are equivalent, then we will write $X_1 \cong X_2$. Of course, $\mathscr{D}(A)$ with the norm $\|\cdot\|_{\mathscr{D}(A)}$ satisfies $\mathscr{D}(A) \hookrightarrow X$. In fact, if $x \in \mathscr{D}(A)$, then for every $y \in Ax$ we have $x = A^{-1}y$, so that $\|x\|_X \leq \|A^{-1}\|_{\mathscr{L}(X)}\|y\|_X \leq C\|y\|_X$. Taking the infimum with respect to $y \in Ax$, we thus find $\|x\|_X \leq C\|x\|_{\mathscr{D}(A)}$ for every $x \in \mathscr{D}(A)$. If Y is a Banach space, we denote by $C((0,\infty);Y)$ the set of all continuos functions from $(0,\infty)$ to Y, and for a Y-valued strongly measurable function $g(\xi)$, $\xi \in (0,\infty)$, we set $\|g(\xi)\|_{L_q^*(Y)} = (\int_0^\infty \|g(\xi)\|_Y^q(\mathrm{d}\xi/\xi))^{1/q}$, $q \in [1,\infty)$, and $\|g(\xi)\|_{L_\infty^*(Y)} = \sup_{\xi\in(0,\infty)}\|g(\xi)\|_Y$. Let $p_0, p_1 \in [1,\infty)$ or let $p_0 = p_1 = \infty$, and for $\gamma \in (0,1)$ define $p^{-1} = (1-\gamma)p_0^{-1} + \gamma p_1^{-1}$ if $p_0, p_1 \in [1,\infty)$ and $p = \infty$ if $p_0 = p_1 = \infty$. Let us set

$$\begin{aligned}
&(X, \mathscr{D}(A))_{\gamma,p} \\
&\quad = \Big\{x \in X : x = v_0(\xi) + v_1(\xi), \xi \in (0,\infty), \\
&\qquad v_0 \in C((0,\infty);X), v_1 \in C((0,\infty);\mathscr{D}(A)), \\
&\qquad \left\|\xi^{\gamma}v_0(\xi)\right\|_{L_{p_0}^*(X)} + \left\|\xi^{\gamma-1}v_1(\xi)\right\|_{L_{p_1}^*(\mathscr{D}(A))} < \infty\Big\}, \\
&\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \\
&\quad = \inf_{v_0,v_1}\left\{\left\|\xi^{\gamma}v_0(\xi)\right\|_{L_{p_0}^*(X)} + \left\|\xi^{\gamma-1}v_1(\xi)\right\|_{L_{p_1}^*(\mathscr{D}(A))}\right\}.
\end{aligned} \tag{19}$$

This characterization of the spaces $(X, \mathscr{D}(A))_{\gamma,p}$ is that obtained by the so-called "mean-methods", and it is equivalent to that performed by the "K-method" (cf. [27, Theorem 1.5.2 and Remark 1.5.2/2]) and the "trace-method" (cf. [27, Theorem 1.8.2]). Then, due to [27, Theorem 1.3.3], for every $\gamma \in (0,1)$ and $p \in [1,\infty]$ the space $(X,\mathscr{D}(A))_{\gamma,p}$ is an exact real interpolation space of exponent γ between X and $\mathscr{D}(A)$. Observe that by exchanging the role of X and $\mathscr{D}(A)$ and performing the transformation $\xi = \tau^{-1}$, we get $(X,\mathscr{D}(A))_{\gamma,p} = (\mathscr{D}(A),X)_{1-\gamma,p}$. Also, if $\mathscr{D}(A) = X$, then $(X,\mathscr{D}(A))_{\gamma,p} \cong X$ (cf. [27, Theorem 1.3.3(f)]). The definition of the spaces $(X,\mathscr{D}(A))_{\gamma,p}$ is meaningful even for the limiting cases $(\gamma,p) = (i,\infty)$, $i = 0,1$, whereas $(X,\mathscr{D}(A))_{i,p}$, $i = 0,1$, $p \in [1,\infty)$, reduces to the zero element of X. In particular (cf. [28, pp. 10–15]), denoting by $\widetilde{Y}^X$ the completion of $\mathscr{D}(A)$ relative to X and endowing it with the norm $\|\cdot\|_{\widetilde{Y}^X}$ in [28, p. 14], we get $(X,\mathscr{D}(A))_{0,\infty} \cong X$ and $(X,\mathscr{D}(A))_{1,\infty} \cong \widetilde{Y}^X$. Let $\gamma_1 \in (0,1)$ and let $p_j \in [1,\infty]$, $j = 1,2$. Then, for $\gamma_2 \in (0,\gamma_1)$ and $q_j \in [1,p_j]$, $j = 1,2$, the following chain of embeddings holds:

$$\begin{aligned}\mathscr{D}(A) &\hookrightarrow (X,\mathscr{D}(A))_{1,\infty} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,1}\\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,q_1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,p_1}\\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,q_2}\\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,p_2} \hookrightarrow \overline{\mathscr{D}(A)}.\end{aligned} \tag{20}$$

Let $\gamma \in [0,1]$. Recall that a Banach space E is said to be of class $J(\gamma,X,\mathscr{D}(A)) \cap K(\gamma,X,\mathscr{D}(A))$ and shortened to $E \in J(\gamma) \cap K(\gamma)$, if E is an intermediate space between $(X,\mathscr{D}(A))_{\gamma,\infty}$ and $(X,\mathscr{D}(A))_{\gamma,1}$, that is, if $(X,\mathscr{D}(A))_{\gamma,1} \hookrightarrow E \hookrightarrow (X,\mathscr{D}(A))_{\gamma,\infty}$. From (20) it thus follows that $(X,\mathscr{D}(A))_{\gamma,p} \in J(\gamma) \cap K(\gamma)$, for every $\gamma \in (0,1)$ and $p \in [1,\infty]$. Moreover, since $(X,\mathscr{D}(A))_{i,1} = \{0\}$, $i = 0,1$, and $(X,\mathscr{D}(A))_{0,\infty} \cong X$, we have $\mathscr{D}(A) \in J(1)\cap K(1)$ and $X \in J(0)\cap K(0)$. Then (cf. [28, p. 12], [27, Theorem 1.10.2], and [9, Section 1.2.3]), for $\gamma_j \in (0,1)$ and $p_j \in [1,\infty]$, $j = 0,1,2$, the reiteration theorem yields

$$\begin{gathered}\left((X,\mathscr{D}(A))_{\gamma_1,p_1},(X,\mathscr{D}(A))_{\gamma_2,p_2}\right)_{\gamma_0,p_0}\\ \cong (X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1+\gamma_0\gamma_2,p_0},\\ \left((X,\mathscr{D}(A))_{\gamma_1,p_1},\mathscr{D}(A)\right)_{\gamma_0,p_0} \cong (X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1+\gamma_0,p_0},\\ \left(X,(X,\mathscr{D}(A))_{\gamma_2,p_2}\right)_{\gamma_0,p_0} \cong (X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0}.\end{gathered} \tag{21}$$

Finally (cf. [29, Theorem 1.II and Remark 1.III]), we recall that if X_1 and X_2 are two complex Banach spaces and $T \in \mathscr{L}(X_1;X_2)$ is such that $T \in \mathscr{L}(Y_{1_k};Y_{2_k})$, $Y_{j_k} \subseteq X_j$, $j,k = 1,2$, then $T \in \mathscr{L}((Y_{1_1},Y_{1_2})_{\gamma_0,p_0};(Y_{2_1},Y_{2_2})_{\gamma_0,p_0})$, $\gamma_0 \in (0,1)$, $p_0 \in [1,\infty]$, and

$$\|T\|_{\mathscr{L}((Y_{1_1},Y_{1_2})_{\gamma_0,p_0};(Y_{2_1},Y_{2_2})_{\gamma_0,p_0})} \le \|T\|^{1-\gamma_0}_{\mathscr{L}(Y_{1_1};Y_{2_1})}\|T\|^{\gamma_0}_{\mathscr{L}(Y_{1_2};Y_{2_2})}. \tag{22}$$

As a consequence of this general result and the identity

$$\left((X,\mathscr{D}(A))_{\gamma_1,p_1},X\right)_{\gamma_0,p_0} = \left(X,(X,\mathscr{D}(A))_{\gamma_1,p_1}\right)_{1-\gamma_0,p_0}, \tag{23}$$

from the third in (21) we find that if $T \in \mathscr{L}(X)$ is such that $T \in \mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma_1,p_1})$ and $T \in \mathscr{L}((X,\mathscr{D}(A))_{\gamma_2,p_2};X)$, then $T \in \mathscr{L}((X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0};(X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1,p_0})$, $\gamma_j \in (0,1)$, $p_j \in [1,\infty]$, $j = 0,1,2$, and the following estimate holds:

$$\begin{aligned}&\|T\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0};(X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1,p_0})}\\ &\quad\le \|T\|^{1-\gamma_0}_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma_1,p_1})}\|T\|^{\gamma_0}_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma_2,p_2};X)}.\end{aligned} \tag{24}$$

Notice that here $\gamma_0\gamma_2+(1-\gamma_0)\gamma_1 \in (\min\{\gamma_1,\gamma_2\},\max\{\gamma_1,\gamma_2\}) \subsetneq (0,1)$ for every $\gamma_0 \in (0,1)$. Therefore, if we let $\gamma = \gamma_0\gamma_2$ and let $\delta = (1-\gamma_0)\gamma_1$, then $\gamma+\delta < 1$, $\gamma_1 = \delta/(1-\gamma_0) > \delta$, and $\gamma_2 = \gamma/\gamma_0 > \gamma$. Hence, in order that the additional inequalities $\gamma_j < 1$, $j = 1,2$, are satisfied, we have to choose $\gamma_0 \in (\gamma, 1-\delta)$. As we will see this simple observation will be the key for the proof of the second estimates (90) in the following Proposition 16.

We recall that for every fixed $x \in \mathscr{D}(A)$ the map $T(\lambda) = \lambda x$ satisfies $\|T\|_{\mathscr{L}(\mathbf{C};X)} = \|x\|_X$, $\|T\|_{\mathscr{L}(\mathbf{C};\mathscr{D}(A))} = \|x\|_{\mathscr{D}(A)}$ and $\|T\|_{\mathscr{L}(\mathbf{C},(X,\mathscr{D}(A))_{\gamma,p})} = \|x\|_{(X,\mathscr{D}(A))_{\gamma,p}}$. Then (22) with $X_1 = Y_{1_1} = Y_{1_2} = \mathbf{C}$, $X_2 = Y_{2_1} = X$ and $Y_{2_2} = \mathscr{D}(A)$ yields the interpolation inequality:

$$\begin{gathered}\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \le c_0\|x\|^{1-\gamma}_X\|x\|^{\gamma}_{\mathscr{D}(A)},\\ x \in \mathscr{D}(A), \quad \gamma \in (0,1), \quad p \in [1,\infty],\end{gathered} \tag{25}$$

with c_0 being the positive constant depending on γ and p such that $\|\lambda\|_{(\mathbf{C},\mathbf{C})_{\gamma,p}} \le c_0|\lambda|$.

As another application of (22) and for further needs, we also recall that if A satisfies (H1), then $A^\circ(zI-A)^{-1}$ satisfies the estimate (cf. [24, formulae (4.16) and (4.17)]).

Consider

$$\begin{gathered}\left\|A^\circ(zI-A)^{-1}\right\|_{\mathscr{L}(X)} \le (C+1)(|z|+1)^{1-\beta}, \quad \forall z \in \Sigma_\alpha,\\ \left\|A^\circ(zI-A)^{-1}\right\|_{\mathscr{L}(\mathscr{D}(A);X)} \le C(|z|+1)^{-\beta}, \quad \forall z \in \Sigma_\alpha.\end{gathered} \tag{26}$$

From (26), using (22) with $X_j = Y_{j_1} = Y_{2_2} = X$, $j = 1,2$, and $Y_{1_2} = \mathscr{D}(A)$, it then follows for every $\gamma \in (0,1)$ and $p \in [1,\infty]$

$$\begin{aligned}&\left\|A^\circ(zI-A)^{-1}\right\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};X)}\\ &\quad\le c_1(C+1)^{1-\gamma}C^{\gamma}(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha,\end{aligned} \tag{27}$$

where c_1 is the positive constant depending on γ and p such that $\|x\|_X \le c_1\|x\|_{(X,X)_{\gamma,p}}$.

For $\gamma \in (0,1)$ and $p \in [1,\infty]$ we now define the Banach spaces $X_A^{\gamma,p}$ by

$$\begin{gathered}X_A^{\gamma,p} = \left\{x \in X : [x]_{X_A^{\gamma,p}} := \left\|\xi^\gamma A^\circ(\xi I-A)^{-1}x\right\|_{L^*_p(X)} < \infty\right\},\\ \|x\|_{X_A^{\gamma,p}} = \|x\|_X + [x]_{X_A^{\gamma,p}}.\end{gathered} \tag{28}$$

It is a well-known fact that if A is single-valued and $\beta = 1$ in (H1), then $(X,\mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$ (cf. [30, Theorem 3.1] and [27, Theorem 1.14.2]). On the contrary, if $\beta \in (0,1)$, then such an equivalence is no longer true, as first observed in [13, Theorem 2] for single-valued operators and, in the case $p = \infty$, in [2, Theorem 1.12] for the m. l. ones. Recently, extending [13] to m. l. operators and [2] to $p \in [1,\infty]$, in [24, Proposition 4.3] it has been shown that the following embedding relations hold:

$$X_A^{\gamma,p} \hookrightarrow (X, \mathscr{D}(A))_{\gamma,p}, \quad \gamma \in (0,1), \ p \in [1,\infty], \tag{29}$$

$$(X, \mathscr{D}(A))_{\gamma,p} \hookrightarrow X_A^{\gamma+\beta-1,p}, \quad \gamma \in (1-\beta, 1), \ p \in [1,\infty]. \tag{30}$$

Then, as in the single-valued case, $(X, \mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$ if $\beta = 1$ in (H1). More precisely (see the proof of [24, Proposition 4.3]), if $x \in X_A^{\gamma,p}$, $\gamma \in (0,1)$, $p \in [1,\infty]$, then

$$\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq 2\|x\|_{X_A^{\gamma,p}}, \tag{31}$$

whereas if $x \in (X, \mathscr{D}(A))_{\gamma,p}$, $\gamma \in (1-\beta, 1)$, $p \in [1,\infty]$, then

$$\|x\|_{X_A^{\gamma+\beta-1,p}} \leq c_2 \|x\|_{(X,\mathscr{D}(A))_{\gamma,p}}, \tag{32}$$

with c_2 being a positive constant depending on β, γ and p.

By setting $\delta = \gamma + \beta - 1$, $\gamma \in (1-\beta, 1)$, from (30) it follows

$$\begin{gathered} \mathscr{D}(A) \hookrightarrow (X, \mathscr{D}(A))_{1+\delta-\beta,p} \hookrightarrow X_A^{\delta,p} \hookrightarrow X, \\ \delta \in (0,\beta), \quad p \in [1,\infty]. \end{gathered} \tag{33}$$

Then, if $\beta \in (0,1)$, the spaces $X_A^{\delta,p}$, $\delta \in (0,1)$, $p \in [1,\infty]$, are intermediate spaces between X and $\mathscr{D}(A)$ only for $\delta \in (0,\beta)$, whereas, when $\delta \in [\beta, 1)$, they may be smaller than $\mathscr{D}(A)$. In any case, when $\beta \in (0,1)$, it is not known if the spaces $X_A^{\delta,p}$, $\delta \in (0,\beta)$, $p \in [1,\infty]$, are only intermediate or just interpolation spaces between X and $\mathscr{D}(A)$.

Notice that $[X_A^{\gamma,p} \cap A0] = \{0\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. Indeed, assume that there exists $x \neq 0$ such that $x \in [X_A^{\gamma,p} \cap A0]$ for some $\gamma \in (0,1)$ and $p \in [1,\infty]$. Then, since $x \in A0 = \mathscr{N}((zI - A)^{-1})$, $z \in \rho(A)$, we have $A^\circ(\xi I - A)^{-1}x = \xi(\xi - A)^{-1}x - x = -x$ for every $\xi > 0$ and $[x]_{X_A^{\gamma,p}} = \|\xi^\gamma\|_{L_p^*(X)}\|x\|_X = \infty$, contradicting $x \in X_A^{\gamma,p}$. This property plays a key role in the proof of many of the results in [24]. Further, due to (30), it implies that $[\mathscr{D}(A) \cap A0] = [(X, \mathscr{D}(A))_{\gamma,p} \cap A0] = \{0\}$, $\gamma \in (1-\beta, 1)$, $p \in [1,\infty]$. On the contrary, since $\{0\}$ may be a proper subset of $[(X, \mathscr{D}(A))_{\gamma,p} \cap A0]$ for $\gamma \in (0, 1-\beta]$, $\beta < 1$, in general it is not true that $[\overline{\mathscr{D}(A)} \cap A0] = \{0\}$. This is true, instead, if $\beta = 1$. In this case the topological direct sum $X_0 = \overline{\mathscr{D}(A)} \oplus A0$ is a closed subspace of X, and if X is reflexive, it coincides with the whole X (cf. [3, Theorems 2.4 and 2.6]).

For every $\gamma \in (0,1)$ and $p \in [1,\infty]$ from (27), (29), and (31) it follows

$$\begin{aligned} &\left\|A^\circ(zI - A)^{-1}\right\|_{\mathscr{L}(X_A^{\gamma,p};X)} \\ &\quad \leq 2c_1(C+1)^{1-\gamma}C^\gamma(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha. \end{aligned} \tag{34}$$

Hence, for $\gamma \in (0,1)$ and $p \in [1,\infty]$ we may rewrite (27) and (34) more compactly as

$$\left\|A^\circ(zI - A)^{-1}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_3(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha, \tag{35}$$

where $Y_\gamma^p \in \{(X, \mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$ and c_3 is equal to $c_1(C+1)^{1-\gamma}C^\gamma$ or $2c_1(C+1)^{1-\gamma}C^\gamma$ according that $Y_\gamma^p = (X, \mathscr{D}(A))_{\gamma,p}$ or $Y_\gamma^p = X_A^{\gamma,p}$.

With the exception of the case $\beta = 1$, in general it is not clear if embeddings analogous to (20) hold even for the spaces $X_A^{\gamma,p}$. In fact, using (20), (29), and (30) we can only prove that if $\gamma \in (1-\beta, 1)$ and $1 \leq q \leq p \leq \infty$, then

$$X_A^{\gamma,q} \hookrightarrow (X, \mathscr{D}(A))_{\gamma,q} \hookrightarrow (X, \mathscr{D}(A))_{\gamma,q} \hookrightarrow X_A^{\gamma+\beta-1,p}, \tag{36}$$

whereas if $1 - \beta < \gamma_2 < \gamma_1 < 1$ and $p_1, p_2 \in [1,\infty]$, then

$$X_A^{\gamma_1,p_1} \hookrightarrow (X, \mathscr{D}(A))_{\gamma_1,p_1} \hookrightarrow (X, \mathscr{D}(A))_{\gamma_2,p_2} \hookrightarrow X_A^{\gamma_2+\beta-1,p_2}. \tag{37}$$

What can be proved without invoking (20), (29), and (30) and using only the definition of the norm $\|\cdot\|_{X_A^{\gamma,p}}$ is instead the following result, which extends to the spaces $X_A^{\gamma,p}$ the embeddings $(X, \mathscr{D}(A))_{\gamma_1,p} \hookrightarrow (X, \mathscr{D}(A))_{\gamma_2,p}$, and $(X, \mathscr{D}(A))_{\gamma_1,\infty} \hookrightarrow (X, \mathscr{D}(A))_{\gamma_2,p}$, $0 < \gamma_2 < \gamma_1 < 1$, $p \in [1,\infty]$ (cf. (20) with $(p_1, p_2) = (p, p)$ and $(p_1, p_2) = (\infty, p)$).

Proposition 2. *Let A be an m. l. operator satisfying the resolvent condition (H1). Then the following embeddings hold for every $0 < \gamma_2 < \gamma_1 < 1$ and $p \in [1,\infty]$:*

$$X_A^{\gamma_1,p} \hookrightarrow X_A^{\gamma_2,p}, \tag{38}$$

$$X_A^{\gamma_1,\infty} \hookrightarrow X_A^{\gamma_2,p}. \tag{39}$$

Proof. If $\beta = 1$ in (H1), then there is nothing to prove since $(X, \mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$ and both (38) and (39) follow from (20). Therefore, without loss of generality, we assume that $\beta \in (0, \alpha]$ is such that $\beta < \alpha$ if $\alpha = 1$. We begin by proving (38). Let first $p \in [1,\infty)$. For every $x \in X_A^{\gamma_1,p}$, $0 < \gamma_2 < \gamma_1 < 1$, we write

$$[x]_{X_A^{\gamma_2,p}}^p = I_1 + I_2, \tag{40}$$

where

$$I_j = \int_{a_j}^{b_j} \left\|\xi^{\gamma_2} A^\circ(\xi I - A)^{-1}x\right\|_X^p \frac{d\xi}{\xi}, \quad j = 1, 2, \tag{41}$$

$(a_1, b_1, a_2, b_2) = (0, 1, 1, \infty)$. Using the first inequality in (26) we find

$$\begin{aligned} I_1 &\leq (C+1)^p \|x\|_X^p \int_0^1 \xi^{\gamma_2 p - 1}(\xi+1)^{(1-\beta)p} d\xi \\ &\leq 2^{(1-\beta)p}(C+1)^p \|x\|_X^p \int_0^1 \xi^{\gamma_2 p - 1} d\xi \leq \left[c_4 \|x\|_{X_A^{\gamma_1,p}}\right]^p, \end{aligned} \tag{42}$$

where $c_4 = 2^{1-\beta}(C+1)(\gamma_2 p)^{-1/p}$. Concerning I_2, instead, using $\gamma_2 - \gamma_1 < 0$, we get

$$\begin{aligned} I_2 &= \int_1^\infty \xi^{(\gamma_2-\gamma_1)p} \left\|\xi^{\gamma_1} A^\circ(\xi I - A)^{-1}x\right\|_X^p \frac{d\xi}{\xi} \\ &\leq \int_1^\infty \left\|\xi^{\gamma_1} A^\circ(\xi I - A)^{-1}x\right\|_X^p \frac{d\xi}{\xi} \leq [x]_{X_A^{\gamma_1,p}}^p \leq \|x\|_{X_A^{\gamma_1,p}}^p. \end{aligned} \tag{43}$$

Summing up (40)–(43) and setting $c_5 = [(c_4)^p + 1]^{1/p}$, it thus follows $\|x\|_{X_A^{\gamma_2,p}} = \|x\|_X + [x]_{X_A^{\gamma_2,p}} \le (1+c_5)\|x\|_{X_A^{\gamma_1,p}}$, completing the proof of (38) in the case $p \in [1,\infty)$. Let $p = \infty$. For every $x \in X_A^{\gamma_1,\infty}$, $0 < \gamma_2 < \gamma_1 < 1$, we write

$$[x]_{X_A^{\gamma_2,\infty}} = \max\{I_3, I_4\}, \tag{44}$$

where $I_j = \sup_{\xi\in U_j}\|\xi^{\gamma_2}A^\circ(\xi I - A)^{-1}x\|_X$, $j = 3,4$, $U_3 = (0,1)$, $U_4 = [1,\infty)$. Again, the first inequality in (26) yields

$$\begin{aligned} I_3 &\le (C+1)\|x\|_X \sup_{\xi\in(0,1)}\left[\xi^{\gamma_2}(\xi+1)^{1-\beta}\right] \\ &\le 2^{1-\beta}(C+1)\|x\|_{X_A^{\gamma_1,\infty}}. \end{aligned} \tag{45}$$

Instead, using $\gamma_2 - \gamma_1 < 0$, we have

$$\begin{aligned} I_4 &= \sup_{\xi\in[1,\infty)} \xi^{\gamma_2-\gamma_1}\left\|\xi^{\gamma_1}A^\circ(\xi I - A)^{-1}x\right\|_X \le [x]_{X_A^{\gamma_1,\infty}} \\ &\le \|x\|_{X_A^{\gamma_1,\infty}}. \end{aligned} \tag{46}$$

Summing up (44)–(46) and setting $c_6 = 2^{1-\beta}(C+1)$, we thus find $\|x\|_{X_A^{\gamma_2,\infty}} = \|x\|_X + [x]_{X_A^{\gamma_2,\infty}} \le (1+c_6)\|x\|_{X_A^{\gamma_1,\infty}}$. This completes the proof of (38) for the case $p = \infty$. We now prove (39). Due to (38) with $p = \infty$, it suffices to assume that $p \in [1,\infty)$. As above, for every $x \in X_A^{\gamma_1,\infty}$, $0 < \gamma_2 < \gamma_1 < 1$, we write $[x]^p_{X_A^{\gamma_2,p}} = I_1 + I_2$, where I_1 and I_2 are defined by (41). Hence, the same computations as in (42) yield

$$I_1 \le \left[c_4\|x\|_{X_A^{\gamma_1,\infty}}\right]^p. \tag{47}$$

As far as I_2 is concerned, instead, we have

$$I_2 \le [x]^p_{X_A^{\gamma_1,\infty}} \int_1^\infty \xi^{(\gamma_2-\gamma_1)p-1}\mathrm{d}\xi \le \left[c_7\|x\|_{X_A^{\gamma_1,\infty}}\right]^p, \tag{48}$$

where $c_7 = [(\gamma_1-\gamma_2)p]^{-1/p}$. Summing up (47) and (48) and setting $c_8 = [(c_4)^p + (c_7)^p]^{1/p}$, we deduce $\|x\|_{X_A^{\gamma_2,p}} \le (1 + c_8)\|x\|_{X_A^{\gamma_1,\infty}}$. The proof is complete. □

Remark 3. Notice that (37) with $p_1 = p_2 = p$ yields $X_A^{\gamma_1,p} \hookrightarrow X_A^{\gamma_2+\beta-1,p}$, $1-\beta < \gamma_2 < \gamma_1 < 1$, and this latter embedding is less accurate than (38).

Remark 4. The main problem for extending (20) to the spaces $X_A^{\gamma,p}$ in the case $\beta < 1$ is that it is not clear if it holds $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $1 \le q < p \le \infty$. In fact, the embedding

$$\begin{gathered} (X,\mathscr{D}(A))_{\gamma,q} \hookrightarrow (X,\mathscr{D}(A))_{\gamma,q}, \\ \gamma\in(0,1), \quad 1\le q<p\le\infty, \end{gathered} \tag{49}$$

is a consequence of the property of the functional K entering the definition of the interpolation spaces $(X,\mathscr{D}(A))_{\gamma,p}$ through the "K-method", and in particular of its monotonicity (see the proof of [27, Theorem 1.3.3(c), (d)]). With embedding (49) at hands, to derive (20) it thus suffices to prove that $(X,\mathscr{D}(A))_{\gamma_1,\infty} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,1}$, $0 < \gamma_2 < \gamma_1 < 1$ (see the proof of [27, Theorem 1.3.3(e)] taking there $(A_0, A_1, \theta, \widetilde{\theta}) = (\mathscr{D}(A), X, 1-\gamma_1, 1-\gamma_2)$ and using $(\mathscr{D}(A), X)_{1-\gamma,p} = (X,\mathscr{D}(A))_{\gamma,p}$). If we try to repeat the proof of (49) for the spaces $X_A^{\gamma,p}$, we will be faced with two problems. The first is that we do not know if the function $g(\xi) = \|A^\circ(\xi I - A)^{-1}x\|_X$, $\xi\in(0,\infty)$, $x \in X$, is monotone decreasing, which would allow us to prove $X_A^{\gamma,p} \hookrightarrow X_A^{\gamma,\infty}$, $\gamma\in(0,1)$, $p\in[1,\infty)$. For if $g(\xi)$ was monotone decreasing, then for every $\xi\in(0,\infty)$ and $x \in X_A^{\gamma,p}$, $\gamma\in(0,1)$, $p\in[1,\infty)$, we would find

$$\begin{aligned} \xi^\gamma g(\xi) &= c_9\left(\int_0^\xi \mu^{\gamma p}\frac{\mathrm{d}\mu}{\mu}\right)^{1/p} g(\xi) \\ &\le c_9\left(\int_0^\xi [\mu^\gamma g(\mu)]^p \frac{\mathrm{d}\mu}{\mu}\right)^{1/p} \le c_9[x]_{X_A^{\gamma,p}}, \end{aligned} \tag{50}$$

where $c_9 = (\gamma p)^{-1/p}$. Taking the supremum with respect to $\xi\in(0,\infty)$ in the latter inequality, we would get $[x]_{X_A^{\gamma,\infty}} \le c_9[x]_{X_A^{\gamma,p}}$, proving $X_A^{\gamma,p} \hookrightarrow X_A^{\gamma,\infty}$, $\gamma\in(0,1)$, $p\in[1,\infty)$. The second problem is that the function $\xi^\gamma g(\xi)$ is not necessarily bounded for $x \in X_A^{\gamma,p}$, $\gamma\in(0,1)$, $p\in[1,\infty)$, precluding us to prove $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $\gamma\in(0,1)$, $q\in[1,p)$. Indeed, from (35) we can only find $\xi^\gamma g(\xi) \le c_3\xi^\gamma(\xi+1)^{1-\beta-\gamma}\|x\|_{X_A^{\gamma,p}}$, and when $\beta < 1$, the right-hand side of this inequality goes to infinity as ξ goes to infinity. On the contrary, if $\xi^\gamma g(\xi)$ were bounded, then for every $1 \le q < p < \infty$ we would obtain

$$\begin{aligned} [x]^p_{X_A^{\gamma,p}} &= \int_0^\infty [\xi^\gamma g(\xi)]^p\frac{\mathrm{d}\xi}{\xi} \\ &\le \left(\sup_{\xi\in(0,\infty)}\xi^\gamma g(\xi)\right)^{p-q}\int_0^\infty [\xi^\gamma g(\xi)]^q\frac{\mathrm{d}\xi}{\xi} \\ &= [x]^{p-q}_{X_A^{\gamma,\infty}}[x]^p_{X_A^{\gamma,p}}. \end{aligned} \tag{51}$$

If now in addition $g(\xi)$ were also monotone decreasing, in order that $[x]_{X_A^{\gamma,\infty}} \le c_9[x]_{X_A^{\gamma,q}}$, from the latter inequality we would get $[x]_{X_A^{\gamma,p}} \le (c_9)^{(p-q)/p}[x]_{X_A^{\gamma,q}}$, completing the proof of $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $\gamma\in(0,1)$, $1\le q<p<\infty$. Due to the former computations, we can thus conclude that in the case $\beta < 1$ the quoted problems are the main obstacles which prevent us to extend (49) and, as its consequence, (20) to the spaces $X_A^{\gamma,p}$.

Remark 5. Let $0 < \gamma_2 < \gamma_1 < 1$ be fixed and for every $p \in [1,\infty]$ and let us set $A_p = X_A^{\gamma_2,p}$ and $B_p = X_A^{\gamma_1,p}$. We thus have the two families of sets $\mathscr{A} = \{A_p\}_{p\in[1,\infty]}$ and $\mathscr{B} = \{B_p\}_{p\in[1,\infty]}$. Let first $\beta = 1$. In this case, since $(X,\mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$, from (20) we deduce that the sets A_p and B_p are related by the following inclusions in which $1 < q_1 < q_2 < \infty$:

$$B_1 \subseteq B_{q_1} \subseteq B_{q_2} \subseteq B_\infty \subseteq A_1 \subseteq A_{q_1} \subseteq A_{q_2} \subseteq A_\infty. \tag{52}$$

Now let $\beta < 1$. As observed in Remark 4, in this case the embedding $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $1 \le q < p \le \infty$, may be not satisfied and the chain of inclusions (52) could not take place. However, (38) and (39) hold true and for every $p \in [1,\infty]$, and we have $B_p \subseteq A_p$ and $B_\infty \subseteq A_p$.

We have already pointed out that $\{e^{tA}\}_{t\geq 0}$ may be not strongly continuous in the X-norm on $\overline{\mathscr{D}(A)}$. On the contrary, the following result (cf. [24, Proposition 5.2] for the proof) shows that the things are finer on $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. Later, we will need this fact.

Proposition 6. *Let A be as in Proposition 2. If $\gamma \in (1-\beta,1)$; then $\{e^{tA}\}_{t\geq 0}$ is strongly continuous in the X-norm on $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$ for every $p \in [1,\infty]$.*

We conclude the section listing some estimates for the operators $[(-A)^\theta]^\circ e^{tA}$ defined by (11) with respect to the spaces $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. First, in [19, Lemma 3.1] it is shown that $[(-A)^\theta]^\circ e^{tA}x \in \mathscr{D}(A)$ for every $x \in X$ and that the estimate $\|[(-A)^\theta]^\circ e^{tA}x\|_{\mathscr{D}(A)} \leq \|[(-A)^{\theta+1}]^\circ e^{tA}x\|_X$ is satisfied. Hence, using (14), we get

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;\mathscr{D}(A))} \leq \widetilde{c}_{\alpha,\beta,\theta+1} t^{(\beta-\Re\mathrm{e}\,\theta-2)/\alpha}, \qquad \Re\mathrm{e}\,\theta \geq 0, \quad t>0. \tag{53}$$

Combining (14) and (53) with (25) and letting $c_{10} = c_0 (c_{\alpha,\beta,\theta})^{1-\gamma}(c_{\alpha,\beta,\theta+1})^\gamma$, it thus follows (cf. [19, Proposition 3.1]) that for every $\gamma \in (0,1)$ and $p \in [1,\infty]$ the following estimate holds:

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})} \leq c_{10} t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}, \qquad \Re\mathrm{e}\,\theta \geq 0, \quad t>0. \tag{54}$$

Remark 7. We stress that if $\beta<1$, then we can not derive an estimate for the $\mathscr{L}(X;X_A^{\gamma,p})$-norm of $[(-A)^\theta]^\circ e^{tA}$ simply by replacing $(X,\mathscr{D}(A))_{\gamma,p}$ with $X_A^{\gamma,p}$ in (54). This is for two reasons. First, when $\gamma \in [\beta,1)$, we are not assured that $[(-A)^\theta]^\circ e^{tA}x \in X_A^{\gamma,p}$ for every $x \in X$. For if $\gamma \in [\beta,1)$, then the space $X_A^{\gamma,p}$ may be smaller than the domain $\mathscr{D}(A)$ to which $[(-A)^\theta]^\circ e^{tA}x$ belongs by virtue of [19, Lemma 3.1]. The second reason is that, even limiting to $\gamma \in (0,\beta)$ in order that $\mathscr{D}(A) \hookrightarrow X_A^{\gamma,p}$, from (31) we only get $\|[(-A)^\theta]^\circ e^{tA} x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq 2\|[(-A)^\theta]^\circ e^{tA}x\|_{X_A^{\gamma,p}}$, $x \in X$, and we do not know if the right-hand side can be bounded from above by some constant times $t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}\|x\|_X$. Of course, we can employ (32), but in this way all that we can reach is the estimate

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;X_A^{\gamma+\beta-1,p})} \leq c_{11} t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}, \qquad \Re\mathrm{e}\,\theta \geq 0, \quad t>0, \tag{55}$$

where $c_{11} = c_2 c_{10}$, $\gamma \in (1-\beta,1)$ and $p \in [1,\infty]$. Letting $\delta = \gamma+\beta-1$, (55) can be rewritten equivalently as

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;X_A^{\delta,p})} \leq c_{11} t^{(2\beta-\delta-\Re\mathrm{e}\,\theta-2)/\alpha}, \qquad \Re\mathrm{e}\,\theta \geq 0, \quad t>0, \tag{56}$$

where $\delta \in (0,\beta)$ and $p \in [1,\infty]$. When $\beta<1$, there are good motivations to believe that estimate (56) is not the best one. In fact, for instance, when $(\theta,p)=(0,\infty)$, (56) leads us to an estimate which is rougher than the estimate

$$\left\|e^{tA}\right\|_{\mathscr{L}(X;X_A^{\delta,\infty})} \leq c_{12} t^{(\beta-\delta-1)/\alpha}, \quad \delta \in (0,1), \ t>0, \tag{57}$$

as shown in [2, Proposition 3.2], with c_{12} being a positive constant depending on α, β, and δ. Also, (57) ensures that $e^{tA}x$, $x \in X$, belongs to $X_A^{\delta,\infty}$ for every $\delta \in (0,1)$ and not only for $\delta \in (0,\beta)$ as (56) suggests. Furthermore, due to (31), estimate (57) yields (54) with $(\theta,\gamma,p)=(0,\delta,\infty)$. This leads us to believe that (57) can be improved and that estimate (54) holds the same if $X_A^{\gamma,\infty}$ is taken in place of $(X,\mathscr{D}(A))_{\gamma,\infty}$.

Now let $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. As far as the estimates for the $\mathscr{L}(Y_\gamma^p;X)$-norm of operators $[(-A)^\theta]^\circ e^{tA}$ are concerned, instead, at the moment only the following estimates for the case $\theta=1$ are available (cf. [24, Lemma 5.1]):

$$\left\|\left[(-A)^1\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{13} t^{(\beta+\gamma-2)/\alpha}, \qquad t>0, \quad \gamma \in (0,1), \quad p \in [1,\infty], \tag{58}$$

with c_{13} being a positive constant depending on α,β,γ, and p. Estimates (58) are successfully applied in [24, Corollary 5.4] to prove that if $\alpha+\beta>1$, then the map $t \to e^{tA}$ is Hölder continuous from $[0,\infty)$ to $\mathscr{L}(Y_\gamma^p;X)$, $\gamma \in (2-\alpha-\beta,1)$, $p \in [1,\infty]$, with Hölder exponent $\sigma = (\alpha+\beta+\gamma-2)/\alpha$. In Section 3 we will extend (58), proving some estimates for the $\mathscr{L}(Y_\gamma^p;X)$-norm of $[(-A)^\theta]^\circ e^{tA}$, $\Re\mathrm{e}\,\theta \geq 1$, which reduce to (58) in the case $\theta=1$.

Remark 8. Observe that an estimate for the norm $\|[(-A)^\theta]^\circ e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re\mathrm{e}\,\theta \geq 1$, $t>0$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$, can be obtained combining (14), (15), and (58). Indeed, using (15), for every $\Re\mathrm{e}\,\theta \geq 1$, $t>0$ and $x \in Y_\gamma^p$, we have

$$\begin{aligned}\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_X &= \left\|\left[(-A)^{\theta-1}\right]^\circ e^{(t/2)A}\left[(-A)^1\right]^\circ e^{(t/2)A}x\right\|_X \\ &\leq \left\|\left[(-A)^{\theta-1}\right]^\circ e^{(t/2)A}\right\|_{\mathscr{L}(X)}\left\|\left[(-A)^1\right]^\circ e^{(t/2)A}x\right\|_X.\end{aligned} \tag{59}$$

Therefore, due to (14) and (58), from (59) we deduce that

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{14} t^{(2\beta+\gamma-\Re\mathrm{e}\,\theta-2)/\alpha}, \qquad \Re\mathrm{e}\,\theta \geq 1, \quad t>0, \tag{60}$$

where $\gamma \in (0,1)$, $p \in [1,\infty]$ and $c_{14} = 2^{(2+\Re e\,\theta-\gamma-2\beta)/\alpha}\tilde{c}_{\alpha,\beta,\theta-1}c_{13}$. As we will see in the next section estimate (60) is not optimal, in the sense that the negative exponent $(2\beta+\gamma-\Re e\,\theta-2)/\alpha$ can be refined; of course, unless $\beta = 1$. The main reason to believe that (60) can be improved is that its derivation consists of two steps: the first in which $[(-A)^{\theta}]^{\circ}e^{tA}$ is decomposed with the help of (15), and the second in which (60) is obtained combining estimates of very different nature, such as (14) and (58). It is thus to be expected that in this double step derivation some regularity goes missing and that a better result can be reached analyzing more detailedly $[(-A)^{\theta}]^{\circ}e^{tA}x$ for $x \in Y^{p}_{\gamma}$.

3. Behaviour of $[(-A)^{\theta}]^{\circ}e^{tA}$ in $(X,\mathscr{D}(A))_{\gamma,p}$ and $X^{\gamma,p}_{A}$

According to Remark 7 we begin by improving (54), showing that the same estimate holds with $(X,\mathscr{D}(A))_{\gamma,p}$ being replaced by $X^{\gamma,\infty}_{A}$ if $p=\infty$ and by $X^{\beta\gamma,p}_{A}$ if $p \in [1,\infty)$. Throughout this and the next section, A will be an m. l. operator in X having nonempty domain $\mathscr{D}(A)$ and satisfying the resolvent condition (H1) of Section 2.

Proposition 9. *Let $\Re e\theta \geq 0$, $\gamma \in (0,1)$ and let $p \in [1,\infty]$. Then, there exist positive constants c_j, $j = 15, 16$, depending on α, β, γ, θ, and p such that*

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}\right\|_{\mathscr{L}(X;X^{\gamma,\infty}_{A})} \leq c_{15}t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0, \quad p=\infty, \tag{61}$$

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}\right\|_{\mathscr{L}(X;X^{\beta\gamma,p}_{A})} \leq c_{16}t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0, \quad p\in[1,\infty). \tag{62}$$

Proof. If $\beta = 1$, then $(X,\mathscr{D}(A))_{\gamma,p} \cong X^{\gamma,p}_{A}$ and (61) and (62) with $c_j = c_2c_{10}$, $j = 15, 16$, follow by taking $\beta = 1$ in (32) and (54). Therefore, without the loss of generality, we assume that $\beta \in (0,\alpha]$ is such that $\beta < \alpha$ if $\alpha = 1$. Let $\theta \in \mathbf{C}$, $\Re e\,\theta \geq 0$, $\gamma \in (0,1)$, and $p \in [1,\infty)$ be fixed and let x be an arbitrary element of X. Then, for every $t>0$ we have

$$\begin{aligned}&\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X^{\gamma,\infty}_{A}}\\&= \left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X} + \left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L^{*}_{\infty}(X)},\end{aligned} \tag{63}$$

$$\begin{aligned}&\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X^{\beta\gamma,p}_{A}}\\&= \left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X} + \left\|\xi^{\beta\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L^{*}_{p}(X)}.\end{aligned} \tag{64}$$

Of course, from estimate (54) we find

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X} \leq c_{\gamma,p}c_{10}\|x\|_{X}t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0, \tag{65}$$

with $c_{\gamma,p}$ being such that $\|y\|_{X} \leq c_{\gamma,p}\|y\|_{(X,\mathscr{D}(A))_{\gamma,p}}$, $y \in (X,\mathscr{D}(A))_{\gamma,p}$, $p \in [1,\infty]$. It thus suffices to investigate only the second terms on the right-hand side of (63) and (64). We begin by proving (61). First, using the second identity in (6), for every $\xi \in (0,\infty)$ we get

$$\begin{aligned}&\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\\
&= \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}A^{\circ}(\xi I-A)^{-1}(\lambda I-A)^{-1}x\,d\lambda\\
&= \xi^{\gamma}\left[\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}d\lambda\right]A^{\circ}(\xi I-A)^{-1}x\\
&\quad - \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}A^{\circ}(\lambda I-A)^{-1}x\,d\lambda\\
&= -\frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}\left[\lambda(\lambda I-A)^{-1}-I\right]x\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta+1}e^{t\lambda}(\lambda-\xi)^{-1}(\lambda I-A)^{-1}x\,d\lambda.\end{aligned} \tag{66}$$

Here we have used twice the equality $\int_{\Gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}d\lambda = 0$, $\xi \in (0,\infty)$, which follows from Cauchy's formula after having enclosed Γ on the left with an arc of the circle $\{z \in \mathbf{C} : |z+c| = R\}$, $R>0$, and letting R to infinity. From (66), using $\|(\lambda I-A)^{-1}\|_{\mathscr{L}(X)} \leq C(|\lambda|+1)^{-\beta} \leq C|\lambda|^{-\beta}$, $\lambda \in \Sigma_{\alpha}$, it follows that

$$\begin{aligned}&\left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X}\\
&\leq C(2\pi)^{-1}\|x\|_{X}\\
&\quad\times\int_{\Gamma}\xi^{\gamma}|\lambda|^{1+\Re e\,\theta-\beta}e^{-\Im m\,\theta\arg(-\lambda)}e^{t\Re e\,\lambda}|\lambda-\xi|^{-1}\,|d\lambda|\\
&\leq C(2\pi)^{-1}e^{(\pi/2)|\Im m\,\theta|}\|x\|_{X}\\
&\quad\times\int_{\Gamma}\left(\frac{\xi}{|\lambda|}\right)^{\gamma}|\lambda|^{\gamma+\Re e\,\theta-\beta}e^{t\Re e\,\lambda}\left|1-\left(\frac{\xi}{\lambda}\right)\right|^{-1}|d\lambda|.\end{aligned} \tag{67}$$

Now, since $\Re e\,\lambda \leq -c < 0$ for every $\lambda \in \Gamma$ and since $\xi \in (0,\infty)$, we have

$$\begin{aligned}\left|1-\left(\frac{\xi}{\lambda}\right)\right| &= \left|1-\left(\frac{\xi\bar{\lambda}}{|\lambda|^{2}}\right)\right|\\
&= \left[1+\left(\frac{\xi}{|\lambda|}\right)^{2}-\frac{2\xi\Re e\,\lambda}{|\lambda|^{2}}\right]^{1/2}\\
&\geq \left[1+\left(\frac{\xi}{|\lambda|}\right)^{2}\right]^{1/2}.\end{aligned} \tag{68}$$

Therefore, for every $\lambda \in \Gamma$ and $\xi \in (0,\infty)$ the following inequality holds:

$$\left(\frac{\xi}{|\lambda|}\right)^{\gamma}\left|1-\left(\frac{\xi}{\lambda}\right)\right|^{-1} \le \left(\frac{\xi}{|\lambda|}\right)^{\gamma}\left[1+\left(\frac{\xi}{|\lambda|}\right)^{2}\right]^{-1/2} \le \gamma^{\gamma/2}(1-\gamma)^{(1-\gamma)/2} =: c_\gamma, \tag{69}$$

where we have used the fact that the function $f(s) = s^\gamma(1+s^2)^{-1/2}$, $s \ge 0$, $\gamma \in (0,1)$, attains its maximum value c_γ at the point $s_\gamma = \gamma^{1/2}(1-\gamma)^{-1/2}$. Coming back to (67) and setting $c_{17} = C(2\pi)^{-1}e^{(\pi/2)|\Im m\,\theta|}c_\gamma$, we thus find (here we use also that on Γ it holds $|\lambda| \ge c$, so that $\Re e\,\lambda = -c(|\Im m\,\lambda|+1)^\alpha \ge -c(1+c^{-1})^\alpha|\lambda|^\alpha$):

$$\begin{aligned}\left\|\xi^\gamma A^\circ(\xi I - A)^{-1}\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_X &\le c_{17}\|x\|_X\int_\Gamma |\lambda|^{\gamma+\Re e\,\theta-\beta}e^{t\Re e\,\lambda}\,|d\lambda| \\ &\le c_{17}\|x\|_X\int_\Gamma |\lambda|^{\gamma+\Re e\,\theta-\beta}e^{-c(1+c^{-1})^\alpha t|\lambda|^\alpha}\,|d\lambda| \\ &\le 2c_{17}\|x\|_X\int_0^\infty \mu^{\gamma+\Re e\,\theta-\beta}e^{-c_\alpha t\mu^\alpha}\,d\mu,\end{aligned} \tag{70}$$

where $c_\alpha = c(1+c^{-1})^\alpha$. Finally, taking the supremum with respect to $\xi \in (0,\infty)$ in (70) and performing the transformation $c_\alpha t\mu^\alpha = s$ in the integral on the right, we obtain

$$\left\|\xi^\gamma A^\circ(\xi I - A)^{-1}\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{L_\infty^*(X)} \le c_{18}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \tag{71}$$

where $c_{18} = 2c_{17}\alpha^{-1}c_\alpha^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}E((\gamma+\Re e\,\theta+1-\beta)/\alpha)$, $E(\chi)$, $\chi > 0$, being the Euler gamma function $\int_0^\infty s^{\chi-1}e^{-s}ds$. Then, summing up (65) and (71), from (63) it follows that

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{X_A^{\gamma,\infty}} \le \left(c_{\gamma,\infty}c_{10} + c_{18}\right)\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad \Re e\,\theta \ge 0, \quad t > 0. \tag{72}$$

Since $x \in X$ was arbitrary, this completes the proof of (61) with $c_{15} = c_{\gamma,\infty}c_{10} + c_{18}$. Let us now prove (62). For every $p \in [1,\infty)$ we write

$$\left\|\xi^{\beta\gamma}A^\circ(\xi I - A)^{-1}\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{L_p^*(X)}^p = I_1 + I_2, \tag{73}$$

where $I_j = \int_{a_j}^{b_j}\|\xi^{\beta\gamma}A^\circ(\xi I - A)^{-1}[(-A)^\theta]^\circ e^{tA}x\|_X^p(d\xi/\xi)$, $j = 1,2$, $(a_1,b_1,a_2,b_2) = (0,1,1,\infty)$. First, (35) with $Y_\gamma^p = (X,\mathcal{D}(A))_{\gamma,p}$ yields

$$I_1 \le \left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{(X,\mathcal{D}(A))_{\gamma,p}}^p\int_0^1 \xi^{\beta\gamma p-1}\left[c_3(\xi+1)^{1-\beta-\gamma}\right]^p d\xi. \tag{74}$$

Therefore, since $(\xi+1)^{1-\beta-\gamma} \le c_{\beta,\gamma}$ for every $\xi \in (0,1]$, where $c_{\beta,\gamma} = 2^{1-\beta-\gamma}$ or $c_{\beta,\gamma} = 1$ according that $\gamma \in (0,1-\beta)$ or $\gamma \in [1-\beta,1)$, from (54), we deduce that

$$\begin{aligned}I_1 &\le \left[c_{\beta,\gamma}c_3\right]^p\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{(X,\mathcal{D}(A))_{\gamma,p}}^p\int_0^1 \xi^{\beta\gamma p-1}d\xi \\ &= \left[c_{19}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^p,\end{aligned} \tag{75}$$

with $c_{19} = c_{\beta,\gamma}c_3c_{10}(\beta\gamma p)^{-1/p}$. As far as I_2 is concerned, exploiting (71) and recalling that we have assumed $\beta < 1$, we obtain

$$\begin{aligned}I_2 &= \int_1^\infty \xi^{(\beta-1)\gamma p}\left\|\xi^\gamma A^\circ(\xi I - A)^{-1}\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_X^p\frac{d\xi}{\xi} \\ &\le \left[c_{18}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^p\int_1^\infty \xi^{(\beta-1)\gamma p-1}\,d\xi \\ &\le \left[c_{20}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^p,\end{aligned} \tag{76}$$

where $c_{20} = c_{18}[(1-\beta)\gamma p]^{-1/p}$. Summing up (73)–(76), it thus follows that

$$\left\|\xi^{\beta\gamma}A^\circ(\xi I - A)^{-1}\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{L_p^*(X)} \le c_{21}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \tag{77}$$

where $c_{21} = [(c_{19})^p + (c_{20})^p]^{1/p}$. Finally, (65) and (77) lead us to

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{X_A^{\beta\gamma,p}} \le \left(c_{\gamma,p}c_{10} + c_{21}\right)\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad \Re e\,\theta \ge 0, \quad t > 0. \tag{78}$$

Since $x \in X$ was arbitrary, this completes the proof of (62) with $c_{16} = c_{\gamma,p}c_{10} + c_{21}$. □

Remark 10. If $\theta = 0$, then (61) is precisely the estimate (57). In this sense our result improves [2] and shows that (54) holds the same with $(X,\mathcal{D}(A))_{\gamma,p}$ being replaced with $X_A^{\gamma,\infty}$ if $p = \infty$ and $X_A^{\beta\gamma,p}$ and if $p \in [1,\infty)$. Also, when $\beta < 1$, (61) and (62) are in two aspects better than the estimate (55) deduced from (54) with the help of (32). First, here we do not need to restrict γ to $(1-\beta,1)$. Further, despite limiting γ to $(1-\beta,1)$, (61) and (62) show that $[(-A)^\theta]^\circ e^{tA}x$, $\Re e\,\theta \ge 0$, $t > 0$, $x \in X$, enjoys more regularity than that predicted by (55). For, since when $\beta < 1$ it holds $0 < \gamma+\beta-1 < \beta\gamma < \gamma$, from (38) and (39) it follows $X_A^{\gamma,\infty} \hookrightarrow X_A^{\beta\gamma,p} \hookrightarrow X_A^{\gamma+\beta-1,p}$, $p \in [1,\infty]$.

Remark 11. We recall that when $\beta < 1$ the spaces $X_A^{\sigma,p}$, $\sigma \in (0,1)$, $p \in [1,\infty]$, are intermediate spaces between X and $\mathcal{D}(A)$ for $\sigma \in (0,\beta)$, but they may be contained in $\mathcal{D}(A)$ for $\sigma \in [\beta,1)$. Therefore, whereas (61) is satisfied for spaces $X_A^{\sigma,\infty}$ eventually smaller than $\mathcal{D}(A)$, for (62) to hold we have to consider only spaces $X_A^{\sigma,p}$, $p \in [1,\infty)$, bigger than $\mathcal{D}(A)$. In fact, letting $\sigma = \beta\gamma$, we have $\sigma \in (0,\beta)$ for every $\gamma \in (0,1)$.

In accordance with Remark 8 we now improve estimate (58).

Proposition 12. *Let* $\Re e\theta \geq 1$, $\gamma \in (0,1)$, $p \in [1,\infty]$ *and let* $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. *Then, there exists a positive constant* c_{22} *depending on* α, β, γ, θ, *and* p *such that*

$$\left\| \left[(-A)^\theta\right]^\circ e^{tA} \right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{22} t^{(\beta+\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0. \tag{79}$$

Proof. First, using the identity $A^\circ(zI-A)^{-1} = z(zI-A)^{-1} - I$, $z \in \Sigma_\alpha$, for every $x \in X$, we rewrite $[(-A)^\theta]^\circ e^{tA}x$, $\Re e\,\theta \geq 0$, in the following way:

$$\begin{aligned}
&\left[(-A)^\theta\right]^\circ e^{tA} x \\
&\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} \lambda(\lambda I - A)^{-1} x \,d\lambda \\
&\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} \left[A^\circ(\lambda I - A)^{-1} x + I\right] x \,d\lambda \\
&\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} A^\circ(\lambda I - A)^{-1} x \,d\lambda, \quad t>0.
\end{aligned} \tag{80}$$

Here we have used $\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} d\lambda = 0$, which follows from the Cauchy formula applied to $(-\lambda)^\theta e^{t\lambda}$ after having enclosed Γ on the left with an arc of the circle $\{z \in \mathbf{C} : |z+c| = R\}$, $R>0$, and letting R to infinity. Let now $\theta \in \mathbf{C}$, $\Re e\,\theta \geq 1$, $\gamma \in (0,1)$, and $p \in [1,\infty]$ be fixed and let x be an arbitrary element of Y_γ^p. From (35) it then follows that

$$\begin{aligned}
&\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \\
&\quad \leq c_{23} \|x\|_{Y_\gamma^p} \int_\Gamma |\lambda|^{\Re e\,\theta-1} e^{t\Re e\,\lambda} (|\lambda|+1)^{1-\beta-\gamma} |d\lambda|, \quad t>0,
\end{aligned} \tag{81}$$

where $c_{23} = (2\pi)^{-1} e^{(\pi/2)|\Im m\,\theta|} c_3$. Now, recalling that $|\lambda| \geq c > 0$ for every $\lambda \in \Gamma$, we have $|\lambda| \leq |\lambda|+1 \leq (1+c^{-1})|\lambda|$, $\lambda \in \Gamma$. As a consequence, the following inequality holds:

$$(|\lambda|+1)^{1-\beta-\gamma} \leq \widetilde{c}_{\beta,\gamma} |\lambda|^{1-\beta-\gamma}, \quad \forall \lambda \in \Gamma, \tag{82}$$

where $\widetilde{c}_{\beta,\gamma} = (1+c^{-1})^{1-\beta-\gamma}$ or $\widetilde{c}_{\beta,\gamma} = 1$ according that $\gamma \in (0, 1-\beta]$ or $\gamma \in (1-\beta,1)$ ($(0,1-\beta] = \emptyset$ if $\beta = 1$). Therefore, setting $c_{24} = 2\widetilde{c}_{\beta,\gamma} c_{23}$, (81) and (82) yield

$$\begin{aligned}
&\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \\
&\quad \leq c_{24} \|x\|_{Y_\gamma^p} \int_0^\infty \mu^{\Re e\,\theta-\beta-\gamma} e^{-c_\alpha t\mu^\alpha} d\mu, \quad t>0,
\end{aligned} \tag{83}$$

with c_α being as in (70). Finally, the transformation $c_\alpha t\mu^\alpha = s$ in the last integral leads us to the following estimate:

$$\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \leq c_{25} \|x\|_{Y_\gamma^p} t^{(\beta+\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0, \tag{84}$$

where $c_{25} = c_{24}\alpha^{-1} c_\alpha^{(\beta+\gamma-\Re e\,\theta-1)/\alpha} E((\Re e\,\theta+1-\beta-\gamma)/\alpha)$, $E(\chi)$, $\chi>0$, is the Euler's gamma function. Notice that here $\Re e\,\theta \geq 1$ implies $\Re e\,\theta + 1 - \beta - \gamma \geq 2-\beta-\gamma > 0$ for every $\beta \in (0,1]$ and $\gamma \in (0,1)$, so that $E((\Re e\,\theta+1-\beta-\gamma)/\alpha)$ makes sense. Since (84) is satisfied for every arbitrary element $x \in Y_\gamma^p$, the proof is complete with $c_{22} = c_{25}$. □

Remark 13. Estimate (79) is better than (60) obtained in Remark 8 using (14), (15), and (58). In fact, for every $\beta \in (0,\alpha]$, $\alpha \in (0,1]$, $\gamma \in (0,1)$ and $\Re e\,\theta \geq 1$, the following inequality holds:

$$\begin{aligned}
\rho_1 &:= \frac{(2\beta+\gamma-\Re e\,\theta-2)}{\alpha} \\
&\leq \frac{(\beta+\gamma-\Re e\,\theta-1)}{\alpha} := \rho_2 < 0.
\end{aligned} \tag{85}$$

Then, $t^{\rho_2} \leq t^{\rho_1}$, $t \in (0,1]$, and (79) is more accurate than (60) for small values of t.

Estimate (79) with $\theta = 1$ yields the following result which we will need in Section 5 to prove the equivalence between problem (170) and the fixed-point equation (179).

Corollary 14. *Let* $\alpha+\beta > 1$ *in (H1). Then, for every* $x \in X$ *the following equalities hold:*

$$A^{-1}\left(e^{tA} - I\right) x = \left(e^{tA} - I\right) A^{-1} x = \int_0^t e^{(t-s)A} x\, ds, \quad t \geq 0. \tag{86}$$

Proof. The assertion is obvious for $t = 0$. Let $t > 0$ and let $x \in X$. Commuting $A^{-1} \in \mathscr{L}(X)$ with the integral sign, from (9) and the resolvent equation, we have $A^{-1}e^{tA}x = e^{tA}A^{-1}x$, which proves the first equality in (86). To prove the second equality, we first write

$$\begin{aligned}
\left(e^{tA} - I\right) A^{-1} x &= \int_0^t \left[D_r e^{rA}\right]_{r=t-s} A^{-1} x \,ds \\
&= -\int_0^t \left[(-A)^1\right]^\circ e^{(t-s)A} A^{-1} x\, ds,
\end{aligned} \tag{87}$$

and we show that the latter integral is convergent. Indeed, since $\alpha+\beta > 1$, we may consider $A^{-1}x \in \mathscr{D}(A)$ as an element of $(X,\mathscr{D}(A))_{\gamma,p}$, where $\gamma \in (2-\alpha-\beta, 1)$ and $p \in [1,\infty]$. With this choice for γ, from (79) with $\theta = 1$ and (25) we obtain (here we use also $\|A^{-1}x\|_{\mathscr{D}(A)} = \inf_{y \in A(A^{-1}x)} \|y\|_X = \inf_{y \in (AA^{-1})x} \|y\|_X = \|x\|_{\mathscr{D}(AA^{-1})} \leq \|x\|_X$, due to $I \subset AA^{-1}$. Then, $\|A^{-1}x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq c_0 \|A^{-1}x\|_X^{1-\gamma} \|A^{-1}x\|_{\mathscr{D}(A)}^\gamma \leq c_0 \|A^{-1}\|_{\mathscr{L}(X)}^{1-\gamma} \|x\|_X$):

$$\begin{aligned}
&\left\| \int_0^t \left[(-A)^1\right]^\circ e^{(t-s)A} A^{-1} x\, ds \right\|_X \\
&\quad \leq c_{22} \left\| A^{-1}x \right\|_{(X,\mathscr{D}(A))_{\gamma,p}} \int_0^t (t-s)^{(\beta+\gamma-2)/\alpha} ds \\
&\quad \leq c_{22} c_{\alpha,\beta,\gamma} c_0 \left\| A^{-1} \right\|_{\mathscr{L}(X)}^{1-\gamma} \|x\|_X t^{(\alpha+\beta+\gamma-2)/\alpha},
\end{aligned} \tag{88}$$

where $c_{\alpha,\beta,\gamma} = \alpha(\alpha+\beta+\gamma-2)^{-1}$. We now recall that (cf. [24, formula (3.21)])

$$\left[(-A)^1\right]^{\circ} e^{tA}(-A)^{-\zeta} = \left[(-A)^{1-\zeta}\right]^{\circ} e^{tA}, \quad \Re e\, \zeta \in (1-\beta, 1], \quad t > 0, \tag{89}$$

with $(-A)^{-\zeta}$ being the negative fractional powers of $-A$ defined by (cf. [24, Section 3]) $(2\pi i)^{-1}\int_{\Gamma}(-\lambda)^{-\zeta}(\lambda I - A)^{-1} d\lambda$, $\Re e\,\zeta > 1-\beta$. To complete the proof it thus suffices to apply (89) with $\zeta = 1$ to (87) and to recall that $[(-A)^0]^{\circ} e^{tA} = e^{tA}$, $t > 0$. Notice that the integral on the right-hand side of (86) is convergent, too. In fact, from (14), it follows that $\| \int_0^t e^{(t-s)A} x\, ds \|_X \leq \tilde{c}_{\alpha,\beta,0} \|x\|_X \int_0^t (t-s)^{(\beta-1)/\alpha} ds = \alpha(\alpha+\beta-1)^{-1} \tilde{c}_{\alpha,\beta,0} \|x\|_X t^{(\alpha+\beta-1)/\alpha}$. □

Remark 15. In particular, from (86) it follows that if $\alpha+\beta > 1$, then $\int_0^t e^{(t-s)A} x\, ds \in \mathscr{D}(A)$ for every $x \in X$ and $(e^{tA} - I)x \subseteq A\int_0^t e^{(t-s)A} x\, ds$. This extends to m. l. operators satisfying (H1) the well-known result for sectorial single-valued linear operators (see, for instance, [9, Proposition 2.1.4(ii)] and [11, Proposition 1.2(ii)]).

With the help of (54) and Proposition 12, we can now derive the following interpolation estimates (90) for the operators $[(-A)^{\theta}]^{\circ} e^{tA}$, $\Re e\,\theta \geq 1$, which are considered as operators from $(X, \mathscr{D}(A))_{\gamma,p}$ to $(X, \mathscr{D}(A))_{\delta,p}$. As we will see in the proof of Proposition 16, here the fact that the spaces $(X, \mathscr{D}(A))_{\sigma,p}$ are real interpolation spaces between X and $\mathscr{D}(A)$ plays a key role. For it allows us to exploit the interpolation inequality (24) in the derivation of our estimates in the case $\gamma+\delta < 1$.

Proposition 16. *Let* $\Re e\theta \geq 1$, $\gamma, \delta \in (0,1)$, *and* $p \in [1,\infty]$. *Then, there exist positive constants* c_j, $j = 26, 27$, *depending on* α, β, γ, δ, θ, *and* p *such that for every* $t > 0$

$$\begin{aligned} &\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};(X,\mathscr{D}(A))_{\delta,p})} \\ &\quad \leq \begin{cases} c_{26} t^{(2\beta+\gamma-\delta-\Re e\,\theta-2)/\alpha}, & \gamma, \delta \in (0,1), \\ c_{27} t^{(\beta+\gamma-\delta-\Re e\,\theta-1)/\alpha}, & \text{if } \gamma+\delta < 1. \end{cases} \end{aligned} \tag{90}$$

Proof. For brevity, we will use the shortenings $Y_{\sigma}^p = (X, \mathscr{D}(A))_{\sigma,p}$, $\sigma \in (0,1)$, $p \in [1,\infty]$. We begin by proving the first estimate in (90). Let $\theta \in \mathbf{C}$, $\Re e\,\theta \geq 1$, $\gamma, \delta \in (0,1)$ and $p \in [1,\infty]$ be fixed and let x be an arbitrary element of Y_{γ}^p. Moreover, let ζ and ζ' be two arbitrary complex numbers such that $\theta = \zeta + \zeta'$ and whose real parts satisfy $\Re e\,\zeta \geq 0$ and $\Re e\,\zeta' \geq 1$. From the decomposition formula (15) it then follows for every $t > 0$:

$$\begin{aligned} &\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} x \right\|_{Y_{\delta}^p} \\ &\quad = \left\| \left[(-A)^{\zeta}\right]^{\circ} e^{(t/2)A} \left[(-A)^{\zeta'}\right]^{\circ} e^{(t/2)A} x \right\|_{Y_{\delta}^p} \\ &\quad \leq \left\| \left[(-A)^{\zeta}\right]^{\circ} e^{(t/2)A} \right\|_{\mathscr{L}(X;Y_{\delta}^p)} \left\| \left[(-A)^{\zeta'}\right]^{\circ} e^{(t/2)A} x \right\|_X \\ &\quad \leq \left\| \left[(-A)^{\zeta}\right]^{\circ} e^{(t/2)A} \right\|_{\mathscr{L}(X;Y_{\delta}^p)} \left\| \left[(-A)^{\zeta'}\right]^{\circ} e^{(t/2)A} \right\|_{\mathscr{L}(Y_{\gamma}^p;X)} \|x\|_{Y_{\gamma}^p}. \end{aligned} \tag{91}$$

Therefore, using (54) and (79) with the triplet (θ, γ, t) being equal to $(\zeta, \delta, t/2)$ and $(\zeta', \gamma, t/2)$, respectively, from (91) and $\Re e\,\theta = \Re e\,\zeta + \Re e\,\zeta'$, we deduce that

$$\begin{aligned} &\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} x \right\|_{Y_{\delta}^p} \\ &\quad \leq c_{10} c_{22} \left(\frac{t}{2}\right)^{(\beta-\delta-\Re e\,\zeta-1)/\alpha} \left(\frac{t}{2}\right)^{(\beta+\gamma-\Re e\,\zeta'-1)/\alpha} \|x\|_{Y_{\gamma}^p} \\ &\quad \leq c_{26} t^{(2\beta+\gamma-\delta-\Re e\,\theta-2)/\alpha} \|x\|_{Y_{\gamma}^p}, \quad t > 0, \end{aligned} \tag{92}$$

where $c_{26} = 2^{(2+\Re e\,\theta+\delta-\gamma-2\beta)/\alpha} c_{10} c_{22}$. This completes the proof of the first estimate in (90), due to the arbitrariness of $x \in Y_{\gamma}^p$. Let us now prove the second estimate in (90). Let $\theta \in \mathbf{C}$, $\Re e\,\theta \geq 1$, $\gamma, \delta \in (0,1)$, $\gamma+\delta < 1$, and $p \in [1,\infty]$ be fixed. Using $\gamma+\delta < 1$, we fix $\gamma_2 \in (\gamma/(1-\delta), 1) \subsetneq (\gamma, 1)$, and we let $\gamma_1 = (\gamma_2\delta)/(\gamma_2-\gamma)$. Clearly, since $\gamma_2 \in (\gamma/(1-\delta), 1)$, we have $\gamma_1 \in (\delta, 1)$. In addition, it holds:

$$1-\delta > \frac{\gamma_1-\delta}{\gamma_1} = \left(\frac{\gamma_2\delta}{\gamma_2-\gamma} - \delta\right)\left(\frac{\gamma_2-\gamma}{\gamma_2\delta}\right) = \frac{\gamma}{\gamma_2} > \gamma. \tag{93}$$

Due to (93), we now set $\gamma_0 = \gamma/\gamma_2 = (\gamma_1-\delta)/\gamma_1 \in (\gamma, 1-\delta)$, so that $\gamma = \gamma_0\gamma_2$ and $\delta = (1-\gamma_0)\gamma_1$. From (24) with $p_0 = p$ it thus follows that

$$\begin{aligned} &\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(Y_{\gamma}^p;Y_{\delta}^p)} \\ &\quad \leq \left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(X;Y_{\gamma_1}^{p_1})}^{1-\gamma_0} \left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(Y_{\gamma_2}^{p_2};X)}^{\gamma_0}, \quad t > 0, \end{aligned} \tag{94}$$

where $p_j \in [1,\infty]$, $j = 1, 2$. Applying (54) and (79) with the pair (γ, p) being replaced with (γ_1, p_1) and (γ_2, p_2), respectively, from (94) we finally obtain

$$\begin{aligned} &\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(Y_{\gamma}^p;Y_{\delta}^p)} \\ &\quad \leq \left[c_{10} t^{(\beta-\gamma_1-\Re e\,\theta-1)/\alpha}\right]^{1-\gamma_0} \left[c_{22} t^{(\beta+\gamma_2-\Re e\,\theta-1)/\alpha}\right]^{\gamma_0} \\ &\quad \leq (c_{10})^{1-\gamma_0} (c_{22})^{\gamma_0} t^{[\beta+\gamma_0\gamma_2-(1-\gamma_0)\gamma_1-\Re e\,\theta-1]/\alpha} \\ &\quad = (c_{10})^{\delta/\gamma_1} (c_{22})^{\gamma/\gamma_2} t^{(\beta+\gamma-\delta-\Re e\,\theta-1)/\alpha}, \quad t > 0. \end{aligned} \tag{95}$$

This completes the proof of the second estimate in (90) with $c_{27} = (c_{10})^{\delta/\gamma_1} (c_{22})^{\gamma/\gamma_2}$. □

Remark 17. We stress that if $\beta < 1$ and $\gamma + \delta < 1$, then the first estimate in (90) is rougher than the second one for small values of t, which justify our special attention to the case $\gamma + \delta < 1$. Indeed, if $\beta < 1$, then for every $\Re e\,\theta \geq 1$ the following inequality holds:

$$\rho_3 := \frac{(2\beta + \gamma - \delta - \Re e\,\theta - 2)}{\alpha} < \frac{(\beta + \gamma - \delta - \Re e\,\theta - 1)}{\alpha} =: \rho_4 < 0, \tag{96}$$

so that $t^{\rho_4} \leq t^{\rho_3}$ for $t \in (0,1]$. In other words, if β and $\gamma + \delta$ are both less than one, then the second estimate in (90) establishes that the norm $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};(X,\mathscr{D}(A))_{\delta,p})}$, $\Re e\,\theta \geq 1$, may blow up as t goes to 0, but with an order of singularity lower than that predicted by the first estimate. In this sense, though less general, the second estimate in (90) is better than the first one.

Remark 18. The reason why the second estimate in (90) yields a better exponent than the first one is the same mentioned in Remark 8. That is, while the first estimate is obtained in two steps: decomposing $[(-A)^{\theta}]^{\circ} e^{tA}$ through (15) and then applying (54) and (79), the second estimate is essentially derived in a single step, using (24).

The following Remark 19 points out why, with the exception of the case when $\beta = 1$ and A is single-valued, to prove (90) we can not proceed as in [9, Proposition 2.2.9].

Remark 19. In the optimal case $\beta = 1$, the exponents in both estimates (90) coincide equals to $\nu = \gamma - \delta - \Re e\,\theta$. Hence, in this special case, the assumption $\gamma + \delta < 1$ does not give any enhancement. Also, if we further assume that $\theta \in \mathbf{N}$, then we restore the same estimates as in [9, Proposition 2.2.9(i)]. In this respect, our result extends [9] to the m. l. case, even though our proof really differs from that in [9]. For, there, the norms in the spaces $(X, \mathscr{D}(A))_{\sigma,p}$ are replaced with the norms in the spaces $\mathscr{D}_A(\sigma, p)$, with the latter being the spaces of all $x \in X$ such that $\|x\|_{\mathscr{D}_A(\sigma,p)} = \|x\|_X + [x]_{\mathscr{D}_A(\sigma,p)} < \infty$, where $[x]_{\mathscr{D}_A(\sigma,p)} = \|\xi^{(2-\beta-\sigma)/\alpha}[(-A)^1]^{\circ} e^{\xi A}\|_{L_p^*(X)}$. It is well known that if $\beta = 1$ and A is single-valued, then $(X, \mathscr{D}(A))_{\sigma,p} \cong \mathscr{D}_A(\sigma, p)$ (cf. [31, Theorem 3], [9, Proposition 2.2.2] and [27, Theorem 1.14.5]). On the contrary, if $(\alpha, \beta) \neq (1, 1)$ and/or A is really an m. l. operator, such equivalence is no longer true and we have

$$\begin{aligned} X_A^{\sigma,p} &\hookrightarrow (X, \mathscr{D}(A))_{\sigma,p} \hookrightarrow \mathscr{D}_A(\alpha\sigma, p), \qquad p \in [1, \infty), \\ X_A^{\sigma,\infty} &\hookrightarrow (X, \mathscr{D}(A))_{\sigma,\infty} \hookrightarrow \mathscr{D}_A(\sigma, \infty), \qquad p = \infty. \end{aligned} \tag{97}$$

Differently from the spaces $X_A^{\sigma,p}$ and as a consequence of $A0 \subseteq \bigcap_{t>0} \mathscr{N}([(-A)^1]^{\circ} e^{tA})$, the spaces $\mathscr{D}_A(\sigma, p)$ contain $A0$. It can thus be shown that if $\alpha + \beta > 1$, then for every $\sigma \in (2 - \alpha - \beta, 1)$ and $\varphi \in (0, (\alpha + \beta + \sigma - 2)/\alpha)$ (here $(\alpha + \beta + \sigma - 2)/\alpha < 1$, since $\sigma < 1 \leq 2 - \beta$) the following embeddings hold:

$$\begin{aligned} &\{0\} \cup [\mathscr{D}_A(\sigma, p) \setminus A0] \hookrightarrow X_A^{\varphi,p} \hookrightarrow (X, \mathscr{D}(A))_{\varphi,p}, \\ &\qquad\qquad p \in [1, \infty), \\ &\{0\} \cup [\mathscr{D}_A(\sigma, \infty) \setminus A0] \hookrightarrow X_A^{(\alpha+\beta+\sigma-2)/\alpha,\infty} \\ &\qquad\qquad \hookrightarrow (X, \mathscr{D}(A))_{(\alpha+\beta+\sigma-2)/\alpha,\infty}, \end{aligned} \tag{98}$$

with $\{0\} \cup [\mathscr{D}_A(\sigma, p) \setminus A0]$ being endowed with the norm of $\mathscr{D}_A(\sigma, p)$. Obviously, due to (29), it suffices to prove the embeddings on the right of (97) and on the left of (98). It is out of the aims of this paper to go into the details of these proofs, and for them we refer the readers to [24, Proposition 6.3]. Here we want only to make clear that, with the exception of the case when $\beta = 1$ and A is single-valued, embeddings (97) and (98) prevent us from carrying out the proof of estimates (90) simply by repeating the computations in [9]. Notice that, due to the property $[X_A^{\sigma,p} \cap A0] = \{0\}$, from the second embeddings in (97) and (98) it follows that if $\alpha+\beta > 1$ and $\sigma \in (2 - \alpha - \beta, 1)$, then

$$X_A^{\sigma,\infty} \hookrightarrow \{0\} \cup [\mathscr{D}_A(\sigma, \infty) \setminus A0] \hookrightarrow X_A^{(\alpha+\beta+\sigma-2)/\alpha,\infty}. \tag{99}$$

Since $(\alpha+\beta+\sigma-2)/\alpha \leq \sigma$ (indeed, $\alpha \leq 1 \leq (2-\beta-\sigma)/(1-\sigma)$ implies $\alpha + \beta + \sigma - 2 \leq \alpha\sigma$), (99) agrees with (38) for $p = \infty$. In addition, if $2\alpha + \beta > 2$ and $\sigma \in ((2 - \alpha - \beta)/\alpha, 1)$, then the first embeddings in (97) and (98) yield for every $\varphi \in (0, (\alpha + \beta + \alpha\sigma - 2)/\alpha)$ the following:

$$X_A^{\sigma,p} \hookrightarrow \{0\} \cup [\mathscr{D}_A(\alpha\sigma, p) \setminus A0] \hookrightarrow X_A^{\varphi,p}, \quad p \in [1, \infty). \tag{100}$$

Since $\varphi < (\alpha + \beta + \alpha\sigma - 2)/\alpha \leq \sigma$, (100) agrees with (38) for $p \in [1, \infty)$. Furthermore, if $\beta = 1$, then from (29), (30), and (99) it follows that $(X, \mathscr{D}(A))_{\sigma,\infty} \cong X_A^{\sigma,\infty} \cong \{0\} \cup [\mathscr{D}_A(\sigma, \infty) \setminus A0]$, $\sigma \in (0, 1)$. This confirms that in the real m. l. case the equivalence between $X_A^{\sigma,p}$, $(X, \mathscr{D}(A))_{\sigma,p}$ and $\mathscr{D}_A(\sigma, p)$ does not hold even when $\beta = 1$.

Using Propositions 9 and 12, we now obtain estimates for the operators $[(-A)^{\theta}]^{\circ} e^{tA}$, $\Re e\,\theta \geq 1$, considered as operators from $X_A^{\gamma,p}$ to $X_A^{\delta,p}$. Clearly, since $\beta < 1$ the spaces $X_A^{\sigma,p}$ may be not real interpolation spaces between X and $\mathscr{D}(A)$, we can not proceed as in the proof of the second estimate in (90) and a weaker result has to be expected.

Proposition 20. *Let $\Re e\theta \geq 1$, $\gamma, \delta \in (0, 1)$, and $p \in [1, \infty]$. Then, there exist positive constants c_j, $j = 28, 29, 30$, depending on α, β, γ, δ, θ, and p such that*

$$\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(X_A^{\gamma,\infty}; X_A^{\delta,\infty})} \leq c_{28} t^{(2\beta+\gamma-\delta-\Re e\theta-2)/\alpha}, \qquad p = \infty, \quad t > 0, \tag{101}$$

$$\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(X_A^{\gamma,p}; X_A^{\beta\delta,p})} \leq c_{29} t^{(2\beta+\gamma-\delta-\Re e\theta-2)/\alpha}, \qquad p \in [1, \infty), \quad t > 0. \tag{102}$$

Moreover, if $\gamma \in (0,1)$ *and* $\delta \in (1-\beta,1)$ *are such that* $\gamma+\delta<1$, *then*

$$\left\| \left[(-A)^{\theta}\right]^{\circ} e^{tA} \right\|_{\mathscr{L}(X_A^{\gamma,p};X_A^{\delta+\beta-1,p})} \le c_{30} t^{(\beta+\gamma-\delta-\Re e-1)/\alpha}, \qquad p \in [1,\infty], \quad t>0. \tag{103}$$

Proof. Due to (61) and (79), in order to prove (101) and (102) it suffices to repeat the same computations as in (91) and (92), with the pair $((X,\mathscr{D}(A))_{\gamma,p},(X,\mathscr{D}(A))_{\delta,p})$ being replaced with $(X_A^{\gamma,\infty},X_A^{\delta,\infty})$ or with $(X_A^{\gamma,p},X_A^{\beta\delta,p})$ provided that $p=\infty$ or $p\in[1,\infty)$. In this way we derive (101) and (102) with $c_{j+13} = 2^{(2+\Re e\,\theta+\delta-\gamma-2\beta)/\alpha} c_j c_{22}$, $j=15,16$. As far as (103) is concerned, we recall that if X_j, $j=1,\dots,4$, are four Banach spaces such that $X_j \hookrightarrow X_{j+2}$, $j=1,2$, and $L\in\mathscr{L}(X_3;X_2)$, then $L\in\mathscr{L}(X_1;X_4)$ with $\|L\|_{\mathscr{L}(X_1;X_4)} \le C_1C_2\|L\|_{\mathscr{L}(X_3;X_2)}$, C_1 and C_2 being the positive constants such that $\|x\|_{X_{j+2}} \le C_j\|x\|_{X_j}$, $x\in X_j$, $j=1,2$. Applying this result to $L=[(-A)^{\theta}]^{\circ}e^{tA}$ with $(X_1,X_2,X_3,X_4) = (X_A^{\gamma,p},(X,\mathscr{D}(A))_{\delta,p},(X,\mathscr{D}(A))_{\gamma,p},X_A^{\delta+\beta-1,p})$, from (29)–(32) and the second estimate in (90) we deduce (103) with $c_{30}=2c_2c_{27}$. This completes the proof. □

Remark 21. The assumption $\gamma+\delta<1$ with $\gamma\in(0,1)$ and $\delta\in(1-\beta,1)$ implies that $\gamma\in(0,1-\delta)\subsetneq(0,\beta)$. Therefore (cf. Remark 11), we conclude that for (103) to hold we have to consider $[(-A)^{\theta}]^{\circ}e^{tA}$, $\Re e\,\theta \ge 1$, as an operator between the intermediate spaces $X_A^{\gamma,p}$ and $X_A^{\varepsilon,p}$, where $\gamma,\varepsilon\in(0,\beta)$, $\varepsilon=\delta+\beta-1$, $\delta\in(1-\beta,1)$, $\gamma+\delta<1$.

4. Hölder Regularity of Some Operator Functions

Here, we study the Hölder regularity of those operator functions that we will need in Section 5. From now on, with $(Z,\|\cdot\|_Z)$ being a complex Banach space, $C([a,b];Z) = C^0([a,b];Z)$ and $C^{\delta}([a,b];Z)$, $\delta\in(0,1)$, $a<b$, denote, respectively, the spaces of all continuous and δ-Hölder continuous functions from $[a,b]$ into Z endowed with the norms $\|g\|_{0,a,b;Z} = \sup_{t\in[a,b]}\|g(t)\|_Z$ and $\|g\|_{\delta,a,b;Z} = \|g\|_{0,a,b;Z} + |g|_{\delta,a,b;Z}$, where $|g|_{\delta,a,b;Z}$ is the seminorm $\sup_{a\le t_1<t_2\le b}(t_2-t_1)^{-\delta}\|g(t_2)-g(t_1)\|_Z$. We endow the subspace $C_0^{\delta}([a,b];Z) = \{g\in C^{\delta}([a,b];Z) : g(a)=0\}$, $\delta\in[0,1)$ with the norm $\|\cdot\|_{\delta,a,b;Z}$. Further, for $k\in\mathbf{N}$ and $\delta\in(0,1)$ we set $C^k([a,b];Z) = \{g\in C([a,b];Z) : D_t^k g\in C([a,b];Z)\}$, $\|g\|_{k,a,b;Z} = \sum_0^k \|D_t^j g\|_{0,a,b;Z}$ ($D_t^0=I$), and $C^{k+\delta}([a,b];Z) = \{g\in C^k([a,b];Z) : D_t^k g\in C^{\delta}([a,b];Z)\}$, $\|g\|_{k+\delta,a,b;Z} = \|g\|_{k,a,b;Z} + |D_t^k g|_{\delta,a,b;Z}$. Recall that if $0\le\delta_2\le\delta_1\le1$, then $C^{\delta_1}([a,b];Z)\hookrightarrow C^{\delta_2}([a,b];Z)$ and $\|g\|_{\delta_2,a,b;Z} \le \max\{1,(b-a)^{\delta_1-\delta_2}\}\|g\|_{\delta_1,a,b;Z}$, $g\in C^{\delta_1}([a,b];X)$. Finally, given three complex Banach spaces $(X_k,\|\cdot\|_{X_k})$, $k=1,2,3$, and a bilinear bounded operator $\mathscr{P}$ from $X_1\times X_2$ to X_3 with norm C_0, that is, $\mathscr{P}\in\mathscr{B}(X_1\times X_2;X_3)$ and $\|\mathscr{P}\|_{\mathscr{B}(X_1\times X_2;X_3)} = \sup_{\|x_k\|_{X_k}=1,k=1,2}\|\mathscr{P}(x_1,x_2)\|_{X_3} = C_0$, we denote by $\mathscr{K}$ the convolution operator

$$\mathscr{K}(v_1,v_2)(t) = \int_0^t \mathscr{P}(v_1(t-r),v_2(r))\,dr, \qquad t\in[0,b], \quad b>0, \tag{104}$$

where $v_k:[0,b]\to X_k$, $k=1,2$. Of course, if $(X_1,X_2)=(\mathbf{C},X_3)$ and if $\mathscr{P}$ is the scalar multiplication in X_3, that is, $\mathscr{P}(z,x)=zx$, $z\in\mathbf{C}$, $x\in X_3$, then $C_0=1$ and $\mathscr{K}$ reduces to the usual convolution operator $\mathscr{K}(v_1,v_2)(t) = \int_0^t v_1(t-r)v_2(r)\,dr$. As usual, for every $q\in[1,\infty]$, we will denote by q' the conjugate exponent of q.

Now let $X_3=X$ and introduce the following linear operators Q_j, $j=1,\dots,6$, where $g_j\in C^{\delta_j}([0,T];X)$, $j=1,2,5$, $g_{l_k}\in C^{\delta_{l_k}}([0,T],X_k)$, $l=3,6$, $k=1,2$, $g_4\in C^{\delta_4}([0,T];\mathbf{C})$, $y\in Y_\gamma^p$, $Y_\gamma^p\in\{(X,\mathscr{D}(A))_{\gamma,p},X_A^{\gamma,p}\}$, $p\in[1,\infty]$, and $t\in[0,T]$, $T>0$ as follows:

$$[Q_1g_1](t) := \int_0^t e^{(t-s)A} g_1(s)\,ds, \tag{105}$$

$$[Q_2g_2](t) := \int_0^t \left[(-A)^1\right]^{\circ} e^{(t-s)A}\left[g_2(s)-g_2(t)\right] ds, \tag{106}$$

$$\left[Q_3\left(g_{3_1},g_{3_2}\right)\right](t) := \left[Q_2\mathscr{K}\left(g_{3_1},g_{3_2}\right)\right](t), \tag{107}$$

$$[Q_4(g_4,y)](t) := [Q_2(g_4y)](t), \tag{108}$$

$$[Q_5g_5](t) := \left[e^{tA}-I\right]g_5(t), \tag{109}$$

$$\left[Q_6\left(g_{6_1},g_{6_2}\right)\right](t) := \left[Q_5\mathscr{K}\left(g_{6_1},g_{6_2}\right)\right](t), \tag{110}$$

with g_4y being the function from $[0,T]$ to Y_γ^p defined by $(g_4y)(t)=g_4(t)y$. We will find conditions on $\delta_j,\delta_{l_k},\delta_4,\gamma\in(0,1)$, $j=1,2,5$, $l=3,6$, $k=1,2$, in order that $Q_jg_j\in C^{\tau_j}([0,T];X)$, $Q_l(g_{l_1},g_{l_2})\in C^{\tau_l}([0,T];X)$ and $Q_4(g_4,y)\in C^{\tau_4}([0,T];X)$ for opportunely chosen $\tau_j,\tau_l,\tau_4\in(0,1)$. We emphasize of the presence of the increment $g_2(s)-g_2(t)$ inside the integral defining Q_2g_2. As we will see, and differently from Q_1, it is just this presence which makes Q_2g_2 well-defined for smooth enough functions g_2. This is the reason why the operator Q_2 as it was defined in [20, formula (4.12)] can make no sense and has to be replaced with that defined by the present (106) (cf. the appendix below). We begin our analysis on the Q_j's with the following result proven in [20, Lemma 4.1]. Since we will need it later, here, removing some misprints in [20], we report its short proof for the reader's convenience.

Lemma 22. *Let* $\alpha+\beta>1$ *in (H1). Then, for every* $\delta_1\in(0,(\alpha+\beta-1)/\alpha)$, *the operator* Q_1 *defined by (105) maps* $C^{\delta_1}([0,T];X)$ *into* $C_0^{\delta_1}([0,T];X)$, *and for every* $t\in[0,T]$ *satisfies the following estimate, where* $p\in(\alpha/(\alpha+\beta-1-\alpha\delta_1),\infty)$ *as follows:*

$$\|Q_1g_1\|_{\delta_1,0,t;X} \le C_1(t)\left(\int_0^t \|g_1\|_{\delta_1,0,s;X}^p ds\right)^{1/p}. \tag{111}$$

Here $C_1(t)$ *is a nondecreasing function of* t *depending also on* α, β, δ_1, *and* p'.

Proof. Let $g_1 \in C^{\delta_1}([0,T];X)$, $\delta_1 \in (0,(\alpha+\beta-1)/\alpha)$, and $t \in [0,T]$. From (14) and the Hölder inequality with $p \in (\alpha/(\alpha+\beta-1-\alpha\delta_1),\infty) \subsetneq (1,\infty)$, for any $\tau \in [0,t]$, we deduce that

$$\begin{aligned}&\|[Q_1 g_1](\tau)\|_X\\ &\quad\le \tilde{c}_{\alpha,\beta,0}\int_0^\tau (\tau-s)^{(\beta-1)/\alpha}\|g_1\|_{0,0,s;X}\mathrm{d}s\\ &\quad\le c_{31}\tau^{[\alpha-(1-\beta)p']/(\alpha p')}\left(\int_0^\tau \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p}\\ &\quad\le c_{31}\tau^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}\tau^{\delta_1}\left(\int_0^\tau \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p},\end{aligned} \tag{112}$$

where $c_{31} = \tilde{c}_{\alpha,\beta,0}\alpha^{1/p'}[\alpha-(1-\beta)p']^{-1/p'}$. Here $\alpha-(1+\alpha\delta_1-\beta)p' > 0$, since $p' \in (1,\alpha/(1+\alpha\delta_1-\beta))$. For $1-1/p > 1-(\alpha+\beta-1-\alpha\delta_1)/\alpha = (1+\alpha\delta_1-\beta)/\alpha$. passing to the supremum with respect to $\tau \in [0,t]$ in (112) we thus find

$$\begin{aligned}&\|Q_1 g_1\|_{0,0,t;X}\\ &\quad\le c_{31}\, t^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}t^{\delta_1}\left(\int_0^t \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p}.\end{aligned} \tag{113}$$

Now let (since $[Q_1g_1](0) = 0$, the case $t_1 = 0$ follows from (112) with $\tau = t_2$) $0 < t_1 < t_2 \le t$. The change of variable $t-s=r$ in (105) leads us to $[Q_1g_1](t_2)-[Q_1g_1](t_1) = \sum_{k=1}^2 I_{k;t_1,t_2,g_1}$, where $I_{1;t_1,t_2,g_1} := \int_{t_1}^{t_2} \mathrm{e}^{rA}g_1(t_2-r)\,\mathrm{d}r$ and $I_{2;t_1,t_2,g_1} := \int_0^{t_1}\mathrm{e}^{rA}[g_1(t_2-r)-g_1(t_1-r)]\,\mathrm{d}r$. Reasoning as in (112) and using the inequality $t_2^\mu - t_1^\mu \le (t_2-t_1)^\mu$, $\mu \in (0,1]$, we get

$$\begin{aligned}&\|I_{1;t_1,t_2,g_1}\|_X\\ &\le c_{31}(t_2-t_1)^{[\alpha-(1-\beta)p']/(\alpha p')}\left(\int_{t_1}^{t_2}\|g_1\|^p_{\delta_1,0,t_2-r;X}\mathrm{d}r\right)^{1/p}\\ &\le c_{31}t_2^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}(t_2-t_1)^{\delta_1}\left(\int_0^t\|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.\end{aligned} \tag{114}$$

Similarly, but taking advantage from $g_1 \in C^{\delta_1}([0,T];X)$, we obtain

$$\begin{aligned}&\|I_{2;t_1,t_2,g_1}\|_X\\ &\le \tilde{c}_{\alpha,\beta,0}(t_2-t_1)^{\delta_1}\int_0^{t_1} r^{(\beta-1)/\alpha}|g_1|_{\delta_1,0,t_2-r;X}\mathrm{d}r\\ &\le c_{31}t_1^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}t_1^{\delta_1}(t_2-t_1)^{\delta_1}\left(\int_0^t\|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.\end{aligned} \tag{115}$$

Thus, letting $\tilde{c}_1(t) = c_{31}t^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}$ from (114) and (115) it follows that

$$\begin{aligned}&\|[Q_1g_1](t_2)-[Q_1g_1](t_1)\|_X\\ &\quad\le \tilde{c}_1(t)\left(t^{\delta_1}+1\right)(t_2-t_1)^{\delta_1}\left(\int_0^t\|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.\end{aligned} \tag{116}$$

Finally, summing up (113) and (116) and using $\int_0^t \|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r = \int_0^t \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s$, we derive (111) with $C_1(t) = \tilde{c}_1(t)(2t^{\delta_1}+1)$. This completes the proof. □

Remark 23. We stress that if we renounce to its Hölder regularity, then for Q_1g_1 to be well-defined it suffices that α and β are as in Lemma 22 and that g_1 is merely in $C([0,T];X)$. In fact (see the last part of the proof of Corollary 14, replacing there x with $g_1(s)$), $\|[Q_1g_1](t)\|_X \le \alpha(\alpha+\beta-1)^{-1}\tilde{c}_{\alpha,\beta,0}\|g_1\|_{0,0,t;X}t^{(\alpha+\beta-1)/\alpha}$, $t \in [0,T]$.

Lemma 24. *Let $3\alpha+\beta > 3$ in (H1). Then, for every $\delta_2 \in ((3-2\alpha-\beta)/\alpha,1)$, the operator Q_2 defined by* (106) *maps $C^{\delta_2}([0,T];X)$ into $C_0^{\nu_2}([0,T];X)$, $\nu_2 = (\alpha\delta_2+2\alpha+\beta-3)/\alpha \in (0,\delta_2]$, and for every $t \in [0,T]$ it satisfies the following estimate:*

$$\|Q_2g_2\|_{\nu_2,0,t;X} \le C_2(t)\,|g_2|_{\delta_2,0,t;X}. \tag{117}$$

Here $C_2(t)$ is a nondecreasing function of t depending also on α, β, and δ_2.

Proof. Denote by $\overline{\alpha}$ the number $(1-\alpha)/\alpha$. In particular, since $3\alpha+\beta>3$ implies $\alpha \in (2/3,1]$, we have $\overline{\alpha} \in [0,1/2)$. Let $t \in [0,T]$, $g_2 \in C^{\delta_2}([0,T];X)$, $\delta_2 \in ((3-2\alpha-\beta)/\alpha,1)$, and $\nu_2 = (\alpha\delta_2+2\alpha+\beta-3)/\alpha \in (0,\delta_2]$. We notice that $(\alpha\delta_2+\beta-2)/\alpha = \nu_2+\overline{\alpha}-1$ and $(\alpha\delta_2+\beta-3)/\alpha = \nu_2-2$. Then, using (14) with $\theta = 1$, for every $\tau \in [0,t]$ we obtain

$$\begin{aligned}&\|[Q_2g_2](\tau)\|_X\\ &\quad\le \tilde{c}_{\alpha,\beta,1}|g_2|_{\delta_2,0,\tau;X}\int_0^\tau (\tau-s)^{(\alpha\delta_2+\beta-2)/\alpha}\mathrm{d}s\\ &\quad= c_{32}|g_2|_{\delta_2,0,\tau;X}\tau^{\nu_2+\overline{\alpha}},\end{aligned} \tag{118}$$

where $c_{32} = \tilde{c}_{\alpha,\beta,1}(\nu_2+\overline{\alpha})^{-1}$. Hence

$$\|Q_2g_2\|_{0,0,t;X} \le c_{32}|g_2|_{\delta_2,0,t;X}t^{\nu_2+\overline{\alpha}}. \tag{119}$$

Now let (since $[Q_2g_2](0) = 0$, the case $t_1 = 0$ follows from (118) with $\tau = t_2$) $0 < t_1 < t_2 \le t$. We have

$[Q_2g_2](t_2) - [Q_2g_2](t_1) = \sum_{k=1}^{3} J_{k;t_1,t_2,g_2}$, where for a function $g : [0,T] \to X$ we set

$$J_{1;t_1,t_2,g} := \int_0^{t_1} \left\{ \left[(-A)^1\right]^\circ e^{(t_2-s)A} - \left[(-A)^1\right]^\circ e^{(t_1-s)A} \right\} \times \left[g(s) - g(t_1)\right] ds,$$
$$J_{2;t_1,t_2,g} := \int_0^{t_1} \left[(-A)^1\right]^\circ e^{(t_2-s)A} \left[g(t_1) - g(t_2)\right] ds,$$
$$J_{3;t_1,t_2,g} := \int_{t_1}^{t_2} \left[(-A)^1\right]^\circ e^{(t_2-s)A} \left[g(s) - g(t_2)\right] ds. \quad (120)$$

First, using (13) with $(s,t,\theta) = (t_1 - s, t_2 - s, 1)$, $s \in (0,t_1)$, and (14) with $\theta = 2$, and letting $(c_{33}, c_{34}) = (\widetilde{c}_{\alpha,\beta,2}(1-\nu_2)^{-1}, c_{33}\nu_2^{-1})$, we get

$$\begin{aligned}\left\|J_{1;t_1,t_2,g_2}\right\|_X &\le \widetilde{c}_{\alpha,\beta,2}|g_2|_{\delta_2,0,t_1;X}\int_0^{t_1}\left[\int_{t_1-s}^{t_2-s}\xi^{(\beta-3)/\alpha}d\xi\right](t_1-s)^{\delta_2}ds\\ &\le \widetilde{c}_{\alpha,\beta,2}|g_2|_{\delta_2,0,t_1;X}\int_0^{t_1}\left[\int_{t_1-s}^{t_2-s}\xi^{(\alpha\delta_2+\beta-3)/\alpha}d\xi\right]ds\\ &= c_{33}|g_2|_{\delta_2,0,t_1;X}\int_0^{t_1}\left[(t_1-s)^{\nu_2-1}-(t_2-s)^{\nu_2-1}\right]ds\\ &= c_{34}|g_2|_{\delta_2,0,t_1;X}\left[t_1^{\nu_2}+(t_2-t_1)^{\nu_2}-t_2^{\nu_2}\right]\\ &\le c_{34}|g_2|_{\delta_2,0,t_2;X}(t_2-t_1)^{\nu_2}.\end{aligned} \quad (121)$$

Let us turn to $J_{2;t_1,t_2,g_2}$. We first observe that the integral $\int_0^{t_1}[(-A)^1]^\circ e^{(t_2-s)A}ds$ is convergent. For, $\| \int_0^{t_1}[(-A)^1]^\circ e^{(t_2-s)A} ds\|_X \le \widetilde{c}_{\alpha,\beta,1}\int_0^{t_1}(t_2-s)^{(\beta-2)/\alpha}ds \le C_{\alpha,\beta,t_1,t_2}$, where C_{α,β,t_1,t_2} is equal to $\widetilde{c}_{\alpha,\beta,1}\ln[t_2(t_2-t_1)^{-1}]$ if $\beta = 1$ and to $\alpha(2-\alpha-\beta)^{-1}$ $\widetilde{c}_{\alpha,\beta,1}[(t_2-t_1)^{(\alpha+\beta-2)/\alpha} - t_2^{(\alpha+\beta-2)/\alpha}]$ if $\beta \in (0,1)$. Thus, we may rewrite it as $-\int_{t_2}^{t_2-t_1}[(-A)^1]^\circ e^{rA}dr = \int_{t_2}^{t_2-t_1} D_r e^{rA}dr = e^{(t_2-t_1)A} - e^{t_2A}$. Consequently,

$$\begin{aligned}\left\|J_{2;t_1,t_2,g_2}\right\|_X &\le \widetilde{c}_{\alpha,\beta,0}\left[(t_2-t_1)^{(\beta-1)/\alpha}+t_2^{(\beta-1)/\alpha}\right]|g_2|_{\delta_2,0,t_2;X}(t_2-t_1)^{\delta_2}\\ &\le \widetilde{c}_{\alpha,\beta,0}\left\{1+\left[t_2(t_2-t_1)^{-1}\right]^{(\beta-1)/\alpha}\right\}\\ &\quad\times|g_2|_{\delta_2,0,t_2;X}(t_2-t_1)^{(\alpha\delta_2+\beta-1)/\alpha}\\ &\le 2\widetilde{c}_{\alpha,\beta,0}|g_2|_{\delta_2,0,t_2;X}(t_2-t_1)^{\nu_2+2\overline{\alpha}},\end{aligned} \quad (122)$$

where we have used $[t_2(t_2-t_1)^{-1}]^{(\beta-1)/\alpha} \le 1$ and $(\alpha\delta_2 + \beta - 1)/\alpha = \nu_2 + 2\overline{\alpha}$. As far as $J_{3;t_1,t_2,g_2}$ is concerned, instead, reasoning as in the derivation of (118) we find

$$\begin{aligned}\left\|J_{3;t_1,t_2,g_2}\right\|_X &\le \widetilde{c}_{\alpha,\beta,1}|g_2|_{\delta_2,0,t_2;X}\int_{t_1}^{t_2}(t_2-s)^{\nu_2+\overline{\alpha}-1}ds\\ &= c_{32}|g_2|_{\delta_2,0,t_2;X}(t_2-t_1)^{\nu_2+\overline{\alpha}}.\end{aligned} \quad (123)$$

Then, summing up (121)–(123) and letting $\widetilde{c}_2(t) = c_{34} + 2\widetilde{c}_{\alpha,\beta,0}t^{2\overline{\alpha}} + c_{32}t^{\overline{\alpha}}$, we obtain

$$\begin{aligned}\left\|[Q_2g_2](t_2) - [Q_2g_2](t_1)\right\|_X &\le \sum_{k=1}^{3}\left\|J_{k;t_1,t_2,g_2}\right\|_X\\ &\le \widetilde{c}_2(t)|g_2|_{\delta_2,0,t;X}(t_2-t_1)^{\nu_2}.\end{aligned} \quad (124)$$

Finally, (119) and (124) yield (117) with $C_2(t) = c_{32}t^{\nu_2+\overline{\alpha}} + \widetilde{c}_2(t)$. □

Remark 25. In particular, Lemma 24 establishes that, with the exception of the case $\beta = 1$ in which $\nu_2 = \delta_2$, Q_2 produces a loss of regularity equal to $\delta_2 - \nu_2 = (3 - 2\alpha - \beta)/\alpha$.

As Corollary 14, the next result will be needed to prove the equivalence between problem (170) and the fixed-point equation (179). From now on, if $A^{-1} \in \mathscr{L}(X)$ and $g \in C^\delta([0,T];X)$, $\delta \in [0,1)$, with $A^{-1}g$ we will always mean the function in $C^\delta([0,T];\mathscr{D}(A))$ defined by $(A^{-1}g)(t) = A^{-1}(g(t))$. Notice that $\|A^{-1}g\|_{\delta,0,t;\mathscr{D}(A)} \le \|g\|_{\delta,0,t;X}$, $t \in [0,T]$.

Corollary 26.

(i) *Let $2\alpha + \beta > 2$ in (H1). Then, for every $g \in C^\delta([0,T];X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$,*

$$A^{-1}[Q_2g](t) = -\int_0^t e^{(t-s)A}\left[g(s) - g(t)\right]ds, \quad t \in [0,T]. \quad (125)$$

(ii) *Let $\alpha + \beta > 1$ in (H1). Then, for every $g \in C([0,T];X)$*

$$\left[Q_2\left(A^{-1}g\right)\right](t) = -\int_0^t e^{(t-s)A}\left[g(s) - g(t)\right]ds, \quad t \in [0,T]. \quad (126)$$

Proof. Of course, it suffices to assume that $t \in (0,T]$. Let us first prove (i). So, let $2\alpha + \beta > 2$, $g \in C^\delta([0,T];X)$, $\delta \in ((2 - \alpha - \beta)/\alpha, 1)$, and $t \in (0,T]$, and we observe that both sides of (125) are well defined. Indeed, replacing the pair (g_2, δ_2) with (g, δ), from (118) we get

$$\begin{aligned}\left\|[Q_2g](t)\right\|_X &\le \widetilde{c}_{\alpha,\beta,1}\alpha(\alpha\delta+\alpha+\beta-2)^{-1}|g|_{\delta,0,t;X}t^{(\alpha\delta+\alpha+\beta-2)/\alpha}.\end{aligned} \quad (127)$$

On the other side, $I_{t,g}=\int_0^t \mathrm{e}^{(t-s)A}[g(s)-g(t)]\mathrm{d}s$ satisfies

$$\begin{aligned}\left\|I_{t,g}\right\|_X &\leq \widetilde{c}_{\alpha,\beta,0}|g|_{\delta,0,t;X}\int_0^t (t-s)^{(\alpha\delta+\beta-1)/\alpha}\mathrm{d}s\\ &\leq c_{35}|g|_{\delta,0,t;X}t^{(\alpha+\alpha\delta+\beta-1)/\alpha},\end{aligned}\tag{128}$$

where $c_{35}=\alpha(\alpha\delta+\alpha+\beta-1)^{-1}\widetilde{c}_{\alpha,\beta,0}$. Then, commuting $A^{-1}\in\mathscr{L}(X)$ with the integral signs, using (80) with $\theta=1$, and taking into account (7), we find

$$\begin{aligned}&A^{-1}\left[Q_2g_2\right](t)\\ &=A^{-1}\int_0^t\left[-\frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{(t-s)\lambda}A^\circ(\lambda I-A)^{-1}\mathrm{d}\lambda\right]\left[g_2(s)-g_2(t)\right]\mathrm{d}s\\ &=-\int_0^t\left[\frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{(t-s)\lambda}A^{-1}A^\circ(\lambda I-A)^{-1}\mathrm{d}\lambda\right]\left[g_2(s)-g_2(t)\right]\mathrm{d}s\\ &=-\int_0^t\left[\frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{(t-s)\lambda}(\lambda I-A)^{-1}\mathrm{d}\lambda\right]\left[g_2(s)-g_2(t)\right]\mathrm{d}s.\end{aligned}\tag{129}$$

Since $(2\pi i)^{-1}\int_\Gamma \mathrm{e}^{(t-s)\lambda}(\lambda I-A)^{-1}\mathrm{d}\lambda=\mathrm{e}^{(t-s)A}$, the proof of (125) is complete. We now prove (ii). Let $\alpha+\beta>1$, $g\in C([0,T];X)$ and $t\in(0,T]$. Then, for every $\gamma\in(2-\alpha-\beta,1)$, the same reasonings made to derive (88), except for replacing x with g$(s)-g(t)$, yield

$$\begin{aligned}&\left\|\left[Q_2\left(A^{-1}g\right)\right](t)\right\|_X\\ &\quad\leq 2c_{22}c_{\alpha,\beta,\gamma}c_0\left\|A^{-1}\right\|_{\mathscr{L}(X)}^{1-\gamma}\|g\|_{0,0,t;X}t^{(\alpha+\beta+\gamma-2)/\alpha}.\end{aligned}\tag{130}$$

Hence, $[Q_2(A^{-1}g)](t)$ being meaningful, we obtain (126) simply applying to it formula (89) with $\zeta=1$ and then using $[(-A)^0]^\circ\mathrm{e}^{(t-s)A}=\mathrm{e}^{(t-s)A}$, $s\in(0,t)$. In particular, a better estimate than (130) holds. For, $[Q_2(A^{-1}g)](t)=-\int_0^t \mathrm{e}^{(t-s)A}[g(s)-g(t)]\mathrm{d}s$ satisfies

$$\begin{aligned}\left\|\left[Q_2\left(A^{-1}g\right)\right](t)\right\|_X &\leq 2\widetilde{c}_{\alpha,\beta,0}\|g\|_{0,0,t;X}\int_0^t(t-s)^{(\beta-1)/\alpha}\mathrm{d}s\\ &\leq 2c_{36}\|g\|_{0,0,t;X}t^{(\alpha+\beta-1)/\alpha},\end{aligned}\tag{131}$$

where $c_{36}=\alpha(\alpha+\beta-1)^{-1}\widetilde{c}_{\alpha,\beta,0}$. The proof is complete. □

Let us now examine the operator Q_3 defined by (107). To this purpose we need the following result which is proved in [20, Corollary 3.2].

Lemma 27. *Let $\delta_{3_k}\in(0,1)$, $k=1,2$, be such that $\sigma_3=\delta_{3_1}+\delta_{3_2}\in(0,1/p')$, $p\in(1/(1-\delta_{3_1}),\infty)$. Then the convolution operator $\mathscr{K}$ defined by* (104) *maps $C^{\delta_{3_1}}([0,T];X_1)\times C^{\delta_{3_2}}([0,T];X_2)$ into $C_0^{\sigma_3}([0,T];X)$, and for every $t\in[0,T]$ satisfies the following estimate:*

$$\begin{aligned}&\left\|\mathscr{K}(g_{3_1},g_{3_2})\right\|_{\sigma_3,0,t;X}\\ &\quad\leq t^{-\sigma_3+1/p'}\widetilde{c}_3(t)\left\|g_{3_1}\right\|_{\delta_{3_1},0,t;X_1}\left(\int_0^t\left\|g_{3_2}\right\|_{\delta_{3_2},0,s;X_2}^p\mathrm{d}s\right)^{1/p}.\end{aligned}\tag{132}$$

Here $\widetilde{c}_3(t)$ is a nondecreasing function of t depending also on δ_{3_1} and δ_{3_2}. Further, in the cases $\delta_{3_1}\in(0,1)$, $\delta_{3_2}=0$, and $\delta_{3_1}=\delta_{3_2}=0$, the following estimates hold, respectively, as follows:

$$\begin{aligned}&\left\|\mathscr{K}\left(g_{3_1},g_{3_2}\right)\right\|_{\delta_{3_1},0,t;X}\\ &\quad\leq C_0t^{1-\delta_{3_1}}\left(1+t^{\delta_{3_1}}\right)\left\|g_{3_1}\right\|_{\delta_{3_1},0,t;X_1}\left\|g_{3_2}\right\|_{0,0,t;X_2},\end{aligned}\tag{133}$$

$$\left\|\mathscr{K}\left(g_{3_1},g_{3_2}\right)\right\|_{0,0,t;X}\leq C_0t\left\|g_{3_1}\right\|_{0,0,t;X_1}\left\|g_{3_2}\right\|_{0,0,t;X_2}.$$

From Lemmas 24 and 27 we obtain the following Lemma 28.

Lemma 28. *Let α and β be as in Lemma 24. Then, for every $\delta_{3_1}\in((3-2\alpha-\beta)/\alpha,1)$ and $\delta_{3_2}\in(0,1)$ such that $\sigma_3=\delta_{3_1}+\delta_{3_2}\in((3-2\alpha-\beta)/\alpha,1/p')$, $p\in(1/(1-\delta_{3_1}),\infty)$, the operator Q_3 defined by* (107) *maps $C^{\delta_{3_1}}([0,T];X_1)\times C^{\delta_{3_2}}([0,T];X_2)$ into $C_0^{\nu_3}([0,T];X)$, $\nu_3=(\alpha\sigma_3+2\alpha+\beta-3)/\alpha$, and for every $t\in[0,T]$ satisfies the following estimate:*

$$\begin{aligned}&\left\|Q_3\left(g_{3_1},g_{3_2}\right)\right\|_{\nu_3,0,t;X}\\ &\quad\leq t^{-\sigma_3+1/p'}C_2(t)\widetilde{c}_3(t)\left\|g_{3_1}\right\|_{\delta_{3_1},0,t;X_1}\left(\int_0^t\left\|g_{3_2}\right\|_{\delta_{3_2},0,s;X_2}^p\mathrm{d}s\right)^{1/p}.\end{aligned}\tag{134}$$

Proof. First, if $\delta_{3_1}\in((3-2\alpha-\beta)/\alpha,1)$ and $p\in(1/(1-\delta_{3_1}),\infty)$, then $1/p'\in(\delta_{3_1},1)\subsetneq((3-2\alpha-\beta)/\alpha,1)$. Consequently, the assumption $\sigma_3=\delta_{3_1}+\delta_{3_2}\in((3-2\alpha-\beta)/\alpha,1/p')$, $\delta_{3_2}\in(0,1)$, makes sense. Now, Lemma 27 yields $\mathscr{K}(g_{3_1},g_{3_2})\in C_0^{\sigma_3}([0,T];X)$ for any pair $(g_{3_1},g_{3_2})\in C^{\delta_{3_1}}([0,T];X_1)\times C^{\delta_{3_2}}([0,T];X_2)$. Then, recalling that $Q_3(g_{3_1},g_{3_2})=Q_2\mathscr{K}(g_{3_1},g_{3_2})$, the assertion follows from Lemma 24, with δ_2 and g_2 being replaced by σ_3 and $\mathscr{K}(g_{3_1},g_{3_2})$, respectively. Finally, (134) follows from (117) and (132). □

We can now restore the loss of regularity produced by Q_2.

Proposition 29. *Let $5\alpha+2\beta>6$ in (H1). Then, for every $\delta_3\in((3-2\alpha-\beta)/\alpha,1/2)$, the operator Q_3 defined by* (107) *maps $C^{\delta_3}([0,T];X_1)\times C^{\delta_3}([0,T];X_2)$ into $C_0^{\delta_3}([0,T];X)$, and* *for every* $t \in [0,T]$ *satisfies the following estimate, where* $p \in (1/(1-2\delta_3),\infty)$ *and* $C_3(t) = C_2(t)\tilde{c}_3(t)\max\{1, t^{(\alpha\delta_3+2\alpha+\beta-3)/\alpha}\}$:

$$\left\|Q_3\left(g_{3_1}, g_{3_2}\right)\right\|_{\delta_3,0,t;X} \leq t^{1-2\delta_3-1/p} C_3(t) \left\|g_{3_1}\right\|_{\delta_3,0,t;X_1} \left(\int_0^t \left\|g_{3_2}\right\|^p_{\delta_3,0,s;X_2} \mathrm{d}s\right)^{1/p}. \quad (135)$$

Proof. Let $\delta_3 \in ((3-2\alpha-\beta)/\alpha, 1/2)$ and let $p \in (1/(1-2\delta_3),\infty) \subsetneq (1/(1-\delta_3),\infty)$. Then, $2\delta_3 \in ((6-4\alpha-2\beta)/\alpha, 1/p') \subseteq ((3-2\alpha-\beta)/\alpha, 1/p')$. We are thus in position to apply Lemma 28 with $\delta_{3_1} = \delta_{3_2} = \delta_3$ from which we deduce that Q_3 maps $C^{\delta_3}([0,T];X_1) \times C^{\delta_3}([0,T];X_2)$ into $C_0^{\nu_3}([0,T];X)$, $\nu_3 = (2\alpha\delta_3 + 2\alpha + \beta - 3)/\alpha$. But, since our choice for δ_3 implies $\nu_3 > \delta_3$, we *a fortiori* have the fact that Q_3 maps $C^{\delta_3}([0,T];X_1) \times C^{\delta_3}([0,T];X_2)$ into $C_0^{\delta_3}([0,T];X)$. Finally, (135) follows from (134) and the estimate $\|g\|_{\gamma,0,t;X} \leq \max\{1, t^{\delta-\gamma}\}\|g\|_{\delta,0,t;X}$, $g \in C^{\delta}([0,T];X)$, $\delta \geq \gamma$. □

The next Lemma 30 concerns the operator Q_4. Its proof is similar to that of Lemma 24, but with the essential difference that the presence of $y \in Y^p_\gamma$ allows us to use estimate (79) in place of (14). As a consequence and provided to choose γ large enough, we will achieve a better result in which any loss of regularity is observed.

Lemma 30. *Let* $2\alpha + \beta > 2$ *in (H1) and* $r \in [1,\infty]$. *Then, for every* $\delta_4 \in (0,1)$ *and* $\gamma \in (3-2\alpha-\beta, 1)$ *the operator* Q_4 *defined by* (108) *maps* $C^{\delta_4}([0,T];\mathbf{C}) \times Y^r_\gamma$, $Y^r_\gamma \in \{(X,\mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$, *into* $C_0^{\delta_4}([0,T];X)$, *and for every* $t \in [0,T]$ *satisfies the following estimate:*

$$\left\|Q_4\left(g_4, y\right)\right\|_{\delta_4,0,t;X} \leq C_4(t)\, t^{(2\alpha+\beta+\gamma-3)/\alpha} |g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y^r_\gamma}. \quad (136)$$

Here $C_4(t)$ *is a nondecreasing function of* t *depending on* α, β, δ_4, γ *and* r.

Proof. Let $t \in [0,T]$, $g_4 \in C^{\delta_4}([0,T];\mathbf{C})$, $\delta_4 \in (0,1)$, and $y \in Y^r_\gamma$, $\gamma \in (3-2\alpha-\beta,1)$, $r \in [1,\infty]$. As in the proof of Lemma 24 we set $\overline{\alpha} = (1-\alpha)/\alpha$ and we observe that, since $2\alpha+\beta > 2$ implies $\alpha \in (1/2,1]$, here $\overline{\alpha} \in [0,1)$. Furthermore, we denote by $\sigma_{\alpha,\beta,\gamma}$ the number $(2\alpha+\beta+\gamma-3)/\alpha \in (0,1)$, so that the exponents $(\beta+\gamma-2)/\alpha$ and $(\beta+\gamma-3)/\alpha$ appearing in (79) with $\theta = 1$ and $\theta = 2$ may be rewritten, as $\sigma_{\alpha,\beta,\gamma} + \overline{\alpha} - 1$ and $\sigma_{\alpha,\beta,\gamma} - 2$, respectively. Then, using (79) with $\theta = 1$, we obtain

$$\begin{aligned} \left\|\left[Q_4\left(g_4, y\right)\right](\tau)\right\|_X &\leq c_{22} |g_4|_{\delta_4,0,\tau;\mathbf{C}} \|y\|_{Y^r_\gamma} \int_0^\tau (\tau - s)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s \qquad (137) \\ &= c_{37} |g_4|_{\delta_4,0,\tau;\mathbf{C}} \|y\|_{Y^r_\gamma} \tau^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}, \quad \forall \tau \in [0,t], \end{aligned}$$

where $c_{37} = c_{22}(\delta_4 + \sigma_{\alpha,\beta,\gamma} + \overline{\alpha})^{-1}$. Hence, taking the supremum with respect to $\tau \in [0,t]$, one has

$$\left\|Q_4\left(g_4, y\right)\right\|_{0,0,t;X} \leq c_{37} |g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y^r_\gamma} t^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}. \quad (138)$$

Now, let (since $[Q_4(g_4,y)](0) = 0$, the case $t_1 = 0$ follows from (137) with $\tau = t_2$) $0 < t_1 < t_2 \leq t$. We have $[Q_4(g_4,y)](t_2) - [Q_4(g_4,y)](t_1) = \sum_{k=1}^3 J_{k;t_1,t_2,g_4 y}$, the $J_{k;t_1,t_2,g}$'s, $g : [0,T] \to X$, being as in (120). Using (13) with $(s,t,\theta) = (t_1 - s, t_2 - s, 1)$, $s \in (0,t_1)$, and (79) with $\theta = 2$, and letting $(c_{38}, c_{39}) = (c_{22}(1-\delta_4)^{-1}, c_{38}\delta_4^{-1})$, we get

$$\begin{aligned} \left\|J_{1;t_1,t_2,g_4 y}\right\|_X &\leq c_{22} |g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y^r_\gamma} \times \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{\sigma_{\alpha,\beta,\gamma}-2} \mathrm{d}\xi\right] (t_1 - s)^{\delta_4} \mathrm{d}s \\ &\leq c_{22} |g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y^r_\gamma} \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{\delta_4+\sigma_{\alpha,\beta,\gamma}-2} \mathrm{d}\xi\right] \mathrm{d}s \\ &\leq c_{22} |g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y^r_\gamma} \times \int_0^{t_1} (t_2 - s)^{\sigma_{\alpha,\beta,\gamma}} \left[\int_{t_1-s}^{t_2-s} \xi^{\delta_4-2} \mathrm{d}\xi\right] \mathrm{d}s \\ &\leq c_{38} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} t_2^{\sigma_{\alpha,\beta,\gamma}} \times \int_0^{t_1} \left[(t_1 - s)^{\delta_4-1} - (t_2 - s)^{\delta_4-1}\right] \mathrm{d}s \\ &= c_{39} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} t_2^{\sigma_{\alpha,\beta,\gamma}} \left[t_1^{\delta_4} + (t_2 - t_1)^{\delta_4} - t_2^{\delta_4}\right] \\ &\leq c_{39} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} t_2^{\sigma_{\alpha,\beta,\gamma}} (t_2 - t_1)^{\delta_4}. \end{aligned} \quad (139)$$

Now, let us examine $J_{k;t_1,t_2,g_4 y}$, $k = 2,3$. First, using (79) with $\theta = 1$, we find

$$\begin{aligned} \left\|J_{2;t_1,t_2,g_4 y}\right\|_X &\leq c_{22} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} \left[\int_0^{t_1} (t_2 - s)^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s\right] (t_2 - t_1)^{\delta_4} \\ &= c_{40} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} \left[t_2^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}} - (t_2 - t_1)^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}\right] (t_2 - t_1)^{\delta_4} \\ &\leq c_{40} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} t_2^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}} (t_2 - t_1)^{\delta_4}. \end{aligned} \quad (140)$$

Instead, the same computations made to derive (137) yield

$$\begin{aligned} \left\|J_{3;t_1,t_2,g_4 y}\right\|_X &\leq c_{22} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} \int_{t_1}^{t_2} (t_2 - s)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s \qquad (141) \\ &= c_{37} |g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y^r_\gamma} (t_2 - t_1)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}. \end{aligned}$$

From (139)–(141) and $\|[Q_4(g_4,y)](t_2) - [Q_4(g_4,y)](t_1)\|_X \leq \sum_{k=1}^3 \|J_{k;t_1,t_2,g_4 y}\|_X$, it follows that

$$\begin{aligned} &\left\|\left[Q_4\left(g_4, y\right)\right](t_2) - \left[Q_4\left(g_4, y\right)\right](t_1)\right\|_X \\ &\quad \leq \tilde{c}_4(t)\, t^{\sigma_{\alpha,\beta,\gamma}} |g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y^r_\gamma} (t_2 - t_1)^{\delta_4}, \end{aligned} \quad (142)$$

where $\widetilde{c}_4(t) = c_{39} + (c_{37} + c_{40})t^{\overline{\alpha}}$. Finally, summing up (138) and (142) we get (136) with $C_4(t) = c_{37}t^{\delta_4+\overline{\alpha}} + \widetilde{c}_4(t)$. The proof is complete. □

Remark 31. Notice that if $Y_\gamma^r = X_A^{\gamma,r}$, then in order to be sure that the conclusions of Lemma 30 hold with y which really belongs to some intermediate space between X and $\mathscr{D}(A)$ we have to choose $\gamma \in (3 - 2\alpha - \beta, \beta)$. This is possible, provided that the stronger assumption $2\alpha + \beta > 3 - \beta \geq 2$ is satisfied. Otherwise, if $2\alpha+\beta \in (2, 3-\beta]$, $\beta < 1$, then $\gamma \in (3-2\alpha-\beta, 1) \subsetneq [\beta, 1)$ and y may be contained in $\mathscr{D}(A)$.

Finally, for the operator Q_5 we have the following result. Again a loss of regularity is exhibited, even though of an amount smaller than that in Lemma 24 (cf. Remark 33).

Lemma 32. *Let $2\alpha + \beta > 2$ in (H1). Then, for every $\delta_5 \in ((2 - \alpha - \beta)/\alpha, 1)$, the operator Q_5 defined by* (109) *maps $C_0^{\delta_5}([0,T]; X)$ into $C_0^{\nu_5}([0,T]; X)$, $\nu_5 = (\alpha\delta_5 + \alpha + \beta - 2)/\alpha \in (0, \delta_5]$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\|Q_5 g_5\|_{\nu_5,0,t;X} \leq C_5(t) |g_5|_{\delta_5,0,t;X}. \tag{143}$$

Here $C_5(t)$ is a nondecreasing function of t depending also on α, β, and δ_5.

Proof. Let $g_5 \in C_0^{\delta_5}([0,T]; X)$, $\delta_5 \in ((2 - \alpha - \beta)/\alpha, 1)$, and $\nu_5 = (\alpha\delta_5 + \alpha + \beta - 2)/\alpha \in (0, \delta_5]$. We still let $\overline{\alpha} = (1 - \alpha)/\alpha$ and as in Lemma 30 we have $\overline{\alpha} \in [0, 1)$. Further, observe that $\delta_5 + (\beta - 1)/\alpha = \nu_5 + \overline{\alpha} \in (0, \delta_5]$. Let $t \in [0,T]$. Then, using (14) and $g_5(0) = 0$, we get

$$\begin{aligned} &\|Q_5 g_5\|_{0,0,t;X} \\ &\quad \leq \sup_{\tau\in[0,t]} \left[\widetilde{c}_{\alpha,\beta,0}\tau^{(\beta-1)/\alpha} + 1\right] |g_5|_{\delta_5,0,\tau;X}\tau^{\delta_5} \\ &\quad \leq \left[\widetilde{c}_{\alpha,\beta,0} + t^{(1-\beta)/\alpha}\right] |g_5|_{\delta_5,0,t;X}t^{\nu_5+\overline{\alpha}}. \end{aligned} \tag{144}$$

Now, let (since $[Q_5 g_5](0) = 0$, the case $t_1 = 0$ follows from (144) and $\|[Q_5 g_5](t_2)\|_X \leq \|Q_5 g_5\|_{0,0,t_2;X}$) $0 < t_1 < t_2 \leq t$. We have $[Q_5 g_5](t_2) - [Q_5 g_5](t_1) = \sum_{k=1}^{3} U_{k;t_1,t_2,g_5}$, where for a function $g : [0,T] \to X$ we let

$$\begin{aligned} U_{1;t_1,t_2,g} &:= e^{t_2 A}\left[g(t_2) - g(t_1)\right], \\ U_{2;t_1,t_2,g} &:= \left(e^{t_2 A} - e^{t_1 A}\right) g(t_1), \\ U_{3;t_1,t_2,g} &:= g(t_1) - g(t_2). \end{aligned} \tag{145}$$

First, since $t_2^{(\beta-1)/\alpha} \leq (t_2 - t_1)^{(\beta-1)/\alpha}$ for every $\beta \in (0, 1]$, we deduce that

$$\begin{aligned} \left\|U_{1;t_1,t_2,g_5}\right\|_X &\leq \widetilde{c}_{\alpha,\beta,0}t_2^{(\beta-1)/\alpha}|g_5|_{\delta_5,0,t_2;X}(t_2 - t_1)^{\delta_5} \\ &\leq \widetilde{c}_{\alpha,\beta,0}|g_5|_{\delta_5,0,t_2;X}(t_2 - t_1)^{\nu_5+\overline{\alpha}}. \end{aligned} \tag{146}$$

As far as $U_{2;t_1,t_2,g_5}$ is concerned, instead, rewriting $e^{t_2 A} - e^{t_1 A}$ as $-\int_{t_1}^{t_2}[(-A)^1]^\circ e^{rA}dr$ and using both $g_5(0) = 0$ and $(\alpha\delta_5 + \beta - 2)/\alpha = \nu_5 - 1$, it follows that

$$\begin{aligned} \left\|U_{2;t_1,t_2,g_5}\right\|_X &\leq \widetilde{c}_{\alpha,\beta,1}|g_5|_{\delta_5,0,t_1;X}t_1^{\delta_5}\int_{t_1}^{t_2} r^{(\beta-2)/\alpha}dr \\ &\leq \widetilde{c}_{\alpha,\beta,1}|g_5|_{\delta_5,0,t_1;X}\int_{t_1}^{t_2} r^{\nu_5-1}dr \\ &\leq \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1}|g_5|_{\delta_5,0,t_1;X}\left(t_2^{\nu_5} - t_1^{\nu_5}\right) \\ &\leq \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1}|g_5|_{\delta_5,0,t_1;X}(t_2 - t_1)^{\nu_5}. \end{aligned} \tag{147}$$

Then, since $\|U_{3;t_1,t_2,g_5}\|_X \leq |g_5|_{\delta_5,0,t_2;X}(t_2 - t_1)^{\delta_5}$, from (146) and (147) we find

$$\begin{aligned} &\|[Q_5 g_5](t_2) - [Q_5 g_5](t_1)\|_X \\ &\quad \leq \sum_{k=1}^{3}\left\|U_{k;t_1,t_2,g_5}\right\|_X \leq \widetilde{c}_5(t)|g_5|_{\delta_5,0,t;X}(t_2 - t_1)^{\nu_5}, \end{aligned} \tag{148}$$

where $\widetilde{c}_5(t) = \widetilde{c}_{\alpha,\beta,0}t^{\overline{\alpha}} + \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1} + t^{\delta_5-\nu_5}$. Summing up (144) and (148) we obtain (143) with $C_5(t) = [\widetilde{c}_{\alpha,\beta,0} + t^{(1-\beta)/\alpha}]t^{\nu_5+\overline{\alpha}} + \widetilde{c}_5(t)$. This completes the proof. □

Remark 33. Thus, with the exception of $\beta = 1$, Q_5 produces a loss of regularity equal to $\delta_5 - \nu_5 = (2 - \alpha - \beta)/\alpha \leq (3 - 2\alpha - \beta)/\alpha$. In this sense Q_5 behaves better than Q_2.

Remark 34. Notice that, under the weaker assumptions $\alpha + \beta > 1$ and $g_5 \in C([0,T]; X)$, (86) with $x = g_5(t)$, $t \in [0,T]$, yields $A^{-1}[Q_5 g_5](t) = [Q_5(A^{-1}g_5)](t) = \int_0^t e^{(t-s)A}g_5(t)ds$.

Similarly as we have done in Proposition 29 for restoring the loss of regularity produced by Q_2, we now show how Lemma 27 allows to restore that produced by Q_5. We begin with the following version of Lemma 28 relative to Q_6, and which is obtained combining Lemma 27 with Lemma 32 instead of Lemma 24.

Lemma 35. *Let α and β be as in Lemma 32. Then, for every $\delta_{6_1} \in ((2-\alpha-\beta)/\alpha, 1)$ and $\delta_{6_2} \in (0, 1)$ such that $\sigma_6 = \delta_{6_1} + \delta_{6_2} \in ((2 - \alpha - \beta)/\alpha, 1/p')$, $p \in (1/(1 - \delta_{6_1}), \infty)$, the operator Q_6 defined by* (110) *maps $C^{\delta_{6_1}}([0,T]; X_1) \times C^{\delta_{6_2}}([0,T]; X_2)$ into $C_0^{\nu_6}([0,T]; X)$, $\nu_6 = (\alpha\sigma_6 + \alpha + \beta - 2)/\alpha$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\begin{aligned} &\left\|Q_6\left(g_{6_1}, g_{6_2}\right)\right\|_{\nu_6,0,t;X} \\ &\quad \leq t^{-\sigma_6+1/p'}C_5(t)\widetilde{c}_3(t)\left\|g_{6_1}\right\|_{\delta_{6_1},0,t;X_1}\left(\int_0^t\left\|g_{6_2}\right\|_{\delta_{6_2},0,s;X_2}^p ds\right)^{1/p}. \end{aligned} \tag{149}$$

Proof. First, if $\delta_{6_1} \in ((2-\alpha-\beta)/\alpha, 1)$ and $p \in (1/(1-\delta_{6_1}), \infty)$, then $1/p' \in (\delta_{6_1}, 1) \subsetneq ((2 - \alpha - \beta)/\alpha, 1)$. Consequently, the assumption $\sigma_6 = \delta_{6_1} + \delta_{6_2} \in ((2-\alpha-\beta)/\alpha, 1/p')$ makes sense, provided to choose $\delta_{6_2} \in (0,1)$ small enough. Lemma 27 then yields $\mathscr{K}(g_{6_1}, g_{6_2}) \in C_0^{\sigma_6}([0,T];X)$ for any pair $(g_{6_1}, g_{6_2}) \in C^{\delta_{6_1}}([0,T];X_1) \times C^{\delta_{6_2}}([0,T];X_2)$. Then, since $Q_6(g_{6_1}, g_{6_2}) = Q_5\mathscr{K}(g_{6_1}, g_{6_2})$, the assertion follows from Lemma 32, with the pair (δ_5, g_5) being replaced by $(\sigma_6, \mathscr{K}(g_{6_1}, g_{6_2}))$. Finally, (149) follows from (143) and (132). □

From Lemma 35 we obtain the analogous of Proposition 29 for Q_6.

Proposition 36. *Let $3\alpha + 2\beta > 4$ in (H1). Then, for every $\delta_6 \in ((2-\alpha-\beta)/\alpha, 1/2)$, the operator Q_6 defined by (110) maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\delta_6}([0,T];X)$, and for every $t \in [0,T]$ satisfies the following estimate, where $p \in (1/(1-2\delta_6), \infty)$ and $C_6(t) = C_5(t)\tilde{c}_3(t)\max\{1, t^{(\alpha\delta_6+\alpha+\beta-2)/\alpha}\}$:*

$$\left\|Q_6\left(g_{6_1}, g_{6_2}\right)\right\|_{\delta_6,0,t;X} \leq t^{1-2\delta_6-1/p} C_6(t) \left\|g_{6_1}\right\|_{\delta_6,0,t;X_1} \left(\int_0^t \left\|g_{6_2}\right\|^p_{\delta_6,0,s;X_2} \mathrm{d}s\right)^{1/p}. \tag{150}$$

Proof. Let $\delta_6 \in ((2-\alpha-\beta)/\alpha, 1/2)$ and $p \in (1/(1-2\delta_6), \infty) \subsetneq (1/(1-\delta_6), \infty)$. Then, $2\delta_6 \in ((4-2\alpha-2\beta)/\alpha, 1/p') \subseteq ((2-\alpha-\beta)/\alpha, 1/p')$ and we can apply Lemma 35 with $\delta_{6_k} = \delta_6$, $k = 1, 2$. We thus deduce that Q_6 maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\nu_6}([0,T];X)$, $\nu_6 = (2\alpha\delta_6 + \alpha + \beta - 2)/\alpha$. But, since $\delta_6 > (2-\alpha-\beta)/\alpha$ implies $\nu_6 > \delta_6$, we *a fortiori* have the fact that Q_6 maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\delta_6}([0,T];X)$. Finally, (150) follows from (149) and $\|Q_6(g_{6_1}, g_{6_2})\|_{\delta_6,0,t;X} \leq \max\{1, t^{\nu_6-\delta_6}\}\|Q_6(g_{6_1}, g_{6_2})\|_{\nu_6,0,t;X}$. □

In Section 6 we will also encounter Q_5 acting on functions which enjoy some space regularity, that is, functions g_5 which are Hölder continuous in time with values on $Y^r_\gamma \in \{(X, \mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$. In this case Lemma 32 can be refined, and the loss of regularity produced by Q_5 is naturally restored by the additional condition of space regularity on g_5. In some sense, the forthcoming Corollary 38 is the analogous of Lemma 30, where the function $g_4 y$ involved in the definition of $Q_4(g_4, y)$ (cf. (108)) was of class $C^{\delta_4}([0,T];Y^r_\gamma)$.

Lemma 37. *Let $\alpha + \beta > 1$ in (H1) and $Y^r_\gamma \in \{(X, \mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$, $\gamma \in (2-\alpha-\beta, 1)$, $r \in [1,\infty]$. Then, for every $\delta_5 \in (0, (\alpha+\beta+\gamma-2)/\alpha]$, the operator Q_5 defined by (109) maps $C^{\delta_5}([0,T];Y^r_\gamma)$ into $C_0^{\delta_5}([0,T],X)$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\|Q_5 g_5\|_{\delta_5,0,t;X} \leq c_{41} t^{(\alpha+\beta+\gamma-2-\alpha\delta_5)/\alpha}\left(2t^{\delta_5}+1\right)\|g_5\|_{\delta_5,0,t;Y^r_\gamma}. \tag{151}$$

Here c_{41} is a positive constant depending on α, β, γ, and r.

Proof. Let $\gamma \in (2-\alpha-\beta, 1) \subseteq (1-\beta, 1)$ and let $\chi_{\alpha,\beta,\gamma}$ be the number $(\alpha+\beta+\gamma-2)/\alpha \in (0,1)$, so that the exponent $(\beta+\gamma-2)/\alpha$ in (79) with $\theta = 1$ is equal to $\chi_{\alpha,\beta,\gamma} - 1$. Let $g_5 \in C^{\delta_5}([0,T];Y^r_\gamma)$, $\delta_5 \in (0, \chi_{\alpha,\beta,\gamma}]$, $r \in [1,\infty]$. Since $[Q_5 g_5](0) = 0$, we assume that $t \in (0,T]$ and we observe that, due to Propositions 6 and 12, $[Q_5 g_5](t)$ is rewritten as follows:

$$\begin{aligned}
[Q_5 g_5](t) &= \left[e^{tA} - I\right] g_5(t) = \lim_{\varepsilon \to 0^+} \left[e^{tA} - e^{\varepsilon A}\right] g_5(t) \\
&= \lim_{\varepsilon \to 0^+} \int_\varepsilon^t D_s e^{sA} g_5(t)\,\mathrm{d}s \\
&= -\lim_{\varepsilon \to 0^+} \int_\varepsilon^t \left[(-A)^1\right]^\circ e^{sA} g_5(t)\,\mathrm{d}s \\
&= -\int_0^t \left[(-A)^1\right]^\circ e^{sA} g_5(t)\,\mathrm{d}s.
\end{aligned} \tag{152}$$

Indeed, for every $\varepsilon \in [0,t)$ and $x \in Y^r_\gamma$, (79) with $\theta = 1$ yields

$$\begin{aligned}
&\left\|\int_\varepsilon^t \left[(-A)^1\right]^\circ e^{sA} x\,\mathrm{d}s\right\|_X \\
&\quad \leq c_{22}\|x\|_{Y^r_\gamma} \int_\varepsilon^t s^{\chi_{\alpha,\beta,\gamma}-1}\mathrm{d}s \leq c_{41}\|x\|_{Y^r_\gamma}(t-\varepsilon)^{\chi_{\alpha,\beta,\gamma}},
\end{aligned} \tag{153}$$

where $c_{41} = c_{22}\chi^{-1}_{\alpha,\beta,\gamma}$. From (152) and (153) with $(\varepsilon, t, x) = (0, \tau, g_5(\tau))$ we thus get

$$\begin{aligned}
\|Q_5 g_5\|_{0,0,t;X} &= \sup_{\tau\in[0,t]} \left\|[Q_5 g_5](\tau)\right\|_X \\
&\leq c_{41}\|g_5\|_{0,0,t;Y^r_\gamma} t^{\chi_{\alpha,\beta,\gamma}}.
\end{aligned} \tag{154}$$

Now, let $0 \leq t_1 < t_2 \leq t$. From (152) it follows that $[Q_5 g_5](t_2) - [Q_5 g_5](t_1) = -\sum_{k=1}^2 V_{k;t_1,t_2,g_5}$, where for every function $g : [0,T] \to Y^p_\gamma$ we have set

$$\begin{aligned}
V_{1;t_1,t_2,g} &:= \int_0^{t_1} \left[(-A)^1\right]^\circ e^{sA}\left[g(t_2) - g(t_1)\right]\mathrm{d}s, \\
V_{2;t_1,t_2,g} &:= \int_{t_1}^{t_2} \left[(-A)^1\right]^\circ e^{sA} g(t_2)\,\mathrm{d}s.
\end{aligned} \tag{155}$$

Hence, using (153) with the triplet (ε, t, x) being replaced by $(0, t_1, g_5(t_2) - g_5(t_1))$ and $(t_1, t_2, g_5(t_2))$, respectively, we deduce that

$$\begin{aligned}
\left\|V_{1;t_1,t_2,g_5}\right\|_X &\leq c_{41}|g_5|_{\delta_5,0,t_2;Y^r_\gamma} t_1^{\chi_{\alpha,\beta,\gamma}}(t_2-t_1)^{\delta_5}, \\
\left\|V_{2;t_1,t_2,g_5}\right\|_X &\leq c_{41}\|g_5\|_{0,0,t_2;Y^r_\gamma}(t_2-t_1)^{\chi_{\alpha,\beta,\gamma}}.
\end{aligned} \tag{156}$$

As a consequence, since $\delta_5 \in (0, \chi_{\alpha,\beta,\gamma}]$,

$$\begin{aligned}
&\left\|[Q_5 g_5](t_2) - [Q_5 g_5](t_1)\right\|_X \\
&\quad \leq c_{41} t^{\chi_{\alpha,\beta,\gamma}-\delta_5}\left(t^{\delta_5}+1\right)\|g_5\|_{\delta_5,0,t;Y^r_\gamma}(t_2-t_1)^{\delta_5}.
\end{aligned} \tag{157}$$

Summing up (154) and (157), we obtain (151). The proof is complete. □

Since in Lemma 37 it is not required that $g_5(0) = 0$, the special case of the constant function $g_5(t) = x \in Y^p_\gamma$, $t \in [0,T]$, is admissible, and we obtain the following result.

Corollary 38. *Let α, β, and Y_γ^r be as in Lemma 37, and let $x \in Y_\gamma^r$, $\gamma \in (2-\alpha-\beta, 1)$, and $r \in [1, \infty]$. Then, for every $\delta_7 \in (0, (\alpha+\beta+\gamma-2)/\alpha]$, the function $[Q_7 x](\cdot) := (e^{\cdot A} - I)x$ belongs to $C_0^{\delta_7}([0,T]; X)$, and for every $t \in [0,T]$ satisfies the estimate*

$$\|Q_7 x\|_{\delta_7,0,t;X} \le c_{41} t^{(\alpha+\beta+\gamma-2-\alpha\delta_7)/\alpha} \left(t^{\delta_7}+1\right) c_{41} \|x\|_{Y_\gamma^r}. \quad (158)$$

Proof. Let $g_5(t) = x$ in the proof of Lemma 37, and observe that V_{1,t_1,t_2,g_5} reduces to the zero element of X. Estimate (158) then follows from (154) and the second estimate in (156). □

For later purposes, we conclude the section with the following remark.

Remark 39. The condition $5\alpha + 2\beta > 6$ in (H1) required in Proposition 29 is the strongest among the conditions for the pair (α, β) required in Corollary 14 and the other results of this section. Indeed,

$$\begin{aligned} 5\alpha + 2\beta > 6 &\Longrightarrow 3\alpha + 2\beta > 6 - 2\alpha \ge 4 \\ &\Longrightarrow 3\alpha + \beta > 4 - \beta \ge 3 \\ &\Longrightarrow 2\alpha + \beta > 3 - \alpha \ge 2 \\ &\Longrightarrow \alpha + \beta > 2 - \alpha \ge 1. \end{aligned} \quad (159)$$

Hence, if $5\alpha + 2\beta > 6$, then Corollary 14 and all the results from Lemma 22 to Corollary 38 are applicable. Next we will make large usage of this fact, but we warn the reader that, for brevity and regarding it as acquired, we will not mention it anymore.

5. Application to Maximal Time Regularity

The results of the previous sections are here applied to correct, refine, and extend the results in [20] concerning the maximal time regularity of the solutions to a class of degenerate abstract evolution equations. Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be two complex Banach spaces, and consider the following degenerate first-order integrodifferential Cauchy problem for $v : I_T \to X$, where $I_T = [0,T]$, $T > 0$, and $n_1, n_2 \in \mathbf{N}$:

$$\begin{aligned} D_t(Mv(t)) = {} & [\lambda_0 M + L] v(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) \\ & + \sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2} + f(t), \quad t \in I_T, \\ Mv(0) = {} & Mv_0. \end{aligned} \quad (160)$$

Here $\mathscr{K}$ is the convolution operator (104) in which $(X_1, X_2, X_3) = (Z, X, X)$, whereas M, L, and L_{i_1}, $i_1 = 1, \ldots, n_1$, are closed single-valued linear operators from X to itself, whose domains fulfill the relation $\mathscr{D}(L) \subseteq \bigcap_{i_1=1}^{n_1} [\mathscr{D}(M) \cap \mathscr{D}(L_{i_1})]$. Further, we assume that

$$L \text{ admits a continuous inverse operator } L^{-1} \in \mathscr{L}(X), \quad \text{i.e., } 0 \in \rho(L), \quad (161)$$

whereas we allow M to have *no* bounded inverse. Hence, in general, $A := LM^{-1}$ is only the m. l. operator defined by

$$\begin{aligned} \mathscr{D}(A) &= \left\{x \in \mathscr{D}\left(M^{-1}\right) : L\left(M^{-1}x\right) \ne \emptyset\right\} \\ &= \left\{x \in \mathscr{R}(M) : M^{-1}x \cap \mathscr{D}(L) \ne \emptyset\right\} \\ &= \{x \in \mathscr{R}(M) : \\ &\qquad \text{there exists } y \in \mathscr{D}(L) \text{ such that } y \in M^{-1}x\} \\ &= \{x \in \mathscr{R}(M) : x = My \text{ for some } y \in \mathscr{D}(L)\} \\ &= M(\mathscr{D}(L)), \\ Ax &= \bigcup_{y \in M^{-1}x \cap \mathscr{D}(L)} Ly \\ &= \{Ly : y \in \mathscr{D}(L) \text{ such that } x = My\}, \\ &\qquad x \in \mathscr{D}(A). \end{aligned} \quad (162)$$

Therefore, problem (160) can *not* be reduced, via the change of unknown $u = Mv$, to an integrodifferential problem related to single-valued linear operators. On the contrary, due to (161) and the closed graph theorem, $ML^{-1}, L_{i_1}L^{-1} \in \mathscr{L}(X)$, $i_1 = 1, \ldots, n_1$. As far as the data vector $(\lambda_0, v_0, k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, y_1, \ldots, y_{n_2}, f)$ is concerned, at the moment, we only assume $\lambda_0 \in \mathbf{C}$, $v_0 \in \mathscr{D}(M)$, $k_{i_1} : I_T \to Z$, $h_{i_2} : I_T \to \mathbf{C}$, $y_{i_2} \in X$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, and $f : I_T \to X$, in order that (160) makes sense in X. This minimal assumptions will be refined later. In general, only *strict* solutions v to (160) shall be investigated, where (cf. [22, 23]) by a strict solution v to (160) we mean that, $\mathscr{D}(L)$ being endowed with the graph norm $\|\cdot\|_{\mathscr{D}(L)} = \|\cdot\|_X + \|L\cdot\|_X$, $v \in C(I_T; \mathscr{D}(L))$, $Mv \in C^1(I_T; X)$, and (160) holds. Clearly, if M^{-1} is really a m. l. operator, then $Mv(0) = Mv_0$ does not necessarily mean $v(0) = v_0$, but only $v(0) - v_0 \in M^{-1}0$. As we will see below, if $v_0 \in \mathscr{D}(L)$ and the data k_{i_1}, h_{i_2}, y_{i_2} and f, $i_l = 1, \ldots, n_l$, $l = 1, 2$, satisfy suitable assumptions, then for a strict solution v to (160) it just holds $v(0) = v_0$. Throughout the section, Y_ψ^q, $\psi \in (0,1)$, $q \in [1, \infty]$, will always denote one between the spaces $(X, \mathscr{D}(A))_{\psi,q}$ and $X_A^{\psi,q}$, A being defined by (162). That is, $Y_\psi^q \in \{(X, \mathscr{D}(A))_{\psi,q}, X_A^{\psi,q}\}$. To avoid confusion, if more than a single Y_ψ^q is involved in some statement, that is, if we write $x_j \in Y_{\psi_j}^q$, $j = 1, \ldots, n$, $n \in \mathbf{N}$, then it is understood that the same choice has been made for all the $Y_{\psi_j}^q$ in the sense that $Y_{\psi_j}^q = (X, \mathscr{D}(A))_{\psi_j,q}$ or $Y_{\psi_j}^q = X_A^{\psi_j,q}$ for every $j = 1, \ldots, n$.

According to [2, Section 1.6], we recall that the M-modified resolvent set $\rho_M(L)$ of L is defined to be the set $\{z \in \mathbf{C} : (zM - L)^{-1} \in \mathscr{L}(X)\}$. The bounded operator $(zM - L)^{-1}$ is called the M modified resolvent of L. It is easy to prove that $\rho_M(L) \subseteq \rho(A)$ and that $M(zM - L)^{-1} = (zI - A)^{-1}$, $z \in \rho_M(L)$ (cf. [2, Theorem 1.14]). With the notion of M-modified resolvent of L at hand, we assume that

(H2) $\rho_M(L)$ contains a region $\Sigma_\alpha = \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge -c(|\mathfrak{Im}\, z| + 1)^\alpha, \mathfrak{Im}\, z \in \mathbf{R}\}$, $\alpha \in (0,1]$, $c > 0$, and for some exponent $\beta \in (0, \alpha]$ and constant $C > 0$ the estimate $\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \leq C(|\lambda|+1)^{-\beta}$ holds for every $\lambda \in \Sigma_\alpha$.

Before we proceed with our analysis we remark that, due to the wide range of choices for the data vector, problem (160) contains many subcases at its interior. So, in spite of the case when at least one between the k_i's is different from zero and problem (160) is really an integrodifferential one, the choice $k_{i_1} = 0$, $i_1 = 1, \ldots, n_1$, yields to consider also various nonintegrodifferential degenerate problems. For instance, those corresponding to $\lambda_0 = k_{i_1} = h_{i_2} = 0$ and $\lambda_0 = k_{i_1} = f = 0$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, respectively:

$$D_t(Mv(t)) = Lv(t) + f(t), \quad t \in I_T, \qquad Mv(0) = Mv_0, \tag{163}$$

$$D_t(Mv(t)) = Lv(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2}, \quad t \in I_T, \qquad Mv(0) = Mv_0. \tag{164}$$

Although (164) differs from (163) only in the fact that f is replaced with $\sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2}$; nevertheless a very different result is achieved when the y_{i_2}'s are assumed to belong to $Y^r_{\gamma_{i_2}}$, at least for opportunely chosen $\gamma_{i_2} \in (0,1)$, $i_2 = 1, \ldots, n_2$. As we will see (cf. Remark 51 and Theorem 56), in this situation the loss of time regularity for the pair (Lv, D_tMv) with respect to that of f, typical of the case $\beta < 1$ in (H2) (see [21, Theorem 9], [2, Theorem 3.26], and [22, Theorem 7.2]), can be restored in order that (Lv, D_tMv) possesses the maximal time regularity which is the minimal between the time regularities of the h_{i_2}'s. The same phenomenon is carried over into the integrodifferential case for the following problems, corresponding to $\lambda_0 = h_{i_2} = 0$, $i_2 = 1, \ldots, n_2$, and $\lambda_0 = f = 0$:

$$D_t(Mv(t)) = Lv(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1}v\right)(t) + f(t), \qquad Mv(0) = Mv_0, \tag{165}$$

$$D_t(Mv(t)) = Lv(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1}v\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2}, \qquad Mv(0) = Mv_0, \tag{166}$$

$t \in I_T$. When $\beta < 1$, the loss of time regularity for the pair (Lv, D_tMv) with respect to that of the vector $(k_1, \ldots, k_{n_1}, f)$ in problem (165) (cf. [22, Theorem 7.1] and [23, Theorem 2.1] for $n_1 = 1$) can be restored in problem (166) assuming that $y_{i_2} \in Y^r_{\gamma_{i_2}}$, $i_2 = 1, \ldots, n_2$. In this context (cf. Remark 51 and Theorem 53) the pair (Lv, D_tMv) has the maximal time regularity which is the minimal between the time regularities of the k_{i_1}'s and h_{i_2}'s.

We stress that, if $\beta = 1$, then no loss of time regularity is observed and all the quoted results agree with the well-known theory of maximal regularity in spaces of continuous functions for the nondegenerate version of (160), corresponding to the case when $M = I$ and L generates an analytic semigroup. Hence, roughly speaking, one can verify the consistency of any result on problem (160) with condition (H2) simply by letting $\beta = 1$ on its statement, and then checking if it is compatible with those for the nondegenerate case. To this purpose, we recall that the question of maximal regularity for the nondegenerate (possibly nonautonomous) version of (160) has been deeply investigated by several authors. See, for instance, [4, 6–8, 10, 32] for problem (165) with $(M, \beta, n_1) = (I, 1, 1)$ and [9, 11] for problem (163) with $(M, \beta) = (I, 1)$.

Finally, assumption (161) excludes the case of $L = 0$ in (160), so that our results cannot be compared with those in [5, 33, 34]. There, assuming that the bilinear bounded operator $\mathscr{P}$ underlying the definition of $\mathscr{K}$ is the scalar multiplication in X, the problem

$$D_t v(t) = \mathscr{K}\left(k_1, L_1 v\right)(t) + f(t), \quad t \in I_T, \quad v(0) = v_0 \tag{167}$$

is treated under the following assumptions: (i) L_1 is a closed densely defined linear operator generating an analytic semigroup; (ii) $k_1 : [0, \infty) \to \mathbf{R}$ is absolutely Laplace transformable. Observe that, if $(k_1, f) = (1, 0)$, then problem (167) reduces to the abstact wave equation $D_t^2 v(t) = L_1 v(t)$, $D_t v(0) = 0$, $v(0) = v_0$, whereas when $M = I$ and $\lambda_0 = k_{i_1} = h_{i_2} = f = 0$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, problem (160) reduces to the abstract heat equation $D_t v(t) = Lv(t)$, $v(0) = v_0$. In other words, whereas [5, 33, 34] are concerned with the hyperbolic case, here we are concerned with the *parabolic* one.

Let us now come back to problem (160). Of course, assumption (H2) implies that the operator A defined by (162) satisfies (H1), so that it generates a semigroup $\{e^{tA}\}_{t \geq 0}$ defined by $e^{0A} = I$ and (9) and satisfying (14). Assuming that $v_0 \in \mathscr{D}(L)$, we let

$$w = L\left(v - v_0\right) \Longleftrightarrow v = L^{-1}w + v_0. \tag{168}$$

Then, by setting

$$\begin{aligned} F_w(t) &= \lambda_0 A^{-1} w(t) \\ &\quad + \sum_{i_1=1}^{n_1} \left[\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t) + \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t)\right] \\ &\quad + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2} + v_1 + f(t), \quad t \in I_T, \end{aligned} \tag{169}$$

where $A^{-1} = ML^{-1} \in \mathscr{L}(X)$, $S_{i_1} = L_{i_1} L^{-1} \in \mathscr{L}(X)$, $i_1 = 1, \ldots, n_1$, and $v_1 = (\lambda_0 M + L)v_0$, we see that v is a strict solution to (160) if and only if w satisfies (indeed, if $v \in C(I_T; \mathscr{D}(L))$, then $\|w(t) - w(s)\|_X = \|L[v(t) - v(s)]\|_X \leq \|v(t) - v(s)\|_{\mathscr{D}(L)} \to 0$ as $s \to t$, $t, s \in I_T$, that is, $w \in C(I_T; X)$. Conversely, if $w \in C(I_T; X)$, then $v = L^{-1}w + v_0 \in \mathscr{D}(L)$ and $\|v(t) - v(s)\|_{\mathscr{D}(L)} \leq (\|L^{-1}\|_{\mathscr{L}(X)} + 1)\|w(t) - w(s)\|_X \to 0$ as $s \to t$, $t, s \in I_T$, that is, $v \in C(I_T; \mathscr{D}(L))$. Finally, since $Mv = A^{-1}w + Mv_0$, we have $Mv \in C^1(I_T; X)$

if and only if $A^{-1}w \in C^1(I_T;X))$ $w \in C(I_T;X)$, $A^{-1}w \in C^1(I_T;X)$, and solves to the following problem:

$$D_t\left(A^{-1}w(t)\right) = w(t) + F_w(t) \in A\left(A^{-1}w(t)\right) + F_w(t), \quad t \in I_T,$$
$$A^{-1}w(0) = 0 \quad \text{(i.e., } w(0) \in A0\text{)}. \tag{170}$$

Now let $2\alpha + \beta > 2$, and assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T;Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T;\mathbf{C})$, and $f \in C^{\mu}(I_T;X)$, where $\eta_{i_1}, \sigma_{i_2}, \mu \in (2-\alpha-\beta/\alpha, 1)$, $i_l = 1,\ldots,n_l$, $l = 1,2$. Then, if $w \in C(I_T;X)$ is a solution to (170) such that $A^{-1}w \in C^1(I_T;X)$, the function F_w satisfies

$$F_w \in C^{\delta}(I_T;X),$$
$$\delta = \min_{i_k=1,\ldots,n_k,\, k=1,2}\left\{\eta_{i_1}, \sigma_{i_2}, \mu\right\} \in \left(\frac{2-\alpha-\beta}{\alpha}, 1\right). \tag{171}$$

Indeed, δ being the smallest Hölder exponent, for every $i_l = 1,\ldots,n_l$, $l = 1,2$, we have $A^{-1}w, h_{i_2}y_{i_2}, f \in C^{\delta}(I_T;X)$ and $\mathscr{K}(k_{i_1}, S_{i_1}w)$, $\mathscr{K}(k_{i_1}, L_{i_1}v_0) \in C_0^{\eta_{i_1}}(I_T;X) \hookrightarrow C_0^{\delta}(I_T;X)$ (cf. Lemma 27 for the case $(\delta_{3_1}, \delta_{3_2}, X_1, X_2) = (\eta_{i_1}, 0, Z, X)$ with the pair (g_{3_1}, g_{3_2}) being replaced by (in fact, since $S_{i_1} = L_{i_1}L^{-1} \in \mathscr{L}(X)$, $i_1 = 1,\ldots,n_1$, if $w \in C(I_T;X)$, then $S_{i_1}w \in C(I_T;X)$, whereas the constant functions $\kappa_{i_1}(t) = L_{i_1}v_0$, $t \in I_T$, $i_1 = 1,\ldots,n_1$, obviously belong to $C(I_T;X)$) $(k_{i_1}, S_{i_1}w)$ and $(k_{i_1}, L_{i_1}v_0)$, resp.). Consequently (cf. [2, Theorem 3.7 and Remark p. 54] with $u_0 = 0$), the solution $A^{-1}w$ to the multivalued evolution problem $D_t(A^{-1}w) \in A(A^{-1}w) + F_w$, $A^{-1}w(0) = 0$ is necessarily of the form

$$A^{-1}w(t) = \left[Q_1F_w\right](t), \quad t \in I_T, \tag{172}$$

with Q_1 being the operator defined by (105). Further (cf. [2, Remark p. 55] with $u_0 = 0$, and where $A^{\circ}e^{tA}$ stands for $D_te^{tA} = -[(-A)^1]^{\circ}e^{tA}$) the derivative of $A^{-1}w$ is given by

$$D_t\left(A^{-1}w(t)\right) = e^{tA}F_w(t) - \left[Q_2F_w\right](t), \quad t \in I_T \setminus \{0\}, \tag{173}$$

with Q_2 being the operator in (106). Notice that Q_2F_w is well defined by virtue of (127) with $g = F_w$. Now let $y_{i_2} \in Y^r_{\gamma_{i_2}}$ and $v_1 + f(0) \in Y^r_{\varphi}$ where $\gamma_{i_2}, \varphi \in (1-\beta, 1)$, $i_2 = 1,\ldots,n_2$, and $r \in [1,\infty]$. Since $A^{-1}w(0) = \mathscr{K}(k_{i_1}, S_{i_1}w)(0) = \mathscr{K}(k_{i_1}, L_{i_1}v_0)(0) = 0$, $i_1 = 1,\ldots,n_1$, from (169) it thus follows that $F_w(0) := x_0$ is independent on w and

$$x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)\,y_{i_2} + v_1 + f(0) \in Y^r_{\gamma},$$
$$\gamma = \min_{i_2=1,\ldots,n_2}\left\{\gamma_{i_2}, \varphi\right\} \in (1-\beta, 1). \tag{174}$$

Indeed (cf. (20) or (38)), we have $Y^r_{\gamma_{i_2}} \hookrightarrow Y^r_{\gamma}$, $i_2 = 1,\ldots,n_2$, and $Y^r_{\varphi} \hookrightarrow Y^r_{\gamma}$, the embeddings being equalities for those between the numbers $\gamma_1,\ldots,\gamma_{n_2}$ and φ which are equal to γ. Then, under these assumptions on the data, formula (173) for $D_t(A^{-1}w(t))$ can be extended until $t = 0$. For, we have $\lim_{t\to 0^+}D_t(A^{-1}w(t)) = x_0 \in A0 + x_0$ and the differential equation in (170) is satisfied even at $t = 0$. To see this, we observe that

$$\left\|D_t\left(A^{-1}w(t)\right) - x_0\right\|_X \le I_1(t) + I_{2,w}(t) + I_{3,w}(t), \quad t \in I_T \setminus \{0\}, \tag{175}$$

where $I_1(t) = \|(e^{tA} - I)x_0\|_X$, $I_{2,w}(t) = \|e^{tA}[F_w(t) - x_0]\|_X$, and $I_{3,w}(t) = \|[Q_2F_w](t)\|_X$. First, from Proposition 6 we get $\lim_{t\to 0^+}I_1(t) = 0$. On the other side, using $F_w \in C^{\delta}(I_T;X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1) \subseteq ((1-\beta)/\alpha, 1)$, we obtain

$$I_{2,w}(t) \le \tilde{c}_{\alpha,\beta,0}\left\|F_w\right\|_{\delta,0,t;X} t^{(\alpha\delta+\beta-1)/\alpha}, \quad t \in I_T \setminus \{0\}, \tag{176}$$

so that $\lim_{t\to 0^+}I_{2,w}(t) = 0$. Finally, (127) with $g = F_w$ yields $\lim_{t\to 0^+}I_{3,w}(t) = 0$, too. Formula (173) thus holds at $t = 0$ with $D_t(A^{-1}w(0)) = \lim_{t\to 0^+}D_t(A^{-1}w(t)) = x_0$.

Remark 40. In [2, Remark p. 55], formula (173) was extended until $t = 0$ only under the more restrictive assumption $x_0 \in X_A^{\gamma,\infty}$, $\gamma \in (1-\beta, 1)$. Indeed [24, Proposition 5.2] was not available at the time of [2] and only the strong continuity of $\{e^{tA}\}_{t\ge 0}$ in the X-norm on the spaces $X_A^{\gamma,\infty}$, $\gamma \in (1-\beta, 1)$, was known (cf. [2, Theorem 3.3]). Notice that in the case of problem (163) the element x_0 reduces to $Lv_0 + f(0)$, so that in the nondegenerate case $(M,\beta) = (I,1)$ we get back the classical assumption $Lv_0 + f(0) \in (X, \mathscr{D}(L))_{\gamma,r}$, $\gamma \in (0,1)$, $r \in [1,\infty]$ (see, for instance, [9, Theorem 4.3.1(iii)] and [11, Theorem 4.5]).

Since (170) implies that $w(t) = D_t(A^{-1}w(t)) - F_w(t)$, from (173) we thus find that

$$w(t) = \left[Q_7x_0\right](t) + \left(e^{tA} - I\right)\left[F_w(t) - x_0\right] - \left[Q_2F_w\right](t), \quad t \in I_T, \tag{177}$$

where, according to the notation in Corollary 38, we have set $[Q_7x_0](t) = (e^{tA} - I)x_0$. In particular, $w(0) = 0$. We conclude that, under the previous assumptions on the pair (α,β) and on the data vector $(k_1,\ldots,k_{n_1}, h_1,\ldots,h_{n_2}, y_1,\ldots,y_{n_2}, f, v_1)$, if $w \in C(I_T;X)$ solves (170), then necessarily $w \in C_0(I_T;X)$. As a consequence (cf. (168)), the strict solution v to (160) satisfies the initial condition just in the sense $v(0) = v_0$.

Introduce the functions $\tilde{f} : I_T \to X$ and $\tilde{h}_{i_2} : I_T \to Y^r_{\gamma_{i_2}}$, $i_2 = 1,\ldots,n_2$, defined by

$$\tilde{f}(t) = f(t) - f(0), \qquad \tilde{h}_{i_2}(t) = \left[h_{i_2}(t) - h_{i_2}(0)\right]y_{i_2}, \quad t \in I_T. \tag{178}$$

Then, replacing F_w with the right-hand side of (169), using (174), and recalling the definitions of the operators Q_j,

$j = 2, \ldots, 6$, in (106)–(110), from (177) we deduce that $w \in C_0(I_T; X)$ solves the fixed-point equation

$$w = w_0 + w_1 + Rw, \tag{179}$$

the functions w_l, $l = 0, 1$, and the operator Rw being defined by

$$w_0 := Q_7 x_0 + \sum_{i_1=1}^{n_1} Q_6\left(k_{i_1}, L_{i_1} v_0\right) + \sum_{i_2=1}^{n_2} Q_5 \widetilde{h}_{i_2} + Q_5 \widetilde{f}, \tag{180}$$

$$w_1 := -\sum_{i_1=1}^{n_1} Q_3\left(k_{i_1}, L_{i_1} v_0\right) - \sum_{i_2=1}^{n_2} Q_4\left(h_{i_2}, y_{i_2}\right) - Q_2 f, \tag{181}$$

$$\begin{aligned} Rw := {} & \lambda_0\left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right] \\ & + \sum_{i_1=1}^{n_1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right) - Q_3\left(k_{i_1}, S_{i_1} w\right)\right]. \end{aligned} \tag{182}$$

Conversely, let $w \in C_0(I_T; X)$ be a solution to the fixed-point equation (179), and assume that the pair (α, β) and the data vector $(k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, y_1, \ldots, y_{n_2}, f, v_1)$ satisfy the assumptions below (170) and (173). Then, as before, $\mathscr{K}(k_{i_1}, S_{i_1} w), \mathscr{K}(k_{i_1}, L_{i_1} v_0) \in C_0^{\delta}(I_T; X)$ and $h_{i_2} y_{i_2}, f \in C^{\delta}(I_T; X)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$ being as in (171). We apply $A^{-1} \in \mathscr{L}(X)$ to both sides of (179), and we show that $A^{-1}w$ satisfies (172) with $F_w \in C(I_T; X)$ as in (169), so that $A^{-1}w$ is a solution to problem (170). To this purpose, we take into account Corollaries 14 and 26. Let $t \in I_T$. First (cf. Remark 34 and recall that $Q_6(\cdot,\cdot) = Q_5 \mathscr{K}(\cdot,\cdot)$), using (86), (174), and (178), we get

$$\begin{aligned} A^{-1} w_0(t) & = \int_0^t e^{(t-s)A}\left[x_0 + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t) + \sum_{i_2=1}^{n_2} \widetilde{h}_{i_2}(t) + \widetilde{f}(t)\right] ds \\ & = \int_0^t e^{(t-s)A}\left[v_1 + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2} + f(t)\right] ds. \end{aligned} \tag{183}$$

Instead, due to the definition of Q_3 and Q_4, using (125) we obtain

$$\begin{aligned} A^{-1} w_1(t) = \int_0^t e^{(t-s)A} \Bigg\{ & \sum_{i_1=1}^{n_1}\left[\mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(s) - \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t)\right] \\ & + \sum_{i_2=1}^{n_2}\left[h_{i_2}(s) - h_{i_2}(t)\right] y_{i_2} + f(s) - f(t)\Bigg\} ds. \end{aligned} \tag{184}$$

Therefore, from (183), (184), and the definition (105) of Q_1 it follows that

$$\begin{aligned} A^{-1}\left[w_0 + w_1\right](t) = \left[Q_1\left(v_1 + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right) + \sum_{i_2=1}^{n_2} h_{i_2} y_{i_2} + f\right)\right](t), \end{aligned} \tag{185}$$

the left-hand side being well-defined due to Remark 23. As far as $A^{-1}[Rw](t)$ is concerned, we first observe that, w being in $C(I_T; X)$, from formula (126) and Remark 34 it follows that $[Q_2(A^{-1}w)](t)$ and $[Q_5(A^{-1}w)](t)$ are both well defined and equal to $-\int_0^t e^{(t-s)A}[w(s) - w(t)]ds$ and $\int_0^t e^{(t-s)A} w(t)\, ds$, respectively. Consequently

$$\begin{aligned} & \left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right](t) \\ & \quad = \int_0^t e^{(t-s)A} w(s)\, ds = \left[Q_1 w\right](t). \end{aligned} \tag{186}$$

Hence, commuting $A^{-1} \in \mathscr{L}(X)$ with both the integral sign and the semigroup, one has

$$A^{-1}\left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right](t) = \left[Q_1\left(A^{-1}w\right)\right](t). \tag{187}$$

Similarly, since Remark 34 and formula (125) yield

$$\begin{aligned} & A^{-1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right)\right](t) = \int_0^t e^{(t-s)A} \mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t)\, ds, \\ & A^{-1}\left[Q_3\left(k_{i_1}, S_{i_1} w\right)\right](t) \\ & \quad = -\int_0^t e^{(t-s)A}\left[\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(s) - \mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t)\right] ds, \end{aligned} \tag{188}$$

we find that

$$\begin{aligned} & A^{-1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right) - Q_3\left(k_{i_1}, S_{i_1} w\right)\right](t) \\ & \quad = \left[Q_1 \mathscr{K}\left(k_{i_1}, S_{i_1} w\right)\right](t), \end{aligned} \tag{189}$$

$i_1 = 1, \ldots, n_1$. In conclusion, from (187) and (189) it follows that

$$A^{-1}[Rw](t) = \left[Q_1\left(\lambda_0 A^{-1} w + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, S_{i_1} w\right)\right)\right](t). \tag{190}$$

Summing up (185) and (190), we finally obtain $A^{-1}w(t) = [Q_1 F_w](t)$, F_w being as in (169). This completes the proof of the equivalence between problem (170) and the fixed point equation (179), provided that the data satisfy the mentioned assumptions.

Remark 41. We can summarize the previous reasonings as follows: problem (160) has been reduced to the fixed-point equation (179) for the new unknow $w = L(v - v_0)$, $v_0 \in \mathscr{D}(L)$. This fixed-point argument is similar to that first successfully applied in [4, 7, 8, 32] to problem (165) with $(M, \beta, n_1) = (I, 1, 1)$ and then generalized in [23] to the degenerate case. A different approach has been followed in [6, 10] for the nondegenerate case and in [22] for the degenerate one. There, assuming that k_1 is absolutely Laplace transformable (cf. [6, 22]) or of bounded variation (cf. [10]), problem (165) with $n_1 = 1$ is solved by constructing its relative resolvent operator. We quote also [35] where the method of constructing the fundamental solution for the equation without the integral term is applied to a class of *concrete* degenerate integrodifferential equations.

From now on, for $5\alpha + 2\beta > 6$, $\beta \in (0, \alpha]$, $\alpha \in (0, 1]$, and $\nu \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $I_{\alpha,\beta,\nu} \subseteq ((3 - 2\alpha - \beta)/\alpha, 1/2) \subseteq (0, 1/2)$ will denote the interval defined by

$$I_{\alpha,\beta,\nu} = \begin{cases} \left(\dfrac{3 - 2\alpha - \beta}{\alpha}, \nu\right], & \text{if } \nu \in \left(\dfrac{3 - 2\alpha - \beta}{\alpha}, \dfrac{1}{2}\right), \\ \left(\dfrac{3 - 2\alpha - \beta}{\alpha}, \dfrac{1}{2}\right), & \text{if } \nu \in \left[\dfrac{1}{2}, 1\right). \end{cases} \tag{191}$$

Clearly, if $\nu, \rho \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $\nu \le \rho$, then $I_{\alpha,\beta,\nu} \subseteq I_{\alpha,\beta,\rho}$.

Lemma 42. *Assume* (161), *and let* $5\alpha + 2\beta > 6$ *in (H2). Assume that* $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $\eta_{i_1} \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $i_1 = 1, \dots, n_1$, *and let* $\eta = \min_{i_1=1,\dots,n_1} \eta_{i_1}$. *Then, for every fixed* $\delta \in I_{\alpha,\beta,\eta}$, *the operator R defined by* (182) *maps continuously* $C^\delta(I_T; X)$ *into* $C_0^\delta(I_T; X)$, *and for every* $t \in I_T$ *satisfies the following estimate, where* $p \in (1/(1 - 2\delta), \infty)$:

$$\|Rw\|_{\delta,0,t;X} \le c_{42}(T) \left(\int_0^t \|w\|_{\delta,0,s;X}^p \, ds \right)^{1/p}, \quad w \in C^\delta(I_T; X). \tag{192}$$

Here $c_{42}(T)$ *is a positive constant depending only on* T, λ_0, α, β, η_{i_1}, δ, p, $\|k_{i_1}\|_{\eta_{i_1},0,T;Z}$ *and* $\|S_{i_1}\|_{\mathscr{L}(X)}$, $i_1 = 1, \dots, n_1$.

Proof. Let $k_{i_1} \in C^{\eta_{i}}(I_T; Z)$, $\eta_{i_1} \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $i_1 = 1, \dots, n_1$, and let us fix an arbitrary number $\delta \in I_{\alpha,\beta,\eta}$, where $\eta = \min_{i_1=1,\dots,n_1} \eta_i$. In particular, since $\delta \le \eta \le \eta_i$, we have $k_{i_1} \in C^\delta(I_T; Z)$ with $\|k_{i_1}\|_{\delta,0,t;Z} \le \max\{1, t^{\eta_{i_1} - \delta}\} \|k_{i_1}\|_{\eta_{i_1},0,t;Z}$, $i_1 = 1, \dots, n_1$. Now let $w \in C^\delta(I_T; X)$ and $t \in I_T$. First, formula (186) being applicable, we rewrite (182) as

$$Rw = \lambda_0 Q_1 w + \sum_{i_1=1}^{n_1} \left[Q_6 \left(k_{i_1}, S_{i_1} w \right) - Q_3 \left(k_i, S_i w \right) \right]. \tag{193}$$

Now, we notice that $5\alpha + 2\beta > 6$ implies that

$$\frac{\alpha + \beta - 1}{\alpha} = \frac{5\alpha + 2\beta - 3\alpha - 2}{2\alpha} > \frac{4 - 3\alpha}{2\alpha} \ge \frac{1}{2}. \tag{194}$$

Since $(1 - \beta)/\alpha \le (2 - \alpha - \beta)/\alpha \le (3 - 2\alpha - \beta)/\alpha$, from (194) it follows that $\delta \in I_{\alpha,\beta,\eta} \subseteq ((3 - 2\alpha - \beta)/\alpha, 1/2) \subseteq ((2 - \alpha - \beta)/\alpha, 1/2) \subsetneq ((1 - \beta)/\alpha, (\alpha + \beta - 1)/\alpha)$, and, consequently,

$$\frac{\alpha}{\alpha + \beta - 1 - \alpha\delta} < \frac{1}{1 - 2\delta}. \tag{195}$$

We conclude (cf. Remark 39) that Lemma 22 and Propositions 29 and 36 are applicable with $\delta \in I_{\alpha,\beta,\eta}$ and $p \in (1/(1 - 2\delta), \infty)$. Then, using estimates (111), (135), and (150) with the pair (g_1, δ_1) and the quintuplets $(g_{l_1}, g_{l_2}, \delta_l, X_1, X_2)$, $l = 3, 6$, being replaced, respectively, by (w, δ) and (indeed, since $S_{i_1} = L_{i_1} L^{-1} \in \mathscr{L}(X)$, if $w \in C^\delta(I_T; X)$, then $S_{i_1} w \in C^\delta(I_T; X)$ with $\|S_{i_1} w\|_{\delta,0,t;X} \le \|S_{i_1}\|_{\mathscr{L}(X)} \|w\|_{\delta,0,t;X}$, $i_1 = 1, \dots, n_1$) $(k_{i_1}, S_{i_1} w, \delta, Z, X)$, $i_1 = 1, \dots, n_1$, from (193) we finally obtain

$$\begin{aligned} \|Rw\|_{\delta,0,t;X} &\le \left\|\lambda_0 Q_1 w\right\|_{\delta,0,t;X} + \sum_{l=3,6,\, i_1=1,\dots,n_1} \left\| Q_l \left(k_{i_1}, S_{i_1} w \right) \right\|_{\delta,0,t;X} \\ &\le c_{42}(T) \left(\int_0^t \|w\|_{\delta,0,s;X}^p \, ds \right)^{1/p}. \end{aligned} \tag{196}$$

Here we have set $c_{42}(T) = |\lambda_0| C_1(T) + T^{1-2\delta-1/p} \sum_{l=3,6,\, i_1=1,\dots,n_1} C_l(T) \|k_{i_1}\|_{\delta,0,T;Z} \|S_{i_1}\|_{\mathscr{L}(X)}$, where $C_l(T)$, $l = 1, 3, 6$, are the values at $t = T$ of the functions $C_l(t)$ in Lemma 22 and Propositions 29 and 36. This completes the proof. □

Remark 43. Assume that in Lemma 42 the Hölder exponents $\eta_{i_1} \in ((3 - 2\alpha - \beta)/\alpha, 1)$ are such that $\eta = \min_{i_1=1,\dots,n_1} \eta_i$ belongs to $((3 - 2\alpha - \beta)/\alpha, 1/2)$. In this case (cf. (191)), the choice $\delta = \eta$ is admissible, and the meaning of Lemma 42 is that the operator R defined by (182) preserves the minimal of the time regularities of $k_1, \dots, k_{n_1}$.

Corollary 44. *Let the assumptions of Lemma 42 be satisfied, and let* η *and* R *be as there. Then, for every fixed* $\delta \in I_{\alpha,\beta,\eta}$, *the sequence* $\{R^n\}_{n=0}^\infty$ ($R^0 = I$, $R^n = RR^{n-1}$, $n \in \mathbf{N}$) *satisfies the following estimates, where* $w \in C^\delta(I_T; X)$ *and* $p \in (1/(1 - 2\delta), \infty)$:

$$\left\|R^n w\right\|_{\delta,0,t;X} \le \left[c_{42}(T)\right]^n \left(\frac{t^n}{n!} \right)^{1/p} \|w\|_{\delta,0,T;X}, \qquad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \tag{197}$$

Proof. Reasoning as in [23, p. 468], we prove (197) by induction. Since for every fixed $\delta \in I_{\alpha,\beta,\eta}$ the operator R maps $C^\delta(I_T; X)$ in $C_0^\delta(I_T; X)$, replacing w with $R^n w$ in (192) and introducing the sequence of scalar nonnegative nondecreasing functions $\{\varphi_n\}_{n=0}^\infty$ defined by $\varphi_n(t) = \|R^n w\|_{\delta,0,t;X}$, $t \in I_T$, from (192) we obtain

$$\varphi_{n+1}(t) \le c_{42}(T) \left(\int_0^t \left|\varphi_n(s)\right|^p ds \right)^{1/p}, \qquad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \tag{198}$$

Then, applying to (198) an induction argument in which the first step of the induction follows from (192), we immediately deduce the following estimates:

$$\varphi_n(t) \le [c_{42}(T)]^n \left(\frac{t^n}{n!}\right)^{1/p} \|w\|_{\delta,0,T;X}, \qquad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \tag{199}$$

The proof is complete. □

Lemma 45. *Let $5\alpha + 2\beta > 6$ in (H2) and $v_0 \in \bigcap_{i_1=1}^{n_1} \mathscr{D}(L_{i_1})$. Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, and $y_{i_2} \in Y^r_{\gamma_{i_2}}$, where $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2} \in (3-2\alpha-\beta, 1)$, $i_l = 1, \dots, n_l$, $l = 1, 2$, and $r \in [1, \infty]$. Let $\tau_1 = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau_1}$, the function w_1 defined by (181) belongs to $C_0^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Let us fix $\delta \in I_{\alpha,\beta,\tau_1}$, $\tau_1 = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. Of course, $k_{i_1} \in C^\delta(I_T; Z)$ and $h_{i_2} \in C^\delta(I_T; \mathbf{C})$, $i_l = 1, \dots, n_l$, $l = 1, 2$. Then, Proposition 29 and Lemma 30 applied with the quintuplets $(g_{3_1}, g_{3_2}, \delta_3, X_1, X_2)$ and the quadruplet $(g_4, y, \delta_4, \gamma)$ being replaced, respectively, by (the constant functions $\kappa_{i_1}(t) = L_{i_1} v_0$, $t \in I_T$, $i = 1, \dots, n_1$, being obviously of class $C^\delta(I_T; X)$) $(k_{i_1}, L_{i_1} v_0, \delta, Z, X)$ and $(h_{i_2}, y_{i_2}, \delta, \gamma_{i_2})$, imply that $Q_3(k_{i_1}, L_{i_1} v_0), Q_4(h_{i_2}, y_{i_2}) \in C_0^\delta(I_T; X)$, $i_l = 1, \dots, n_l$, $l = 1, 2$. Now, since $\delta \in I_{\alpha,\beta,\tau_1} \subseteq ((3-2\alpha-\beta)/\alpha, 1/2) \subseteq (0, 1/2)$, the number $\delta + \mu_{\alpha,\beta}$ satisfies

$$\frac{3-2\alpha-\beta}{\alpha} < \delta \le \delta + \mu_{\alpha,\beta} < \frac{6-3\alpha-2\beta}{2\alpha} < 1, \tag{200}$$

and assumption $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, is meaningful. Lemma 24 with $(g_2, \delta_2) = (f, \mu)$ then yields $Q_2 f \in C_0^{\nu_{\alpha,\beta,\mu}}(I_T; X)$, $\nu_{\alpha,\beta,\mu} = (\alpha\mu + 2\alpha + \beta - 3)/\alpha$. Since $\nu_{\alpha,\beta,\mu} \ge \nu_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} = \delta$, we get $Q_2 f \in C_0^\delta(I_T; X)$, too. Summing up, we get the assertion. □

Before considering the function w_0 in (180), we introduce the following notation. In the sequel, for $3\alpha + 2\beta > 4$, $\beta \in (0, \alpha]$, $\alpha \in (0, 1]$, and $\nu \in ((2-\alpha-\beta)/\alpha, 1)$, $J_{\alpha,\beta,\nu} \subseteq ((2-\alpha-\beta)/\alpha, 1/2) \subseteq (0, 1/2)$ will denote the interval

$$J_{\alpha,\beta,\nu} = \begin{cases} \left(\dfrac{2-\alpha-\beta}{\alpha}, \nu\right], & \text{if } \nu \in \left(\dfrac{2-\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), \\ \left(\dfrac{2-\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), & \text{if } \nu \in \left[\dfrac{1}{2}, 1\right). \end{cases} \tag{201}$$

Notice that, since $(2-\alpha-\beta)/\alpha \le (3-2\alpha-\beta)/\alpha$, if the stronger condition $5\alpha + 2\beta > 6$ is satisfied, then (191) and (201) yield $I_{\alpha,\beta,\nu} \subseteq J_{\alpha,\beta,\nu}$ for every fixed $\nu \in ((3-2\alpha-\beta)/\alpha, 1)$. The introduction of the intervals $J_{\alpha,\beta,\nu}$ is justified by Lemma 46, which requires a weaker condition on the pair (α, β) than the one in Lemmas 42 and 45.

Lemma 46. *Let $3\alpha + 2\beta > 4$ in (H2), and let $v_0 \in \mathscr{D}(L)$. Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; X)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $v_1 + f(0) \in Y^r_\varphi$, where $\eta_{i_1}, \sigma_{i_2} \in ((2-\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (4-2\alpha-2\beta, 1)$, $i_l = 1, \dots, n_l$, $l = 1, 2$, $r \in [1, \infty]$, and $v_1 = (\lambda_0 M + L)v_0$. Let $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau_0 = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in J_{\alpha,\beta,\tau_0}$, the function w_0 defined by (180) belongs to $C_0^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \varrho_{\alpha,\beta}, 1)$, $\varrho_{\alpha,\beta} = (2-\alpha-\beta)/\alpha$.*

Proof. Observe that (cf. (159)) all the results from Lemma 32 to Corollary 38 will be applicable. First, since $2\alpha + 2\beta > 4 - \alpha \ge 3$, the choice $\gamma_{i_2}, \varphi \in (4-2\alpha-2\beta, 1)$, $i_2 = 1, \dots, n_2$, is meaningful. Moreover, since $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\} \in (4-2\alpha-2\beta, 1)$, the number $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$ satisfies $\chi_{\alpha,\beta,\gamma} \in ((2-\alpha-\beta)/\alpha, 1)$. Hence, $\tau_0 = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\} \in ((2-\alpha-\beta)/\alpha, 1)$, too, and J_{α,β,τ_0} is well defined. Now, let $\delta \in J_{\alpha,\beta,\tau_0}$ be fixed. Due to (20) or (38), the element x_0 defined by (174) belongs to Y^r_γ, whereas the functions $\widetilde{h}_{i_2}$ defined by (178) are of class $C_0^\delta(I_T; Y^r_{\gamma_{i_2}}) \hookrightarrow C_0^\delta(I_T; Y^r_\gamma)$. Then, since $\gamma \in (4-2\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$, from Lemma 37 and Corollary 38 applied with the pairs (g_5, δ_5) and (x, δ_7) being replaced by $(\widetilde{h}_{i_2}, \delta)$ and (x_0, δ), respectively, we deduce that $Q_5\widetilde{h}_{i_2}, Q_7 x_0 \in C_0^\delta(I_T; X)$, $i_2 = 1, \dots, n_2$. In addition, since the k_{i_1}'s and the constant functions $\kappa_{i_1}(t) = L_{i_1} v_0$ belong to $C^\delta(I_T; X)$, from Proposition 36 applied with $(g_{6_1}, g_{6_2}, X_1, X_2) = (k_{i_1}, L_{i_1} v_0, Z, X)$, it follows that $Q_6(k_{i_1}, L_{i_1} v_0) \in C_0^\delta(I_T; X)$, $i_1 = 1, \dots, n_1$. Finally, since $\delta \in J_{\alpha,\beta,\tau_0} \subseteq ((2-\alpha-\beta)/\alpha, 1/2)$, the number $\delta + \varrho_{\alpha,\beta}$ satisfies

$$\frac{2-\alpha-\beta}{\alpha} < \delta \le \delta + \varrho_{\alpha,\beta} < \frac{4-\alpha-2\beta}{2\alpha} < 1, \tag{202}$$

and the assumption $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \varrho_{\alpha,\beta}, 1)$, makes sense. Then, the function $\widetilde{f} = f - f(0)$ being of class $C_0^\mu(I_T; X)$, Lemma 32 applied with $(g_5, \delta_5) = (\widetilde{f}, \mu)$ yields $Q_5\widetilde{f} \in C_0^{\widetilde{\nu}_{\alpha,\beta,\mu}}(I_T; X)$, $\widetilde{\nu}_{\alpha,\beta,\mu} = (\alpha\mu + \alpha + \beta - 2)/\alpha$. Since $\widetilde{\nu}_{\alpha,\beta,\mu} \ge \widetilde{\nu}_{\alpha,\beta,\delta+\varrho_{\alpha,\beta}} = \delta$, we conclude that $Q_5\widetilde{f} \in C_0^\delta(I_T; X)$, too. Summing up, we get the assertion. □

Remark 47. We stress that, if $\beta \in (0, 1)$ in (H2), then $0 < \rho_{\alpha,\beta} \le \mu_{\alpha,\beta}$, so that in both Lemmas 45 and 46 we have to assume that $f \in C^\mu(I_T; X)$ with $\mu > \delta$. This is necessary in order to restore the loss of regularity produced by the operators Q_2 and Q_5.

We can now prove the main results of the section.

Theorem 48. *Assume (161) and $v_0 \in \mathscr{D}(L)$, and let $5\alpha+2\beta > 6$ in (H2). Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $v_1 + f(0) \in Y^r_\varphi$, where $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$, $i_l = 1, \dots, n_l$, $l = 1, 2$, $r \in [1, \infty]$, and $v_1 = (\lambda_0 M + L)v_0$. Let $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$ problem (160) admits a unique strict solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_t M v \in C^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Of course, due to (159), the assumption $\gamma_{i_2}, \varphi \in (5 - 3\alpha - 2\beta, 1)$, $i_2 = 1, \ldots, n_2$, makes sense. In addition, since $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in (5 - 3\alpha - 2\beta, 1)$, we have $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha \in ((3-2\alpha-\beta)/\alpha, 1)$. Therefore, by virtue of the choice of the Hölder exponents η_{i_1} and σ_{i_2}, the number $\tau = \min_{i_l=1,\ldots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$ belongs to $((3 - 2\alpha - \beta)/\alpha, 1)$ too, and the interval $I_{\alpha,\beta,\tau}$ is well defined. Further, the numbers η, τ_1, and τ_0 being as in the statements of Lemmas 42, 45, and 46, respectively, we have $\tau = \tau_0 \leq \tau_1 \leq \eta$. As a consequence, since $I_{\alpha,\beta,\tau} \subseteq I_{\alpha,\beta,\tau_1} \subseteq I_{\alpha,\beta,\eta}$ and $I_{\alpha,\beta,\tau} \subseteq J_{\alpha,\beta,\tau}$, all the mentioned lemmas are applicable with $\delta \in I_{\alpha,\beta,\tau}$. To this purpose, we stress that since $((3-2\alpha-\beta)/\alpha, 1) \subseteq ((2-\alpha-\beta)/\alpha, 1)$ and $(5-3\alpha-2\beta, 1) \subseteq (4-2\alpha-2\beta, 1) \subseteq (3-2\alpha-\beta, 1)$, the conditions for the applicability of both Lemmas 45 and 46 are fulfilled. Hence, now let $\delta \in I_{\alpha,\beta,\tau}$ being fixed. First, due to Lemma 42, the operator $\widetilde{R} = R|_{C_0^\delta(I_T;X)}$, $\widetilde{R}g = Rg$, $g \in C_0^\delta(I_T; X)$, *a fortiori* maps $C_0^\delta(I_T; X)$ into itself. Then, $C_0^\delta(I_T; X)$ being endowed with the same norm $\|\cdot\|_{\delta,0,T;X}$ of $C^\delta(I_T; X)$, from (197) we obtain the estimates

$$\left\|\widetilde{R}^n\right\|_{\mathscr{L}(C_0^\delta(I_T;X))} \leq \left[c_{42}(T)\right]^n \left(\frac{T^n}{n!}\right)^{1/p}, \quad n \in \mathbf{N} \cup \{0\}, \quad p \in \left(\frac{1}{1-2\delta}, \infty\right). \tag{203}$$

In particular, (203) yields that $\sum_{n=0}^{\infty} \widetilde{R}^n$ converges in $\mathscr{L}(C_0^\delta(I_T; X))$. From generalized Neumann's Theorem it thus follows that $1 \in \rho(\widetilde{R})$, the inverse $(I - \widetilde{R})^{-1} \in \mathscr{L}(C_0^\delta(I_T; X))$ being precisely $\sum_{n=0}^{\infty} \widetilde{R}^n$. Since Lemmas 45 and 46 (both applied with (observe here that if $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, then the exponent $\widetilde{\nu}_{\alpha,\beta,\mu}$ in the last part of the proof of Lemma 46 satisfies $\widetilde{\nu}_{\alpha,\beta,\mu} \geq \widetilde{\nu}_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} \geq \widetilde{\nu}_{\alpha,\beta,\delta+\varrho_{\alpha,\beta}} = \delta$. For, $\nu_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} = (\alpha\delta + 1 - \alpha)/\alpha = \delta + (1-\alpha)/\alpha)f \in C^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1) \subseteq [\delta+\rho_{\alpha,\beta}, 1))$ imply that $w_0, w_1 \in C_0^\delta(I_T; X)$, we conclude that the fixed-point equation (179) admits the unique solution

$$w = \sum_{n=0}^{\infty} \widetilde{R}^n (w_0 + w_1) \in C_0^\delta(I_T; X). \tag{204}$$

Observe now that the data vector $(k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, f, y_1, \ldots, y_{n_2}, v_1 + f(0))$ satisfies all the assumptions which were needed to show the equivalence between the fixed-point equation (179) and problem (170). Indeed, $\delta \leq \tau$ and $\delta \leq \delta + \mu_{\alpha,\beta} \leq \mu$ imply, respectively, that $k_{i_1} \in C^\delta(I_T; Z)$, $h_{i_2} \in C^\delta(I_T; \mathbf{C})$ and $f \in C^\delta(I_T; X)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, whereas, as in Lemma 46, $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ implies that $y_{i_2}, v_1 + f(0) \in Y_\gamma^r$. Therefore, since $A^{-1} \in \mathscr{L}(X)$, if $w \in C_0^\delta(I_T; X)$ is the solution to the fixed-point equation (179), then $A^{-1}w \in C_0^\delta(I_T; X)$, too, and the function F_w defined by (169) satisfies

$$F_w \in C^\delta(I_T; X), \quad x_0 = F_w(0) = \sum_{i_2=1}^{n_2} h_{i_2}(0)\, y_{i_2} + v_1 + f(0) \in Y_\gamma^r, \tag{205}$$

where $\delta \in I_{\alpha,\beta,\tau} \subsetneq (2-\alpha-\beta)/\alpha, 1)$, $\gamma \in (5-3\alpha-2\beta, 1) \subsetneq (1-\beta, 1)$, and $r \in [1, \infty]$. Consequently, recalling (168), we have proved that problem (160) has a unique strict global solution $v = L^{-1}w + v_0 \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = L^{-1}w(0) + v_0 = v_0$ and such that $Lv = w + Lv_0 \in C^\delta(I_T; X)$. As far as the regularity of $D_t M v$ is concerned, instead, it suffices to observe that (168), (170), $w \in C_0^\delta(I_T; X)$, and $F_w \in C^\delta(I_T; X)$ yield

$$D_t M v = D_t A^{-1} w = w + F_w \in C^\delta(I_T; X). \tag{206}$$

The proof is complete. □

Remark 49. Theorem 48 improves the faulty Thereoms 5.6 and 5.7 in [20] in two aspects. First, the assumption $3\alpha + 8\beta > 10$ is weakened to $5\alpha + 2\beta > 6$. In fact, $3\alpha + 8\beta > 10$ implies that $5\alpha + 2\beta = 3\alpha + 8\beta + 2\alpha - 6\beta > 10 - 4\alpha \geq 6$. Hence, in the special case $\alpha = 1$, the constraint $\beta > 7/8$ in [20] reduces to the definitely weaker $\beta > 1/2$. Second, in [20], only for $n_1 = n_2 = 1$ and opportunely chosen $\gamma < \beta$, the data y_1 and $v_1 + f(0)$ were assumed to belong to the intermediate spaces $X_A^{\gamma,r}$, whereas here, removing the assumption $\gamma < \beta$ and considering the general case $n_1, n_2 \in \mathbf{N}$, we allow $y_1, \ldots, y_{n_2}$ and $v_1 + f(0)$ to belong also to the interpolation spaces $(X, \mathscr{D}(A))_{\gamma,r}$. To emphasize how much these aspects are decisive, let $\alpha = 1$ in Theorem 48. Then, if $\beta \in (1/2, 2/3]$ and the choice $X_A^{\psi,r}$ is understood for Y_ψ^r, we have $\gamma_{i_2}, \varphi \in (2 - 2\beta, 1) \subsetneq [\beta, 1)$, and the spaces $X_A^{\gamma_{i_2},r}$ and $X_A^{\varphi,r}$, $i_2 = 1, \ldots, n_2$, may be smaller than $\mathscr{D}(A)$. However, the choice $Y_\psi^r = (X, \mathscr{D}(A))_{\psi,r}$ being admissible, in this situation too we can solve problem (160) with the data in spaces larger than $\mathscr{D}(A)$. Further, since $2/3 < 7/8$, in this case the results in [20] would not be applicable. These observations lead us to conclude that the more delicate approach followed in this paper with respect to that in [20, Sections 4 and 5], and especially the sharper results of the present Sections 3 and 4, yield a valuable refinement in the treatment of questions of maximal time regularity for the strict solutions to (160); of course, unless that the not too much significant case $\beta = 1$ is assumed in (H2).

Remark 50. The assumption $5\alpha + 2\beta > 6$ in (H2) implies that $\beta \in ((6 - 5\alpha)/2, \alpha] \subseteq (1/2, 1]$ and $\alpha \in (6/7, 1]$. In particular, if $\alpha = 1$, then Theorem 48 holds with $\beta \in (1/2, 1]$, $\eta_{i_1}, \sigma_{i_2} \in (1 - \beta, 1)$, $\gamma_{i_2}, \varphi \in (2 - 2\beta, 1)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, and $\mu_{1,\beta} = 1 - \beta$. Hence, $\gamma \in (2-2\beta, 1)$, $\chi_{1,\beta,\gamma} = \beta + \gamma - 1 \in (1-\beta, \beta)$, and $\delta \in I_{1,\beta,\tau}$ with $\tau \in (1 - \beta, \beta)$, where

$$I_{1,\beta,\tau} = (1 - \beta, \tau], \quad \text{if } \tau \in \left(1 - \beta, \frac{1}{2}\right), \qquad I_{1,\beta,\tau} = \left(1 - \beta, \frac{1}{2}\right), \quad \text{if } \tau \in \left[\frac{1}{2}, \beta\right). \tag{207}$$

Clearly, if $\beta = 1$, then $5\alpha + 2\beta > 6$ is redundant, and Theorem 48 holds with $\eta_{i_1}, \sigma_{i_2}, \gamma_{i_2}, \varphi \in (0, 1)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, $\mu_{1,1} = 0$, $\gamma = \chi_{1,1,\gamma} \in (0, 1)$, and $\delta \in I_{1,1,\tau}$, $\tau \in (0, 1)$, where $I_{1,1,\tau} = (0, \tau]$ if $\tau \in (0, 1/2)$ and $I_{1,1,\tau} = (0, 1/2)$ if $\tau \in [1/2, 1)$.

Remark 51. Observe that, if the η_{i_1}'s and σ_{i_2}'s are assumed to vary in the smaller interval $U_{\alpha,\beta} := ((3-2\alpha-\beta)/\alpha, (\alpha+\beta-1)/\alpha)$, then φ and the γ_{i_2}'s can be chosen such that $\tau = \min_{i_l=1,\ldots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. To this purpose, letting $\rho = \max_{i_l=1,\ldots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\} \in U_{\alpha,\beta}$, it suffices to take $\gamma_{i_2}, \varphi \in V_{\alpha,\beta,\rho}$, $i_2 = 1,\ldots,n_2$, where $V_{\alpha,\beta,\rho} := [2+\alpha\rho-\alpha-\beta, 1) \subsetneq (5-3\alpha-2\beta, 1)$. Then $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in V_{\alpha,\beta,\rho}$ and $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha \geq \rho$. In other words, provided that the data vector $(y_1,\ldots,y_{n_2}, v_1+f(0))$ is smooth enough, the pair (Lv, D_tMv) has the maximal time regularities which is the minimal between the time regularities of the k_{i_1}'s and h_{i_2}'s.

We conclude with the results which follow from Theorem 48 for problems (163)–(166).

Theorem 52. *Assume* (161) *and $v_0 \in \mathscr{D}(L)$, and let $5\alpha+2\beta > 6$ in (H2). Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$ and $Lv_0 + f(0) \in Y^r_\gamma$, where $\eta_{i_1} \in ((3-2\alpha-\beta)/\alpha, 1)$, $i_1 = 1,\ldots,n_1$, $\gamma \in (5-3\alpha-2\beta, 1)$, and $r \in [1,\infty]$. Let $\tau = \min_{i_1=1,\ldots,n_1}\{\eta_{i_1}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$ problem* (165) *admits a unique strict solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_tMv \in C^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Repeat the proofs of Lemmas 42, 45, and 46, Corollary 44, and Theorem 48, letting there $\lambda_0 = h_{i_2} = 0$, $i_2 = 1,\ldots,n_2$. To this purpose, observe that (169) and (174) reduce to $F_w(t) = \sum_{i_1=1}^{n_1}[\mathscr{K}(k_{i_1}, S_{i_1}w)(t) + \mathscr{K}(k_{i_1}, L_{i_1}v_0)(t)] + Lv_0 + f(t)$ and $x_0 = Lv_0 + f(0)$. Consequently, (180)–(182) change to $w_0 = Q_7x_0 + \sum_{i_1=1}^{n_1} Q_6(k_{i_1}, L_{i_1}v_0) + Q_5\tilde{f}$, $w_1 = -\sum_{i_1=1}^{n_1} Q_3(k_{i_1}, L_{i_1}v_0) - Q_2f$, and $Rw = \sum_{i_1=1}^{n_1}[Q_6(k_{i_1}, S_{i_1}w) - Q_3(k_{i_1}, S_{i_1}w)]$. □

Theorem 53. *Assume* (161) *and $v_0 \in \mathscr{D}(L)$, and let $5\alpha+2\beta > 6$ in (H2). Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $Lv_0 \in Y^r_\varphi$, where $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$, $i_l = 1,\ldots,n_l$, $l = 1,2$, and $r \in [1,\infty]$. Let $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau = \min_{i_l=1,\ldots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$ problem* (166) *admits a unique strict solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_tMv \in C^\delta(I_T; X)$.*

Proof. Let $\lambda_0 = f = 0$ in the proofs of Lemmas 42, 45, and 46, Corollary 44, and Theorem 48. In this case, (169) and (174) reduce to $F_w(t) = \sum_{i_1=1}^{n_1}[\mathscr{K}(k_{i_1}, S_{i_1}w)(t) + \mathscr{K}(k_{i_1}, L_{i_1}v_0)(t)] + \sum_{i_2=1}^{n_2} h_{i_2}(t)y_{i_2} + Lv_0$ and $x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)y_{i_2} + Lv_0$. Hence, (180)–(182) change to $w_0 = Q_7x_0 + \sum_{i_1=1}^{n_1} Q_6(k_{i_1}, L_{i_1}v_0) + \sum_{i_2=1}^{n_2} Q_5\tilde{h}_{i_2}$, $w_1 = -\sum_{i_1=1}^{n_1} Q_3(k_{i_1}, L_{i_1}v_0) - \sum_{i_2=1}^{n_2} Q_4(h_{i_2}, y_{i_2})$, and $Rw = \sum_{i_1=1}^{n_1}[Q_6(k_{i_1}, S_{i_1}w) - Q_3(k_{i_1}, S_{i_1}w)]$. □

Let us now turn to the degenerate differential problems (163) and (164).

Theorem 54. *Assume* (161) *and $v_0 \in \mathscr{D}(L)$, and let $5\alpha+2\beta > 6$ in (H2). Assume that $Lv_0 + f(0) \in Y^r_\gamma$, $\gamma \in (5-3\alpha-2\beta, 1)$, $r \in [1,\infty]$, and let $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ problem* (163) *admits a unique strict global solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_tMv \in C^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Let $\lambda_0 = k_{i_1} = h_{i_2} = 0$, $i_l = 1,\ldots,n_l$, $l = 1,2$, in problem (160) and formulae (169), (174) and, (179)–(182). Then, $F_w(t) = Lv_0 + f(t)$, $x_0 = Lv_0 + f(0)$ and $w = w_0 + w_1 = Q_7x_0 + Q_5\tilde{f} - Q_2f$. Consequently, Lemma 42 and Corollary 44 are unneeded, and the proof of Theorem 48 simplifies as follows. First, due to $\gamma \in (5-3\alpha-2\beta, 1)$ we have $\chi_{\alpha,\beta,\gamma} \in ((3-2\alpha-\beta)/\alpha, 1)$, and the interval $I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ is well defined. Hence, let $\delta \in I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ being fixed. Since (cf. (200)) $f \in C^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1) \subsetneq ((3-2\alpha-\beta)/\alpha, 1)$, reasoning as in the last part of the proof of Lemma 45 we get $Q_2f \in C^\delta_0(I_T; X)$. Moreover (see the proof of Lemma 46), since $x_0 \in Y^r_\gamma$, $\gamma \in (5-3\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$ and $\tilde{f} \in C^\mu_0(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1) \subseteq [\delta+\varrho_{\alpha,\beta}, 1)$, $\varrho_{\alpha,\beta} = (2-\alpha-\beta)/\alpha$, Corollary 38 and Lemma 32 applied with $(x, \delta_7) = (x_0, \delta)$ and $(g_5, \delta_5) = (\tilde{f}, \delta+\varrho_{\alpha,\beta})$ yield Q_7x_0, $Q_5\tilde{f} \in C^\delta_0(I_T; X)$. Summing up, we find that $w \in C^\delta_0(I_T; X)$. The assertion then follows from $v = L^{-1}w + v_0$ and (cf. (206)) $D_tMv = w + Lv_0 + f$. □

Remark 55. We refer to [19, Theorem 5.3] for a result of both time *and* space regularity for problem (163). There, provided that ψ and δ are opportunely chosen and the data satisfy assumptions similar to those in Theorem 54, it is shown that $D_tMv \in C^\delta(I_T; (X, \mathscr{D}(A))_{\psi,r})$, and that the higher is the order ψ of the interpolation space where we look for space regularity, the lower is the Hölder exponent δ of regularity in time. Notice that $Lv = D_tMv - f$ has no space regularity, unless f has too.

Theorem 56. *Assume* (161) *and $v_0 \in \mathscr{D}(L)$, and let $5\alpha+2\beta > 6$ in (H2). Assume that $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $Lv_0 \in Y^r_\varphi$, where $\sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$, $i_2 = 1,\ldots,n_2$, and $r \in [1,\infty]$. Let $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau = \min_{i_2=1,\ldots,n_2}\{\sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$, problem* (164) *admits a unique strict global solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_tMv \in C^\delta(I_T; X)$.*

Proof. Let $\lambda_0 = k_{i_1} = f = 0$, $i_1 = 1,\ldots,n_1$, in problem (160) and formulae (169), (174), and (179)–(182). Then, $F_w(t) = \sum_{i_2=1}^{n_2} h_{i_2}(t)y_{i_2} + Lv_0$, $x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)y_{i_2} + Lv_0$ and $w = w_0 + w_1 = Q_7x_0 + \sum_{i_2=1}^{n_2} Q_5\tilde{h}_{i_2} - \sum_{i_2=1}^{n_2} Q_4(h_{i_2}, y_{i_2})$. Therefore, as in Theorem 54, we do not need Lemma 42 and Corollary 44, and the proof of Theorem 48 simplifies as follows. Again, $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in (5-3\alpha-2\beta, 1)$ implies that

$\chi_{\alpha,\beta,\gamma} \in ((3-2\alpha-\beta)/\alpha, 1)$, so that $\tau = \min_{i_2=1,\dots,n_2}\{\sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\} \in ((3-2\alpha-\beta)/\alpha, 1)$, and the interval $I_{\alpha,\beta,\tau}$ is well defined. Let $\delta \in I_{\alpha,\beta,\tau}$ be fixed. First (see the proof of Lemma 45), since $\gamma_{i_2} \in (5-3\alpha-2\beta, 1) \subseteq (3-2\alpha-\beta, 1)$, Lemma 30 applied with $(g_4, y, \delta_4, \gamma) = (h_{i_2}, y_{i_2}, \delta, \gamma_{i_2})$ yields $Q_4(h_{i_2}, y_{i_2}) \in C_0^{\delta}(I_T; X)$, $i_2 = 1, \dots, n_2$. On the other side (see the proof of Lemma 46), since $x_0 \in Y_\gamma^r$ and $\widetilde{h}_{i_2} \in C_0^{\delta}(I_T; Y_{\gamma_{i_2}}^r) \hookrightarrow C_0^{\delta}(I_T; Y_\gamma^r)$, $\gamma \in (5-3\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$, from Lemma 37 and Corollary 38 applied with $(g_5, \delta_5) = (\widetilde{h}_{i_2}, \delta)$ and $(x, \delta_7) = (x_0, \delta)$ we deduce that $Q_5\widetilde{h}_{i_2}, Q_7 x_0 \in C_0^{\delta}(I_T; X)$, $i_2 = 1, \dots, n_2$. Summing up, we find that $w \in C_0^{\delta}(I_T; X)$, and the assertion again follows from $v = L^{-1}w + v_0$ and (cf. (206)) $D_t M v = w + L v_0 + \sum_{i_2=1}^{n_2} h_{i_2} y_{i_2}$. □

6. An Application to a Concrete Case

Theorem 48 is here applied to determine the right functional framework where to search for the solution of an inverse problem arising in the theory of heat conduction for materials with memory. To this purpose, let $\Omega \subsetneq \mathbf{R}^N$, $N \in \mathbf{N}$, be a bounded domain with boundary $\partial\Omega$ of class $C^{1,1}$ (cf. [36, p. 94]). If Ω represents a rigid thermal body with memory, then the linearized theory of heat flow yields the following equations linking the internal energy e, the heat flux $\mathbf{q} = (q_1, \dots, q_N)$, and the temperature Θ (cf. [32, 37–40]):

$$
\begin{aligned}
e(t,x) &= e_0 + a(0,x)\Theta(t,x) + \int_0^t D_t a(t-s,x)\Theta(s,x)\,ds,\\
q_j(t,x) &= -\sum_{i=1}^{r_1} b_i(0) C_{i,j}(x; D_x)\Theta(t,x)\\
&\quad - \sum_{i=1}^{r_1}\int_0^t D_t b_i(t-s) C_{i,j}(x; D_x)\Theta(s,x)\,ds,\\
&\qquad j = 1, \dots, N,\\
D_t e(t,x) &= -\mathrm{div}_x \mathbf{q}(t,x) + g(t,x)\\
&= -\sum_{j=1}^{N} D_{x_j} q_j(t,x) + g(t,x).
\end{aligned}
\tag{208}
$$

Here $t \in I_T$, $I_T = [0,T]$, $T > 0$, $x = (x_1, \dots, x_N) \in \Omega$, $r_1 \in \mathbf{N}$, $e_0 \in \mathbf{R}$, and $D_t = \partial/\partial t$, whereas the $C_{i,j}(x; D_x)$'s represent the first-order linear differential operators

$$
\begin{gathered}
C_{i,j}(x; D_x) = \sum_{k=1}^{N} c_{i,j,k}(x) D_{x_k}, \quad x \in \overline{\Omega},\\
i = 1, \dots, r_1, \quad j = 1, \dots, N,
\end{gathered}
\tag{209}
$$

where $c_{i,j,k} \in C^1(\overline{\Omega}; \mathbf{R})$ and $D_{x_k} = \partial/\partial x_k$, $i = 1, \dots, r_1$, $j, k = 1, \dots, N$. According to the terminology of [39, 40], the functions a, b_i, $i = 1, \dots, r_1$, and g are called, respectively, the *energy-temperature relaxation function*, the *heat conduction relaxation functions,* and the *heat supply function* and we assume that they satisfy the following conditions:

$$
\begin{gathered}
D_t^k a(\cdot, x) \in C(I_T; \mathbf{R}), \quad k = 0, 1, 2,\\
a(0,x) \ge 0, \quad x \in \Omega,
\end{gathered}
\tag{210}
$$

$$
\begin{gathered}
D_t^k b_i \in C(I_T; \mathbf{R}), \quad k = 0, 1, \ i = 1, \dots, r_1,\\
g \in C^1(I_T \times \Omega; \mathbf{R}).
\end{gathered}
\tag{211}
$$

Notice that, different from [32, 37–40], here the energy-temperature relaxation function a is assumed to depend also on the spatial variable $x \in \Omega$. In physical terms, this is equivalent to say that Ω represents a rigid *inhomogeneous* material with memory. Furthermore, in contrast with the quoted papers where only the cases $r_1 = 1$ and $C_{1,j}(x; D_x) = D_{x_j}$ are treated, here we have assumed that the history record of Ω is kept by an arbitrary number $r_1 \in \mathbf{N}$ of heat conduction relaxation functions and that the $C_{i,j}$'s are the more general first-order differential operators defined in (209).

By setting

$$
\widetilde{a}_{j,k} = \sum_{i=1}^{r_1} b_i(0) c_{i,j,k} \in C^1(\overline{\Omega}; \mathbf{R}), \quad j,k = 1, \dots, N, \tag{212}
$$

from (208) and (209), it thus follows that the temperature Θ must satisfy the following equation:

$$
\begin{aligned}
&a(0,x) D_t\Theta(t,x) + D_t a(0,x)\Theta(t,x)\\
&\qquad + \int_0^t D_t^2 a(t-s,x)\Theta(s,x)\,ds - g(t,x)\\
&\quad = \sum_{j,k=1}^{N} D_{x_j}\left[\widetilde{a}_{j,k}(x) D_{x_k}\Theta(t,x)\right]\\
&\qquad + \sum_{i=1}^{r_1}\int_0^t D_t b_i(t-s)\sum_{j=1}^{N} D_{x_j} C_{i,j}(x; D_x)\Theta(s,x)\,ds.
\end{aligned}
\tag{213}
$$

Let us now assume that a is of the following special form:

$$
a(t,x) = \sum_{n=1}^{2} m_n(x) u_n(t), \quad (t,x) \in I_T \times \Omega, \tag{214}
$$

where the functions m_n and u_n, $n = 1, 2$, satisfy the following conditions (cf. (210)):

$$
\begin{gathered}
m_n \in L_\infty(\Omega), \quad n = 1, 2,\\
m_1 \ge 0, \qquad m_2 > 0,
\end{gathered}
\tag{215}
$$

$$
\begin{gathered}
u_n \in C^2(I_T; \mathbf{R}), \quad n = 1, 2,\\
u_2(0) = 0, \qquad u_1(0) > 0, \qquad D_t u_2(0) > 0.
\end{gathered}
\tag{216}
$$

Here, $L_q(\Omega) = L_q(\Omega; \mathbf{R})$, $q \in [1, \infty]$, is the usual L_q space with norm $\|\cdot\|_{q;\Omega}$ (cf. [36, Chapter 7]). Using $m_2, u_1(0)$, $D_t u_2(0) > 0$, for $t \in I_T$ and $x \in \Omega$ we now set

$$a_0(x) = -[u_1(0)]^{-1} m_2(x) D_t u_2(0) < 0, \tag{217}$$

$$a_{j,k}(x) = [u_1(0)]^{-1} \tilde{a}_{j,k}(x), \quad j,k = 1, \ldots, N, \tag{218}$$

$$L(x; D_x) = \sum_{j,k=1}^{N} D_{x_j}\left[a_{j,k}(x) D_{x_k}\right] + a_0(x), \tag{219}$$

$$L_i(x; D_x) = [u_1(0)]^{-1} \sum_{j=1}^{N} D_{x_j} C_{i,j}(x; D_x), \quad i = 1, \ldots, r_1, \tag{220}$$

$$L_{r_1+n}(x; D_x) = L_{r_1+n}(x) = [u_1(0)]^{-1} m_n(x), \quad n = 1, 2, \tag{221}$$

$$\begin{gathered} k_i(t) = D_t b_i(t), \quad i = 1, \ldots, r_1, \\ k_{r_1+n}(t) = -D_t^2 u_n(t), \quad n = 1, 2, \end{gathered} \tag{222}$$

$$\begin{gathered} \tilde{g}(t,x) = [u_1(0)]^{-1} g(t,x), \\ \lambda_0 = -[u_1(0)]^{-1} D_t u_1(0) \in \mathbf{R}. \end{gathered} \tag{223}$$

Then, since (214)–(216) yield $a(0,x) = m_1(x) u_1(0)$ and $D_t^k a(t,x) = \sum_{n=1}^{2} m_n(x) D_t^k u_n(t)$, $k = 1, 2$, if we multiply both sides of (213) by $[u_1(0)]^{-1}$ and use (218)–(223), we are led to the following basic differential equation for the temperature Θ, where $n_1 = r_1 + 2$:

$$\begin{aligned} D_t&\left[m_1(x)\Theta(t,x)\right] \\ &= \lambda_0 m_1(x)\Theta(t,x) + L(x; D_x)\Theta(t,x) + \tilde{g}(t,x) \\ &\quad + \sum_{i=1}^{n_1} \int_0^t k_i(t-s) L_i(x; D_x)\Theta(s,x)\,ds, \\ &\qquad t \in I_T, \quad x \in \Omega. \end{aligned} \tag{224}$$

We endow this differential equation with the initial condition $\Theta(0,x) = \Theta_0(x)$, $x \in \Omega$, and the Dirichlet boundary condition $\Theta(t,x) = 0$, $t \in I_T$, $x \in \partial\Omega$.

We now suppress the dependence on $x \in \Omega$, and we transform (224) in a degenerate integrodifferential Cauchy problem in a Banach space X. To this purpose, for every fixed $q \in (1, \infty)$ and observing that $m_n \in L_\infty(\Omega)$ implies that $\|m_n u\|_{q;\Omega} \le \|m_n\|_{\infty;\Omega} \|u\|_{q;\Omega}$ for every $u \in L_q(\Omega)$, $n = 1, 2$, we set

$$X = \mathscr{D}(M) = \mathscr{D}\left(L_{r_1+n}\right) = L_q(\Omega), \quad n = 1, 2, \tag{225}$$

$$\mathscr{D}(L) = W_q^2(\Omega) \cap \overset{\circ}{W}{}_q^1(\Omega), \qquad \mathscr{D}(L_i) = W_q^2(\Omega), \quad i = 1, \ldots, r_1, \tag{226}$$

$$\begin{gathered} M, L_{r_1+n} \in \mathscr{L}(X), \qquad Mu = m_1 u, \\ L_{r_1+n} u = L_{r_1+n}(x) u, \quad u \in X, \ n = 1, 2, \end{gathered} \tag{227}$$

$$\begin{gathered} L : \mathscr{D}(L) \subseteq X \longrightarrow X, \\ Lu = L(x; D_x) u, \ u \in \mathscr{D}(L), \end{gathered} \tag{228}$$

$$\begin{gathered} L_i : \mathscr{D}(L_i) \subseteq X \longrightarrow X, \\ L_i u = L_i(x; D_x) u, \quad u \in \mathscr{D}(L_i), \quad i = 1, \ldots, r_1. \end{gathered} \tag{229}$$

Here (cf. [36, Chapter 7]), $W_q^k(\Omega) = W_q^k(\Omega; \mathbf{R})$, $k \in \mathbf{N} \cup \{0\}$, $q \in (1, \infty)$, denotes the usual Sobolev space endowed with the norm $\|\cdot\|_{k,q;\Omega}$ $((W_q^0(\Omega), \|\cdot\|_{0,q;\Omega}) = (L_q(\Omega), \|\cdot\|_{q;\Omega}))$, whereas $\overset{\circ}{W}{}_q^k(\Omega)$ denotes the completion of $C_0^\infty(\Omega; \mathbf{R})$ in $W_q^k(\Omega)$, $C_0^\infty(\Omega; \mathbf{R})$ being the set of all real-valued infinitely differentiable functions having compact support in Ω. We further assume that there exists positive constant Λ_i, $i = 0, \ldots, r_1$, such that for every $(x, \xi) \in \overline{\Omega} \times \mathbf{R}^N$ the following inequalities hold:

$$\begin{gathered} \sum_{j,k=1}^{N} c_{i,j,k}(x)\xi_j \xi_k \ge \Lambda_i |\xi|^2, \quad i = 1, \ldots, r_1, \\ \sum_{j,k=1}^{r_1} b_i(0)\Lambda_i \ge \Lambda_0, \end{gathered} \tag{230}$$

where $|\xi|^2 = \sum_{l=1}^{N} \xi_l^2$. Therefore, from (212), (218), and (230) we get

$$\begin{aligned} \sum_{j,k=1}^{N} a_{j,k}(x)\xi_j\xi_k &= [u_1(0)]^{-1} \sum_{i=1}^{r_1} b_i(0) \sum_{j,k=1}^{N} c_{i,j,k}\xi_j\xi_k \\ &\ge [u_1(0)]^{-1} \Lambda_0 |\xi|^2. \end{aligned} \tag{231}$$

From (225)–(231) it follows that M, L, and L_i, $i = 1, \ldots, n_1$, are closed linear operators from X to itself, and the relation $\mathscr{D}(L) \subsetneq \bigcap_{i=1}^{n_1} [\mathscr{D}(M) \cap \mathscr{D}(L_i)] = W_q^2(\Omega)$ holds. In addition, due to (212), (217), (218), and (231), from [36, Theorem 9.15 and Lemma 9.17], it follows that for every fixed $q \in (1, \infty)$ the operator L admits an inverse operator $L^{-1} \in \mathscr{L}(X; W_q^2(\Omega))$. Hence, a fortiori, $L^{-1} \in \mathscr{L}(X)$ and so condition (161) is satisfied (observe also that $L^{-1} \in \mathscr{L}(X; W_q^2(\Omega))$ implies that the norms $\|\cdot\|_{2,q;\Omega}$ and $\|\cdot\|_{\mathscr{D}(L)} = \|\cdot\|_{q;\Omega} + \|L\cdot\|_{q;\Omega}$ are equivalent on $\mathscr{D}(L)$. In fact, if $v \in \mathscr{D}(L)$, then $\|v\|_{2,q;\Omega} = \|L^{-1}Lv\|_{2,q;\Omega} \le \|L^{-1}\|_{\mathscr{L}(X;W_q^2(\Omega))}\|v\|_{\mathscr{D}(L)} \le \widetilde{C}\|L^{-1}\|_{\mathscr{L}(X;W_q^2(\Omega))}\|v\|_{2,q;\Omega}$, $\widetilde{C}$ being a positive constant depending on $\max_{j,k=1,\ldots,N}\|a_{j,k}\|_{C^1(\overline{\Omega};\mathbf{R})}$). The closed graph theorem then yield $ML^{-1}, L_iL^{-1} \in \mathscr{L}(X)$, $i = 1,\ldots,n_1$. Moreover (cf. [19, formula (77)], and [41, formula (2.16)]), the following estimate holds (of course, here $X = L_q(\Omega;\mathbf{R})$ is replaced with the more general $X = L_q(\Omega;\mathbf{C})$):

$$\left\|M(\lambda M - L)^{-1}\right\|_{\mathscr{L}(X)} \le C(|\lambda| + 1)^{-\beta}, \quad \forall\lambda \in \Sigma_1,\ \beta = \frac{1}{q}, \tag{232}$$

where $\Sigma_1 = \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge -c(|\mathfrak{Im}\, z| + 1),\ \mathfrak{Im}\, z \in \mathbf{R}\}$, c being a suitable positive constant depending on q and $\|m_1\|_{\infty;\Omega}$. Hence, condition (H2) is satisfied with $X = L^q(\Omega)$ and $(\alpha,\beta) = (1, 1/q)$. Notice that, since m_1 may have zeros in Ω, M^{-1} is in general a m. l. operator, so that $A = LM^{-1}$ is determined by (cf. (162)):

$$\begin{gathered}\mathscr{D}(A) = M(\mathscr{D}(L)) = \{m_1 v : v \in \mathscr{D}(L)\}, \\ Au = \{Lv : v \in \mathscr{D}(L) \text{ such that } u = m_1 v\}, \quad u \in \mathscr{D}(A).\end{gathered} \tag{233}$$

Using the convolution operator $\mathscr{K}$ in (104) in which for the bilinear operator $\mathscr{P}$ we take the scalar multiplication in X, from (224)–(229) we finally obtain that the temperature $\Theta(t) = \Theta(t,\cdot)$ solves the following degenerate integrodifferential Cauchy problem in X:

$$\begin{aligned} D_t(M\Theta(t)) &= [\lambda_0 M + L]\Theta(t) \\ &\quad + \sum_{i=1}^{n_1}\mathscr{K}(k_i, L_i\Theta)(t) + \widetilde{g}(t), \quad t \in I_T, \\ &\Theta(0) = \Theta_0. \end{aligned} \tag{234}$$

Now, assume for a moment that we are interested in solving the *inverse* problem of recovering both the temperature Θ and the memory kernels $k_1,\ldots,k_{r_1}$ in (234). Clearly, due to (222), if we recover $k_1,\ldots,k_{r_1}$, then the heat conduction relaxation functions $b_1,\ldots,b_{r_1}$ will be known too, unless of the r_1 arbitrary constants $b_i(0)$, $i = 1,\ldots,r_1$. Indeed, $b_1(t) = b_i(0) + \int_0^t k_i(s)\mathrm{d}s$, $t \in I_T$. To solve such an inverse problem, we need r_1 additional informations other than the initial condition $\Theta(0) = \Theta_0$, which, in general, suffices only to guarantee the well-posedness of the *direct* problem of recovering Θ in (234). Suppose then that the following additional pieces of information are given:

$$\Psi_j[M\Theta(t)] = g_j(t), \quad t \in I_T,\ j = 1,\ldots,r_1, \tag{235}$$

where $\Psi_j \in X^* = \mathscr{L}(X;\mathbf{R})$ and $g_j \in C^{2+\nu_j}(I_T;\mathbf{R})$, $\nu_j \in (0,1)$, $j = 1,\ldots,r_1$. We will search for a solution vector $(\Theta, k_1,\ldots,k_{r_1})$ of the inverse problem (234) and (235) such that $\Theta \in C^{1+\delta}(I_T;\mathscr{D}(L))$ and $k_j \in C^{\eta_j}(I_T;\mathbf{R})$, $j = 1,\ldots,r_1$, with the Hölder exponents δ and η_j, $j = 1,\ldots,r_1$, to be made precise in the sequel. We stress that here we will not solve completely the mentioned inverse problem. For, its detailed treatment would lead us out of the aims of this paper. Our intention here is only to highlight how the main results of Section 5 allow to determine the correct functional framework in which the solution of the inverse problem has to be searched. However, a complete treatment of the inverse problem will be the object of a future paper.

Assuming that $\Theta \in C^{1+\delta}(I_T;\mathscr{D}(L))$ solves (234), we introduce the new unknown

$$v(t,x) = D_t\Theta(t,x) \Longleftrightarrow \Theta(t,x) = \Theta_0(x) + \int_0^t v(s,x)\,\mathrm{d}s. \tag{236}$$

Then, differentiating (234) with respect to time and using

$$\begin{aligned} &D_t\mathscr{K}(k_i, L_i\Theta)(t) \\ &\quad = D_t\int_0^t k_i(t-s)L_i\Theta(s)\,\mathrm{d}s = D_t\int_0^t k_i(s)L_i\Theta(t-s)\,\mathrm{d}s \\ &\quad = k_i(t)L_i\Theta(0) + \int_0^t k_i(s)L_iD_t\Theta(t-s)\,\mathrm{d}s \\ &\quad = k_i(t)L_i\Theta_0 + \int_0^t k_i(t-s)L_iv(s)\,\mathrm{d}s, \end{aligned} \tag{237}$$

we find that $v \in C^\delta(I_T;\mathscr{D}(L))$ solves the following degenerate integrodifferential problem:

$$\begin{aligned} &D_t(Mv(t)) \\ &\quad = [\lambda_0 M + L]v(t) + \sum_{i=1}^{n_1}[\mathscr{K}(k_i, L_iv)(t) + k_i(t)y_i] + f(t), \\ &\qquad\qquad t \in I_T, \\ &\qquad Mv(0) = Mv_0, \end{aligned} \tag{238}$$

where $y_i = L_i\Theta_0$, $i = 1,\ldots,n_1$, $f = D_t\widetilde{g}$ and $Mv_0 = [\lambda_0M + L]\Theta_0 + \widetilde{g}(0,\cdot)$ (indeed, since M is the multiplication operator by the function m_1 independent of t, from the differential equation in (234) with $t = 0$ we get $Mv(0) = MD_t\Theta(0) = [\lambda_0M + L]\Theta(0) + \widetilde{g}(0)$). Of course, (238) is the special case $(i_1, i_2, n_2) = (i, i, n_1)$, $h_i = k_i$, $i = 1\ldots,n_1$, of problem (160).

Conversely, assume that $v \in C^\delta(I_T;\mathscr{D}(L))$ solves (238). Then, the function Θ defined by (236) belongs to $C^{1+\delta}(I_T;\mathscr{D}(L))$ and solves (234). Indeed, using the fact that m_1

does not depend on time and that M, L, and L_i, $i = 1, \dots, n_1$, are closed, we obtain

$$\begin{aligned}
& D_t (M\Theta(t)) - [\lambda_0 M + L]\,\Theta(t) \\
&\qquad - \sum_{i=1}^{n_1} \mathscr{K}(k_i, L_i\Theta)(t) - \tilde{g}(t) \\
&\quad = D_t\left[M\left(\Theta_0 + \int_0^t v(s)\,\mathrm{d}s\right)\right] \\
&\qquad - [\lambda_0 M + L]\left[\Theta_0 + \int_0^t v(s)\,\mathrm{d}s\right] \\
&\qquad - \sum_{i=1}^{n_1} \int_0^t k_i(t-s)\,L_i\left[\Theta_0 + \int_0^s v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\qquad - \tilde{g}(0) - \int_0^t D_s\tilde{g}(s)\,\mathrm{d}s \\
&\quad = Mv(t) - [\lambda_0 M + L]\,\Theta_0 \\
&\qquad - \int_0^t [\lambda_0 M + L]\,v(s)\,\mathrm{d}s \\
&\qquad - \sum_{i=1}^{n_1} \int_0^t k_i(t-s)\,L_i\Theta_0\,\mathrm{d}s \\
&\qquad - \sum_{i=1}^{n_1} \int_0^t k_i(t-s)\left[\int_0^s L_i v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\qquad - \tilde{g}(0) - \int_0^t f(s)\,\mathrm{d}s.
\end{aligned} \tag{239}$$

Now, observe that

$$\begin{aligned}
Mv(t) &= Mv_0 + \int_0^t D_s(Mv(s))\,\mathrm{d}s \\
&= [\lambda_0 M + L]\,\Theta_0 + \tilde{g}(0) + \int_0^t D_s(Mv(s))\,\mathrm{d}s,
\end{aligned} \tag{240}$$

$$\int_0^t k_i(t-s)\,L_i\Theta_0\,\mathrm{d}s = \int_0^t k_i(s)\,L_i\Theta_0\,\mathrm{d}s = \int_0^t k_i(s)\,y_i\,\mathrm{d}s, \quad i = 1, \dots, n_1, \tag{241}$$

whereas an application of Fubini's theorem combined with the changes of variables $\xi = s - r$, $r - s = \tau$ and $t - s = \zeta$ easily yields for every $i = 1, \dots, n_1$ the following:

$$\begin{aligned}
& \int_0^t k_i(t-s)\left[\int_0^s L_i v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\quad = \int_0^t k_i(t-s)\left[\int_0^s L_i v(s-r)\,\mathrm{d}r\right]\mathrm{d}s \\
&\quad = \int_0^t \left[\int_s^t k_i(t-r)\,L_i v(r-s)\,\mathrm{d}r\right]\mathrm{d}s \\
&\quad = \int_0^t \left[\int_0^{t-s} k_i(t-s-\tau)\,L_i v(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s \\
&\quad = \int_0^t \mathscr{K}(k_i, L_i v)(t-s)\,\mathrm{d}s \\
&\quad = \int_0^t \mathscr{K}(k_i, L_i v)(\zeta)\,\mathrm{d}\zeta = \int_0^t \mathscr{K}(k_i, L_i v)(s)\,\mathrm{d}s.
\end{aligned} \tag{242}$$

Therefore, replacing (240)–(242) in (239), it follows for every $t \in I_T$ that

$$\begin{aligned}
& D_t(M\Theta(t)) - [\lambda_0 M + L]\,\Theta(t) - \sum_{i=1}^{n_1} \mathscr{K}(k_i, L_i\Theta)(t) - \tilde{g}(t) \\
&\quad = \int_0^t \Bigg\{ D_s(Mv(s)) - [\lambda_0 M + L]\,v(s) \\
&\qquad - \sum_{i=1}^{n_1} [\mathscr{K}(k_i, L_i v)(s) + k_i(s)\,y_i] - f(s) \Bigg\}\,\mathrm{d}s,
\end{aligned} \tag{243}$$

and the latter integral is equal to zero by virtue of (238). Since from (236) it follows that $\Theta(0) = \Theta_0$, we have thus shown that (234) and (238) are *equivalent*. Such an equivalence is the first step in solving the mentioned inverse problem of recovering the vector $(\Theta, k_1, \dots, k_{r_1})$ with the help of the additional information (235).

Let us now apply the linear functional Ψ_j, $j = 1 \dots, r_1$, to (238). Using

$$\begin{aligned}
\Psi_j\left[D_t^k(Mv(t))\right] &= \Psi_j\left[MD_t^{k+1}\Theta(t)\right] = D_t^{k+1}\Psi_j[M\Theta(t)] \\
&= D_t^{k+1} g_j(t), \quad k = 0, 1,
\end{aligned} \tag{244}$$

we thus find the following system of r_1 equations for the r_1 unknown $k_1, \dots, k_{r_1}$:

$$\begin{aligned}
& \sum_{i=1}^{r_1} \Psi_j[y_i]\,k_i(t) \\
&\quad = N_j(t) - \Psi_j[Lv] - \sum_{i=1}^{n_1} \Psi_j[\mathscr{K}(k_i, L_i v)(t)], \\
&\qquad j = 1, \dots, r_1,
\end{aligned} \tag{245}$$

where we have set (recall that $k_{r_1+n} = -D_t u_n$, $n = 1, 2$, are known)

$$\begin{aligned}
N_j(t) &= (D_t - \lambda_0)\,D_t g_j(t) - \Psi_j[f(t)] \\
&\quad - \sum_{n=1}^{2} \Psi_j\left[y_{r_1+n}\right] k_{r_1+n}(t), \quad j = 1, \dots, r_1.
\end{aligned} \tag{246}$$

Therefore, if the matrix $\mathscr{U} := \mathscr{U}^{y_1,\ldots,y_{r_1}}_{\Psi_1,\ldots,\Psi_{r_1}} = (\Psi_i[y_j])_{i,j=1,\ldots,r_1}$ has determinant $\det \mathscr{U} \neq 0$, then from Cramer's formula it follows that the solution $(k_1,\ldots,k_{r_1})$ of (245) is given by

$$\begin{aligned} k_j(t) &= [\det \mathscr{U}]^{-1} \sum_{k=1}^{r_1} \Big\{ N_k(t) - \Psi_k[Lv] \\ &\quad - \sum_{i=1}^{n_1} \Psi_k[\mathscr{K}(k_i, L_i v)(t)] \Big\} \mathscr{U}_{k,j} \\ &=: \widetilde{R}_j(v, k_1,\ldots,k_{r_1})(t), \quad j = 1,\ldots,r_1, \end{aligned} \tag{247}$$

with $\mathscr{U}_{k,j}$, $k, j = 1,\ldots,r_1$, being the cofactor of the element $\Psi_k[y_j]$ of $\mathscr{U}$ (with the convention that $\mathscr{U}_{1,1} = 1$ in the case of $r_1 = 1$). We have thus found a system of r_1 fixed-point equations for the r_1 unknown $k_1,\ldots,k_{r_1}$.

Now, let $Y^r_\psi \in \{(X,\mathscr{D}(A))_{\psi,r}, X^{\psi,r}_A\}$, $\psi \in (0,1)$, $r \in [1,\infty]$, where A is as in (233). Assume that v_0 in the initial condition $Mv(0) = Mv_0$ belongs to $\mathscr{D}(L)$ and that

$$\begin{aligned} &k_i \in C^{\eta_i}(I_T, \mathbf{R}), \qquad f \in C^{\mu}(I_T; X), \qquad \eta_i, \mu \in \left(\frac{1}{q'}, 1\right), \\ &\qquad\qquad i = 1,\ldots,n_1, \\ &y_i \in Y^p_{\gamma_i}, \qquad v_1 + f(0) \in Y^p_{\varphi}, \quad \gamma_i, \varphi \in \left(\frac{1}{q'}, 1\right), \\ &\qquad\qquad p \in [1,\infty], \quad i = 1,\ldots,n_1, \end{aligned} \tag{248}$$

where $v_1 = (\lambda_0 M + L)v_0$ and q' is the conjugate exponent of $q \in (1,\infty)$. Then (cf. (179) with $(i_1, i_2, n_2) = (i, i, n_1)$, $(\alpha, \beta, Z) = (1, 1/q, \mathbf{R})$, and $h_i = k_i$, $i = 1,\ldots,n_1$), problem (238) is equivalent to the fixed-point equation

$$\begin{aligned} w(t) &= R(w, k_1,\ldots,k_{r_1})(t) + \sum_{l=0}^{1} w_l(k_1,\ldots,k_{r_1})(t) \\ &=: T(w, k_1,\ldots,k_{r_1})(t), \end{aligned} \tag{249}$$

where $w = L(v - v_0)$ and

$$\begin{aligned} w_0(k_1,\ldots,k_{r_1}) &= Q_7 x_0 + \sum_{i=1}^{n_1} \left[Q_6(k_i, L_i v_0) + Q_5 \widetilde{k}_i \right] + Q_5 \widetilde{f}, \\ w_1(k_1,\ldots,k_{r_1}) &= -\sum_{i=1}^{n_1} \left[Q_3(k_i, L_i v_0) + Q_4(k_i, y_i) \right] - Q_2 f, \\ R(w, k_1,\ldots,k_{r_1}) &= \lambda_0 \left[Q_5(A^{-1}w) - Q_2(A^{-1}w) \right] \\ &\quad + \sum_{i=1}^{n_1} \left[Q_6(k_i, S_i w) - Q_3(k_i, S_i w) \right]. \end{aligned} \tag{250}$$

Here, the Q_j's, $j = 2,\ldots,6$, are defined by (106)–(110), $S_i = L_i L^{-1}$, and the functions $\widetilde{f}, \widetilde{k}_i$ and $Q_7 x_0$ are defined by $\widetilde{f}(t) = f(t) - f(0)$, $\widetilde{k}_i(t) = [k_i(t) - k_i(0)]y_i$, and $[Q_7 x_0](t) = (e^{tA} - I)x_0$, respectively, where (cf. (174)) $x_0 = \sum_{i=1}^{n_1} k_i(0) y_i + v_1 + f(0)$.

Then, since $v = L^{-1}w + v_0$, if we set $R_j(w, k_1,\ldots,k_{r_1}) = \widetilde{R}_j(L^{-1}w + v_0, k_1,\ldots,k_{r_1})$, $j = 1,\ldots,r_1$, and

$$\begin{aligned} &\Xi(w, k_1,\ldots,k_{r_1}) \\ &\quad = \big(T(w, k_1,\ldots,k_{r_1}), R_1(w, k_1,\ldots,k_{r_1}), \\ &\qquad \ldots, R_{r_i}(w, k_1,\ldots,k_{r_1})\big), \end{aligned} \tag{251}$$

from (247) and (249) we deduce that to solve the inverse problems (234) and (235) for the unknown vector $(\Theta, k_1,\ldots,k_{r_1})$, it suffices to show that the fixed-point equation

$$(w, k_1,\ldots,k_{r_1}) = \Xi(w, k_1,\ldots,k_{r_1}) \tag{252}$$

has a unique solution. In general, this will be done by proving that Ξ is a contraction map in the Banach space

$$\begin{aligned} &Z_{\delta,\eta_1,\ldots,\eta_{r_1}} = C^{\delta}(I_T; X) \times C^{\eta_1}(I_T; \mathbf{R}) \times \cdots \times C^{\eta_{r_1}}(I_T; \mathbf{R}), \\ &\left\| (f_0, f_1,\ldots,f_{r_1}) \right\|_{Z_{\delta,\eta_1,\ldots,\eta_{r_1}}} \\ &\quad = \|f_0\|_{\delta,0,T;X} + \|f_1\|_{\eta_1,0,T;\mathbf{R}} + \cdots + \|f_{r_1}\|_{\eta_{r_1},0,T;\mathbf{R}}, \end{aligned} \tag{253}$$

at least for opportunely chosen Hölder exponents $\delta \in (0,1)$ and $\eta_i \in (1/q', 1)$, $i = 1,\ldots,r_1$, and, eventually, sufficiently small values of $T > 0$. It is just in the choice of δ and the η_i's that the main result of Section 5 plays a key role. The Hölder exponents have to be chosen so that the direct problem (234) in which the k_i's are assumed to be known is well posed. Due to the shown equivalence between problems (234) and (238), the well-posedness of the direct problem (234) is then a consequence of Theorem 48 and formula (236). More precisely, recalling Remark 50 for the case $\alpha = 1$, an application of that theorem yields the following maximal time regularity result for the solution Θ of (234).

Theorem 57. *Let X, $\mathscr{D}(M)$, $\mathscr{D}(L)$, and $\mathscr{D}(L_i)$, $i = 1,\ldots,n_1$, $n_1 = r_1 + 2$, $r_1 \in \mathbf{N}$, be defined by* (225) *and* (226) *with $q \in (1,2)$. Let M, L, and L_i, $i = 1,\ldots,n_1$, be defined by* (227)–(229) *through* (209), (212), *and* (215)–(221), *and let* (230) *and* (231) *be satisfied. Further, let $(A, \mathscr{D}(A))$ be defined by* (233), *and let $Y^r_\psi \in \{(X,\mathscr{D}(A))_{\psi,r}, X^{\psi,r}_A\}$, $\psi \in (0,1)$, $r \in [1,\infty]$. Let $\eta_i \in (1/q', 1)$ and $\gamma_i, \varphi \in (2/q', 1)$, $i = 1,\ldots,n_1$, and assume that*

$$\begin{aligned} &k_i \in C^{\eta_i}(I_T; \mathbf{R}), \quad i = 1,\ldots,n_1, \qquad \Theta_0 \in \mathscr{D}(L), \\ &(\lambda_0 M + L)\Theta_0 + \widetilde{g}(0,\cdot) = Mv_0 \quad \textit{for some } v_0 \in \mathscr{D}(L), \\ &L_i \Theta_0 \in Y^r_{\gamma_i}, \qquad v_1 + D_t \widetilde{g}(0,\cdot) \in Y^r_{\varphi}, \quad i = 1,\ldots,n_1, \\ &\qquad\qquad r \in [1,\infty], \end{aligned} \tag{254}$$

where k_i, $i = 1, \ldots, n_1$, $\tilde{g}$ *and* λ_0 *are defined by* (222) *and* (223) *through* (211) *and* (216), *whereas* $v_1 = (\lambda_0 M + L)v_0$. *Let* $\gamma = \min_{i=1,\ldots,n_1}\{\gamma_i, \varphi\}$ *and* $\tau = \min_{i=1,\ldots,n_1}\{\eta_i, \gamma - 1/q'\}$, *and let* $I_{1,1/q,\tau} \subseteq (1/q', 1/2)$ *be the interval defined by (cf.* (207) *with* $\beta = 1/q$)

$$
\begin{aligned}
I_{1,1/q,\tau} &= \left(\frac{1}{q'}, \tau\right], \quad \text{if } \tau \in \left(\frac{1}{q'}, \frac{1}{2}\right), \\
I_{1,1/q,\tau} &= \left(\frac{1}{q'}, \frac{1}{2}\right), \quad \text{if } \tau \in \left[\frac{1}{2}, 1\right).
\end{aligned}
\tag{255}
$$

Then, for every fixed $\delta \in I_{1,1/q,\tau}$ *problem* (234), *or, equivalently, problem* (224), *admits a unique strict solution* $\Theta \in C^{1+\delta}(I_T; \mathscr{D}(L))$ *satisfying* $D_t\Theta(0) = v_0$ *and such that* $D_t M\Theta, L\Theta \in C^{1+\delta}(I_T; X)$, *provided that* $D_t\tilde{g} \in C^{\mu}(I_T; X)$, $\mu \in [\delta + 1/q', 1)$.

Proof. Apply Theorem 48 with $(i_1, i_2, n_2) = (i, i, n_1)$, $(\alpha, \beta, Z) = (1, 1/q, \mathbf{R})$, and $h_i = k_i$, $i = 1, \ldots, n_1$, to the equivalent problem (238). Since M is the multiplication operator by the function m_1 independent of t, the assertion then follows from $D_t\Theta = v \in C^{\delta}(I_T; \mathscr{D}(L))$, $D_t\Theta(0) = v(0)$, $D_tL\Theta = Lv \in C^{\delta}(I_T; X)$ and $D_t^2 M\Theta = D_t M v \in C^{\delta}(I_T; X)$. □

Larger values of q in Theorem 57 can be obtained assuming more smoothness and some order of vanishing for the function m_1. In fact, let $m_1 \in C^1(\overline{\Omega})$ be such that the following estimate holds for some positive constant K:

$$
|\nabla m_1(x)| := \left\{\sum_{j=1}^{N}\left[D_{x_j} m_1(x)\right]^2\right\}^{1/2} \le K[m_1(x)]^{\vartheta}, \qquad x \in \overline{\Omega}, \quad \vartheta \in (0,1). \tag{256}
$$

Then (232) holds with $\beta = 1/q$ being replaced by (cf. [41, formulae (3.23) and (4.41)]):

$$
\begin{aligned}
\beta &= \frac{1}{2-\vartheta}, \quad \text{if } q \in (2-\vartheta, 2), \\
\beta &= \frac{2}{q(2-\vartheta)}, \quad \text{if } q \in [2, \infty).
\end{aligned}
\tag{257}
$$

(precisely, in [41, formula (3.23)] it is shown that $(|\lambda| + 1)\|Mu\|_{q;\Omega}^{q(2-\vartheta)/2} \le C_q[\|f\|_{q;\Omega}\|Mu\|_{q;\Omega}^{-1+q(2-\vartheta)/2} + \|f\|_{q;\Omega}^{q(2-\vartheta)/2}]$, where $u = (\lambda M - L)^{-1} f$ and $q \in [2, \infty)$. Using (cf. [41, formula (2.15)]) $\|Mu\|_{q;\Omega} \le \|m_1\|_{\infty;\Omega}\|u\|_{q;\Omega} \le C\|m_1\|_{\infty;\Omega}\|f\|_{q;\Omega}$, we thus find that $(|\lambda| + 1)\|Mu\|_{q;\Omega}^{q(2-\vartheta)/2} \le C_q[(C\|m_1\|_{\infty;\Omega})^{-1+q(2-\vartheta)/2} + 1]\|f\|_{q;\Omega}^{q(2-\vartheta)/2}$; that is, $\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \le \{C_q[(C\|m_1\|_{\infty;\Omega})^{-1+q(2-\vartheta)/2} + 1]\}^{2/[q(2-\vartheta)]}(|\lambda| + 1)^{-2/[q(2-\vartheta)]}$). Under (256) we thus find the following better result, where q may be greater than two.

Theorem 58. *Let* (256) *holds, and let* X, $(M, \mathscr{D}(M))$, $(L, \mathscr{D}(L))$, $(L_i, \mathscr{D}(L_i))$, $i = 1, \ldots, n_1$, *be as in Theorem 57, but with* $q \in (2-\vartheta, 2) \cup [2, 4/(2-\vartheta))$. *Let* (254) *be fulfilled, but with* $\eta_i \in (1-\beta, 1)$ *and* $\gamma_i, \varphi \in (2-2\beta, 1)$, $i = 1, \ldots, n_1$, *where* β *is as in* (257). *Let* $\gamma = \min_{i=1,\ldots,n_1}\{\gamma_i, \varphi\}$ *and* $\tau = \min_{i=1,\ldots,n_1}\{\eta_i, \beta + \gamma - 1\}$, *and let* $I_{1,\beta,\tau}$ *be as in* (207). *Then, for every fixed* $\delta \in I_{1,\beta,\tau}$ *problem* (234), *or, equivalently, problem* (224), *admits a unique strict solution* $\Theta \in C^{1+\delta}(I_T; \mathscr{D}(L))$ *satisfying* $D_t\Theta(0) = v_0$ *and such that* $D_t M\Theta, L\Theta \in C^{1+\delta}(I_T; X)$, *provided that* $D_t\tilde{g} \in C^{\mu}(I_T; X)$, $\mu \in [\delta + 1 - \beta, 1)$.

Proof. It suffices to observe that for every $\vartheta \in (0,1)$ and $q \in (2-\vartheta, 2) \cup [2, 4/(2-\vartheta))$, the number β in (257) satisfies $\beta > 1/2$. Hence, proceeding as in the proofs of Theorem 57, except for replacing there $\beta = 1/q$ with β as in (257), we get the assertion. □

Appendix

Here we clarify why the definition of Q_2 in [20] has to be modified in accordance to that in this paper. To avoid confusion with the present notation, we will denote the operator Q_2 in [20] with S_2. Precisely, in [20, formula (4.12)], S_2 was defined as follows:

$$
[S_2 g_2](t) := \int_0^t \left[(-A)^1\right]^{\circ} \mathrm{e}^{(t-s)A} g_2(s)\,\mathrm{d}s, \quad t \in [0, T], \tag{A.1}
$$

and considered as acting on functions $g_2 \in C_0^{\delta_2}([0,T]; X)$, $\delta_2 \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $3\alpha + \beta > 3$. Even though $g_2(0) = 0$, formula (A.1) may have no sense, since

$$
\|S_2 g_2(t)\|_X \le \tilde{c}_{\alpha,\beta,1}|g_2|_{\delta_2,0,t;X}\int_0^t (t-s)^{(\beta-2)/\alpha} s^{\delta_2}\,\mathrm{d}s, \tag{A.2}
$$

and the integral on the right is *not* convergent, the exponent $(\beta-2)/\alpha$ being less or equal than -1. It is for this reason that $g_2(s)$ in (A.1) has to be replaced with the increment $g_2(s) - g_2(t)$ as in formula (106) (see inequality (118)) and to introduce the operator Q_5 as in (109). Of course, as a consequence, the definitions of Q_3 and Q_4 in [20, Lemmas 4.6 and 4.8] as $S_2\mathscr{K}(g_{3_1}, g_{3_2})$ and $S_2(g_4 y)$, respectively, have to be changed too in accordance with the present formulae (107) and (108) containing the increments $\mathscr{K}(g_{3_1}, g_{3_2})(s) - \mathscr{K}(g_{3_1}, g_{3_2})(t)$ and $[g_4(s) - g_4(t)]y$. To this purpose, we want to make clear that, contrarily to [20, Lemma 4.4], the statement and the proof of [20, Lemma 4.8] is correct, since there the function inside the integral on the right-hand side of (A.1) takes its values in an opportune intermediate space $X_A^{\theta,r}$. However, the correctness of that lemma does not suffice to proceed as in [20, Section 5] to solve problem (160) with $n_1 = n_2 = 1$.

For the reader's convenience we thus now indicate how to change the definitions of the functions w_j, $j = 0, 1$, and the operator Rw in [20, formulae (5.8)–(5.10)], and we state the amended version of [20, Theorems 5.6 and 5.7]. First, according to [20] where only this case was treated, let $n_1 = n_2 = 1$ in problem (160), and write k, h, y in place of k_1, h_1 and y_1, respectively. Then, under the same assumptions on the vector (α, β, k, h, f) as those in the present Section 5, it can be shown that problem (160) with $n_1 = n_2 = 1$ is equivalent to the fixed-point equation (179), where (cf. (180)–(182))

$$\begin{aligned} w_0 &= Q_7 x_0 + Q_6 (k, L_1 v_0) + Q_5 \tilde{h} + Q_5 \tilde{f}, \\ w_1 &= -Q_3 (k, L_1 v_0) - Q_4 (h, y) - Q_2 f, \\ Rw &:= \lambda_0 \left[Q_5 \left(A^{-1} w \right) - Q_2 \left(A^{-1} w \right) \right] \\ &\quad + Q_6 (k, Sw) - Q_3 (k, Sw). \end{aligned} \tag{A.3}$$

Here, $x_0 = v_1 + h(0)y + f(0)$, $v_1 = (\lambda_0 M + L)v_0$, is the value at $t = 0$ of the function F_w defined by (169) with $n_l = n_2 = 1$, $Q_7 x_0$, $\tilde{f}$ and $\tilde{h}$ are defined, respectively, by $(e^{tA} - I)x_0$, $f(t) - f(0)$ and $[h(t) - h(0)]y$, S is the operator $L_1 L^{-1} \in \mathscr{L}(X)$, and the Q_j's, $j = 2, \ldots, 6$, are as in (106)–(110). Formulae (A.3) replace the definitions of w_0, w_1 and Rw in [20, formulae (5.8)–(5.10)]. Therefore, from Lemmas 42, 45, and 46 and Corollary 44 with $n_1 = n_2 = 1$ we obtain the following version of Theorem 48.

Theorem A.1. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$, *and let* $5\alpha + 2\beta > 6$ *in (H2). Assume that* $k \in C^{\eta}(I_T; Z)$, $h \in C^{\sigma}(I_T; \mathbf{C})$, $y \in Y_{\theta}^{r}$, *and* $(\lambda_0 M + L)v_0 + f(0) \in Y_{\varphi}^{r}$, *where* $\eta, \sigma \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $\theta, \varphi \in (5 - 3\alpha - 2\beta, 1)$, *and* $r \in [1, \infty]$. *Let* $\gamma = \min\{\theta, \varphi\}$ *and* $\tau = \min\{\eta, \sigma, (\alpha + \beta + \gamma - 2)/\alpha\}$. *Then, for every fixed* $\delta \in I_{\alpha,\beta,\tau}$ *the problem*

$$\begin{aligned} D_t (Mv(t)) &= [\lambda_0 M + L] v(t) + \mathscr{K}(k, L_1 v)(t) \\ &\quad + h(t) y + f(t), \quad t \in I_T, \\ Mv(0) &= Mv_0 \end{aligned} \tag{A.4}$$

admits a unique strict solution $v \in C^{\delta}(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_t Mv \in C^{\delta}(I_T; X)$, *provided that* $f \in C^{\mu}(I_T; X)$, $\mu \in [\delta + (3 - 2\alpha - \beta)/\alpha, 1)$.

Theorem A.1 substitutes [20, Theorem 5.6 and 5.7]. Notice that, differently than [20], here only one statement occurs. In fact, the more suitable procedure followed in this paper makes the separation in [20] of two distinct intervals in which γ may vary totally unneeded. Finally, letting $n_1 = n_2 = 1$ in Theorems 52, 5.14, 54, and 56, we obtain the correct versions of [20, Theorems 5.11, 53, and 5.16] for the subcases of (A.4) corresponding to the choices $\lambda_0 = h = 0$, $\lambda_0 = f = 0$, $\lambda_0 = k = h = 0$, and $\lambda_0 = k = f = 0$, respectively. For saving space, we leave this easy task to the reader.

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