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Research Article

Multivariate Padé Approximation for Solving Nonlinear Partial Differential Equations of Fractional Order

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Two tecHniques were implemented, the Adomian decomposition method (ADM) and multivariate Padé approximation (MPA), for solving nonlinear partial differential equations of fractional order. The fractional derivatives are described in Caputo sense. First, the fractional differential equation has been solved and converted to power series by Adomian decomposition method (ADM), then power series solution of fractional differential equation was put into multivariate Padé series. Finally, numerical results were compared and presented in tables and figures.

1. Introduction

Recently, differential equations of fractional order have gained much interest in engineering, physics, chemistry, and other sciences. It can be said that the fractional derivative has drawn much attention due to its wide application in engineering physics [1–9]. Some approximations and numerical techniques have been used to provide an analytical approximation to linear and nonlinear differential equations and fractional differential equations. Among them, the variational iteration method, homotopy perturbation method [10, 11], and the Adomian decomposition method are relatively new approaches [5–9, 12, 13].

The decomposition method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [12, 13]. Recently, the application of the method is extended for fractional differential equations [6–9, 14].

Many definitions and theorems have been developed for multivariate Padé approximations MPA (see [15] for a survey on multivariate Padé approximation). The multivariate Padé Approximation has been used to obtain approximate solutions of linear or nonlinear differential equations [16–19]. Recently, the application of the unvariate Padé approximation is extended for fractional differential equations [20, 21].

The objective of the present paper is to provide approximate solutions for initial value problems of nonlinear partial differential equations of fractional order by using multivariate Padé approximation.

2. Definitions

For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order, which is the case in most physical processes. The definitions can be seen in [3, 4, 22, 23].

3. Decomposition Method [24]

Consider

$$D^{\alpha}_{*t} u(x,t) = f(u, u_x, u_{xx}) + g(x,t), \quad m-1 < \alpha \le m. \tag{1}$$

The decomposition method requires that a nonlinear fractional differential equation (1) is expressed in terms of operator form as

$$D_{*t}^{\alpha} u(x,t) + Lu(x,t) + Nu(x,t) = g(x,t), \quad x > 0, \tag{2}$$

where L is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operator which also might include other fractional derivatives of order less than α, $D_{*t}^{\alpha} = \partial^{\alpha}/\partial t^{\alpha}$ is the Caputo fractional derivative of order α, and $g(x,t)$ is the source function [24].

Applying the operator J^{α} [3, 4, 22, 23], the inverse of the operator D_{*t}^{α}, to both sides of (5) Odibat and Momani [24] obtained

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha} g(x,t) \\ &\quad - J^{\alpha}[Lu(x,t) + Nu(x,t)], \\ \sum_{n=0}^{\infty} u_n(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha} g(x,t) \\ &\quad - J^{\alpha}\left[L\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right]. \end{aligned} \tag{3}$$

From this, the iterates are determined in [24] by the following recursive way:

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} + J^{\alpha} g(x,t), \\ u_1(x,t) &= -J^{\alpha}(Lu_0 + A_0), \\ u_2(x,t) &= -J^{\alpha}(Lu_1 + A_1), \\ &\vdots \\ u_{n+1}(x,t) &= -J^{\alpha}(Lu_n + A_n). \end{aligned} \tag{4}$$

4. Multivariate Padé Aproximation [25]

Consider the bivariate function $f(x,y)$ with Taylor series development

$$f(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{5}$$

around the origin. We know that a solution of unvariate Padé approximation problem for

$$f(x) = \sum_{i=0}^{\infty} c_i x^i \tag{6}$$

is given by

$$p(x) = \begin{vmatrix} \sum_{i=0}^{m} c_i x^i & x\sum_{i=0}^{m-1} c_i x^i & \dots & x^n \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1} & c_m & \dots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \dots & c_m \end{vmatrix}, \tag{7}$$

$$q(x) = \begin{vmatrix} 1 & x & \dots & x^n \\ c_{m+1} & c_m & \dots & c_{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n} & c_{m+n-1} & \dots & c_m \end{vmatrix}. \tag{8}$$

Let us now multiply jth row in $p(x)$ and $q(x)$ by x^{j+m-1} $(j = 2,\dots,n+1)$ and afterwards divide jth column in $p(x)$ and $q(x)$ by x^{j-1} $(j = 2,\dots,n+1)$. This results in a multiplication of numerator and denominator by x^{mn}. Having done so, we get

$$\frac{p(x)}{q(x)} = \frac{\begin{vmatrix} \sum_{i=0}^{m} c_i x^i & \sum_{i=0}^{m-1} c_i x^i & \cdots & \sum_{i=0}^{m-n} c_i x^i \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ c_{m+1}x^{m+1} & c_m x^m & \cdots & c_{m+1-n}x^{m+1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+n}x^{m+n} & c_{m+n-1}x^{m+n-1} & \cdots & c_m x^m \end{vmatrix}} \tag{9}$$

if $(D = \det D_{m,n} \neq 0)$.

This quotent of determinants can also immediately be written down for a bivariate function $f(x,y)$. The sum $\sum_{i=0}^{k} c_i x^i$ shall be replaced by kth partial sum of the Taylor series development of $f(x,y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree k in $f(x,y)$. Here a bivariate term $c_{ij}x^i y^j$ is said to be of degree $i+j$. If we define

$$\begin{aligned} &p(x,y) \\ &= \begin{vmatrix} \sum_{i+j=0}^{m} c_{ij}x^i y^j & \sum_{i+j=0}^{m-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=0}^{m-n} c_{ij}x^i y^j \\ \sum_{i+j=m+1} c_{ij}x^i y^j & \sum_{i+j=m} c_{ij}x^i y^j & \cdots & \sum_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i+j=m+n} c_{ij}x^i y^j & \sum_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum_{i+j=m} c_{ij}x^i y^j \end{vmatrix}, \end{aligned}$$

$$q(x,y) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum\limits_{i+j=m+1} c_{ij}x^i y^j & \sum\limits_{i+j=m} c_{ij}x^i y^j & \cdots & \sum\limits_{i+j=m+1-n} c_{ij}x^i y^j \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i+j=m+n} c_{ij}x^i y^j & \sum\limits_{i+j=m+n-1} c_{ij}x^i y^j & \cdots & \sum\limits_{i+j=m} c_{ij}x^i y^j \end{vmatrix}. \tag{10}$$

Then it is easy to see that $p(x,y)$ and $q(x,y)$ are of the form

$$p(x,y) = \sum_{i+j=mn}^{mn+m} a_{ij}x^i y^j, \qquad q(x,y) = \sum_{i+j=mn}^{mn+n} b_{ij}x^i y^j. \tag{11}$$

We know that $p(x,y)$ and $q(x,y)$ are called Padé equations [25]. So the multivariate Padé approximant of order (m,n) for $f(x,y)$ is defined as

$$r_{m,n}(x,y) = \frac{p(x,y)}{q(x,y)}. \tag{12}$$

5. Numerical Experiments

In this section, two methods, ADM and MPA, shall be illustrated by two examples. All the results are calculated by using the software Maple12. The full ADM solutions of examples can be seen from Odibat and Momani [24].

Example 1. Consider the nonlinear time-fractional advection partial differential equation [24]

$$D_{*t}^{\alpha} u(x,t) + u(x,t)\, u_x(x,t) = x + xt^2, \quad t > 0,\ x \in R,\ 0 < \alpha \le 1, \tag{13}$$

subject to the initial condition

$$u(x,0) = 0. \tag{14}$$

Odibat and Momani [24] solved the problem using the decomposition method, and they obtained the following recurrence relation [24]:

$$\begin{aligned} u_0(x,t) &= u(x,0) + J^{\alpha}\left(x + xt^2\right) \\ &= x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_{j+1}(x,t) &= -J^{\alpha}\left(A_j\right), \quad j \ge 0, \end{aligned} \tag{15}$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (15), the first few components of the decomposition series are derived in [24] as follows:

$$\begin{aligned} u_0(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_1(x,t) &= -x\left(\frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{4\Gamma(2\alpha+3)\,t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)\,t^{3\alpha+4}}{\Gamma(\alpha+3)^2\Gamma(3\alpha+5)}\right), \\ u_2(x,t) &= 2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\,t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)\,t^{5\alpha+6}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)} + \cdots\right), \end{aligned} \tag{16}$$

and so on; in this manner, the rest of components of the decomposition series can be obtained [24].

The first three terms of the decomposition series are given by [24]

$$\begin{aligned} u(x,t) &= x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \right. \\ &\quad \left. - \frac{4\Gamma(2\alpha+3)\,t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \cdots\right). \end{aligned} \tag{17}$$

For $\alpha = 1$ (16) is

$$u(x,t) = xt + 0.1 \times 10^{-9}t^3 - 0.1333333333xt^5. \tag{18}$$

Now, let us calculate the approximate solution of (18) for $m = 4$ and $n = 2$ by using Multivariate Padé approximation. To obtain multivariate Padé equations of (18) for $m = 4$ and $n = 2$, we use (10). By using (10), we obtain

$$\begin{aligned} p(x,t) &= \begin{vmatrix} xt + 0.1 \times 10^{-9}t^3 & xt & xt \\ 0 & 0.1 \times 10^{-9}t^3 & 0 \\ -0.1333333333xt^5 & 0 & 0.1 \times 10^{-9}t^3 \end{vmatrix} \\ &= 0.1333333333 \times 10^{-10} \\ &\quad \times \left(t^2 + 0.7500000002 \times 10^{-9}\right)x^3t^7, \end{aligned}$$

$$q(x,t) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.1\times 10^{-9}t^3 & 0 \\ -0.1333333333xt^5 & 0 & 0.1\times 10^{-9}t^3 \end{vmatrix} = 0.1333333333\times 10^{-10} \times \left(t^2 + 0.7500000002\times 10^{-9}\right)x^2t^6. \tag{19}$$

So, the multivariate Padé approximation of order $(4,2)$ for (17), that is,

$$[4,2]_{(x,t)} = \frac{\left(t^2 + 0.7500000002\times 10^{-9}\right)xt}{t^2 + 0.7500000002\times 10^{-9}}. \tag{20}$$

For $\alpha = 0.5$ (17) is

$$u(x,t) = 1.128379167xt^{0.5} - 0.9577979850xt^{1.5} + 0.6018022226xt^{2.5} - 0.7005608116xt^{3.5}. \tag{21}$$

For simplicity, let $t^{1/2} = a$; then

$$u(x,a) = 1.128379167xa - 0.9577979850xa^3 + 0.6018022226xa^5 - 0.7005608116xa^7. \tag{22}$$

Using (10) to calculate the multivariate Padé equations for (22) we get

$$p(x,a) = \begin{vmatrix} 1.128379167xa - 0.9577979850xa^3 + 0.6018022226xa^5 & 1.128379167xa - 0.9577979850xa^3 & 1.128379167xa - 0.9577979850xa^3 \\ 0 & 0.6018022226xa^5 & 0 \\ -0.7005608116xa^7 & 0 & 0.6018022226xa^5 \end{vmatrix}$$
$$= -0.4907854507\left(1.201464294a^4 - 1.231832347a^2 - 0.8326662354\right)x^3a^{11},$$
$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0.6018022226xa^5 & 0 \\ -0.7005608116xa^7 & 0 & 0.6018022226xa^5 \end{vmatrix} = 0.4907854507\left(0.7379312378 + 1.718058483a^2\right)x^3a^{10} \tag{23}$$

recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(6,2)$ for (21), that is,

$$[6,2]_{(x,t)} = -\left(1.201464294t^2 - 1.231832347t - 0.8326662354\right)x\sqrt{t} \times \left(0.7379312378 + 1.718058483t\right)^{-1}. \tag{24}$$

For $\alpha = 0.75$ (17) is

$$u(x,t) = 0.00007125345441xt^{7.5} + 0.1764791440\times 10^{-5}xt^{9.5} - 0.1238343301\times 10^{-17}xt^{22.5} - 0.2897967272\times 10^{-19}xt^{24.5}. \tag{25}$$

For simplicity, let $t^{1/2} = a$; then

$$u(x,a) = 0.00007125345441xa^{15} + 0.1764791440\times 10^{-5}xa^{19} - 0.1238343301\times 10^{-17}xa^{45} - 0.2897967272\times 10^{-19}xa^{49}. \tag{26}$$

Using (10) to calculate the multivariate Padé equations and then recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(49,2)$ for (25), that is,

$$[49,2]_{(x,t)} = -0.8398214310\times 10^{-39}x^3t^{113/2} \times \left(-0.00007125345441 - 0.1764791440 \times 10^{-5}t^2 + 0.1238343301\times 10^{-17}t^{15}\right) \times \left(0.8398214310\times 10^{-39}x^2t^{49}\right)^{-1}. \tag{27}$$

Table 1, Figures 1(a), 1(b), 1(c), 2(a), 2(b), 2(c), and 2(d) shows the approximate solutions for (13) obtained for different values of α using the decomposition method (ADM) and the multivariate Padé approximation (MPA). The value of $\alpha = 1$ is for the exact solution $u(x,t) = xt$ [24].

Example 2. Consider the nonlinear time-fractional hyperbolic equation [24]

$$D^{\alpha}_{*t}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t > 0,\ x\in R,\ 1 < \alpha \le 2, \tag{28}$$

subject to the initial condition

$$u(x,0) = x^2, \qquad u_t(x,0) = -2x^2. \tag{29}$$

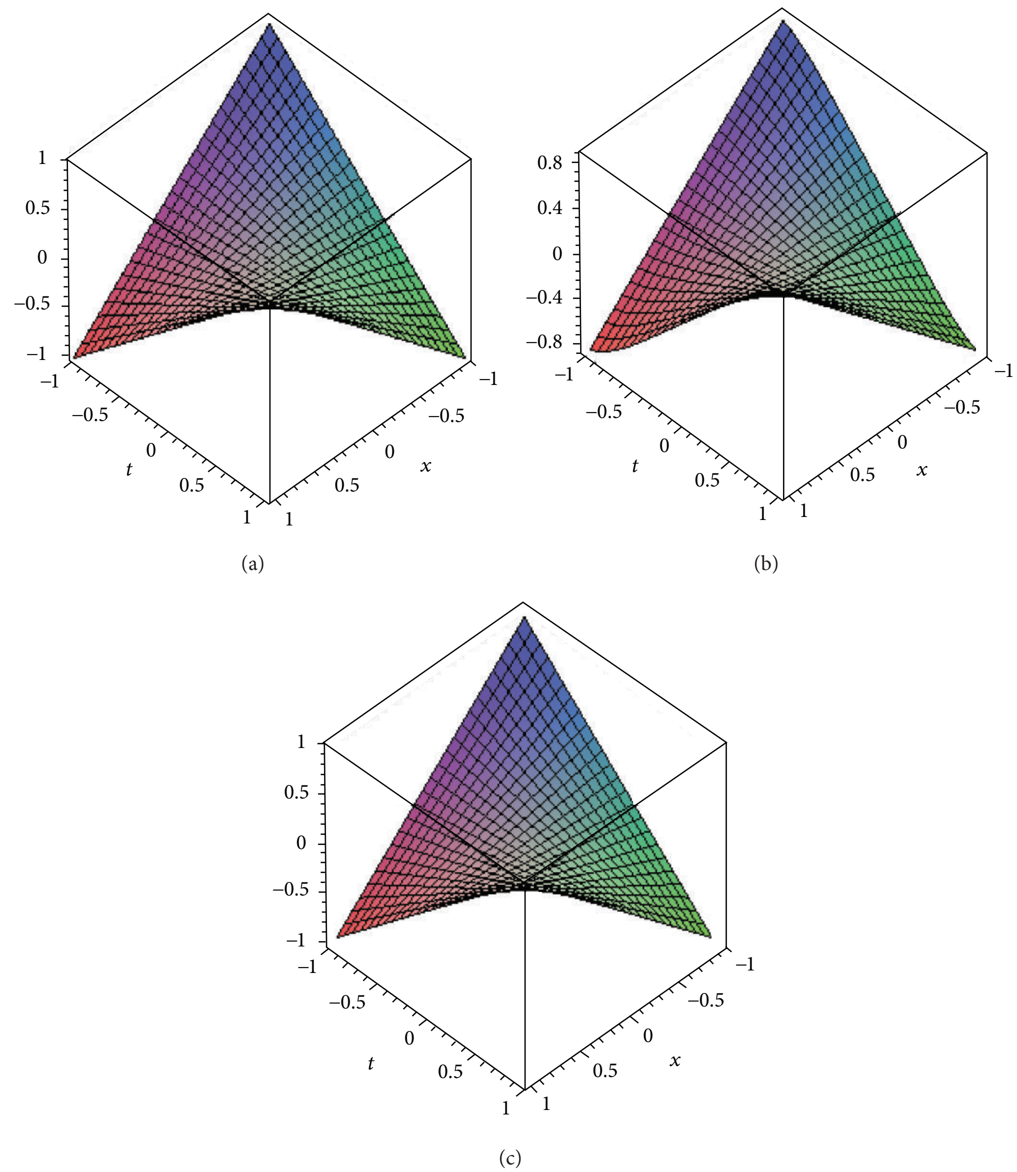


Figure 1: (a) Exact solution of Example 1 for $\alpha = 1$ (b) ADM solution of Example 1 for $\alpha = 1$ (c) Multivariate Padé approximation of ADM solution for $\alpha = 1$ in Example 1.

Odibat and Momani [24] solved the problem using the decomposition method, and they obtained the following recurrence relation in [24]:

$$\begin{gathered} u_0(x,t) = u(x,0) + t u_x(x,0) = x^2(1-2t), \\ u_{j+1}(x,t) = J^{\alpha}\left(A_j\right)_x, \quad j \geq 0, \end{gathered} \tag{30}$$

where A_j are the Adomian polynomials for the nonlinear function $N = uu_x$. In view of (30), the first few components of the decomposition series are derived in [24] as follows:

$$\begin{aligned} u_0(x,t) &= x^2(1-2t), \\ u_1(x,t) &= 6x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right), \\ u_2(x,t) &= 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{2\Gamma(\alpha+2)\,t^{2\alpha+1}}{\Gamma(\alpha+1)\,\Gamma(2\alpha+2)}\right) \\ &\quad + 72x^2\left(\frac{8\Gamma(\alpha+3)\,t^{2\alpha+2}}{\Gamma(\alpha+2)\,\Gamma(2\alpha+3)} - \frac{16\Gamma(\alpha+4)\,t^{2\alpha+3}}{\Gamma(\alpha+3)\,\Gamma(2\alpha+4)}\right), \\ &\;\;\vdots \end{aligned} \tag{31}$$

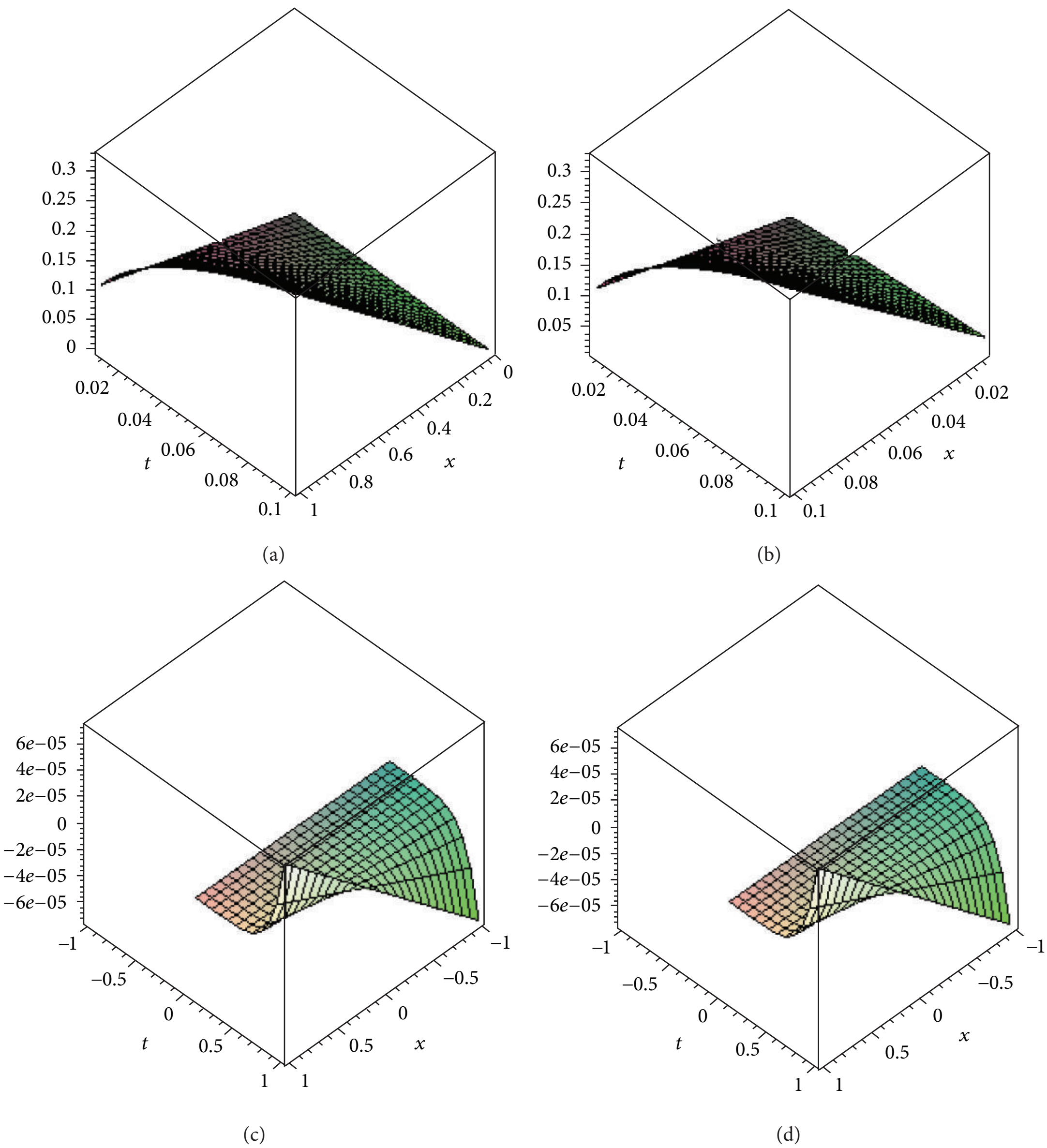


Figure 2: (a) ADM solution of Example 1 for $\alpha = 0.5$ (b) Multivariate Padé approximation of ADM solution for $\alpha = 0.5$ in Example 1 (c) ADM solution of Example 1 for $\alpha = 0.75$ (d) Multivariate Padé approximation of ADM solution for $\alpha = 0.75$ in Example 1.

and so on; in this manner the rest of components of the decomposition series can be obtained.

The first three terms of the decomposition series (7) are given [24] by

$$u(x,t) = x^2(1-2t) + 6x^2 \times \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\right) + 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right). \tag{32}$$

For $\alpha = 2$ (43) is

$$u(x,t) = x^2(1-2t) + 6x^2\left(0.5000000000t^2 - 0.6666666668t^3 + 0.3333333334t^4\right) + 3.000000000x^2t^4. \tag{33}$$

Now, let us calculate the approximate solution of (33) for $m = 4$ and $n = 2$ by using multivariate Padé approximation. To

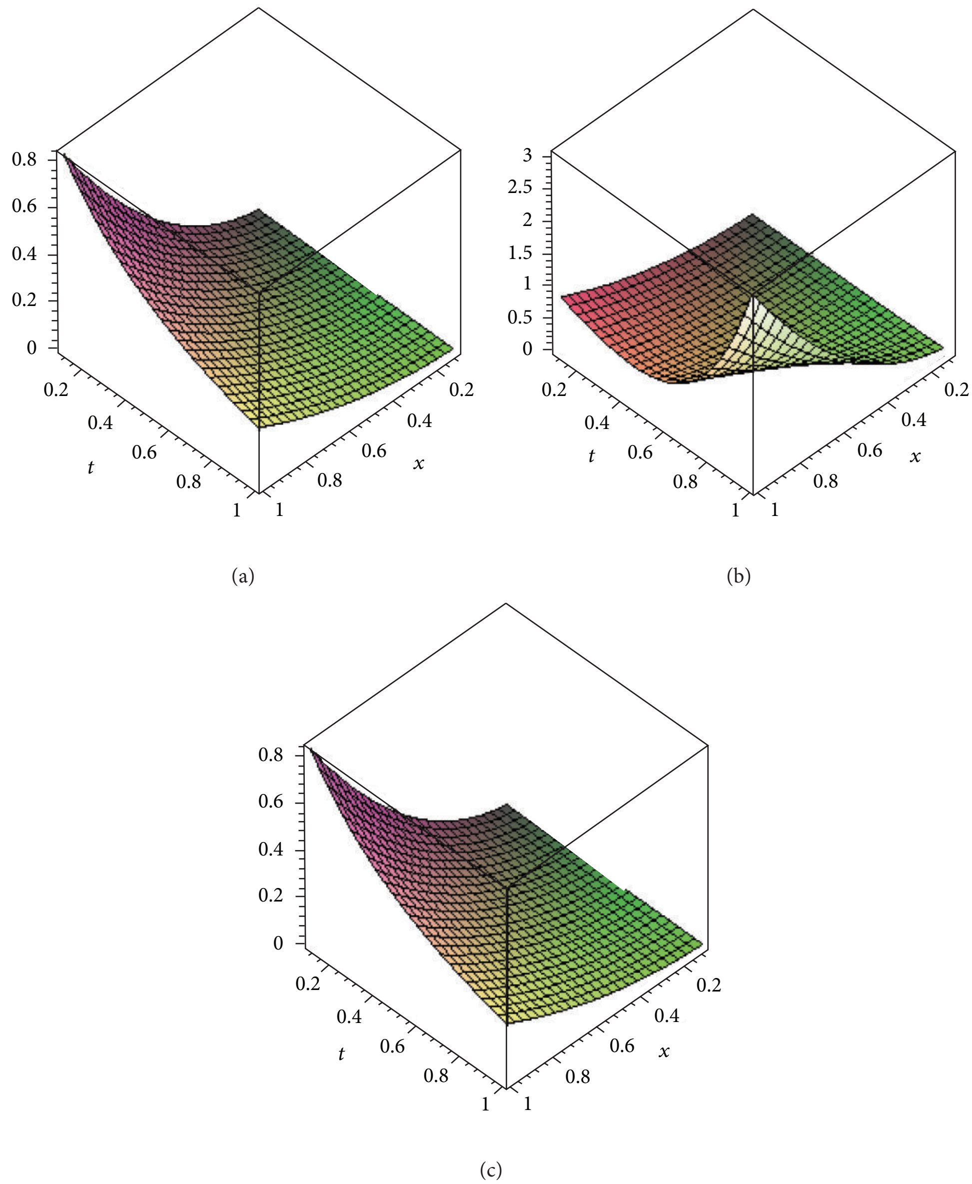


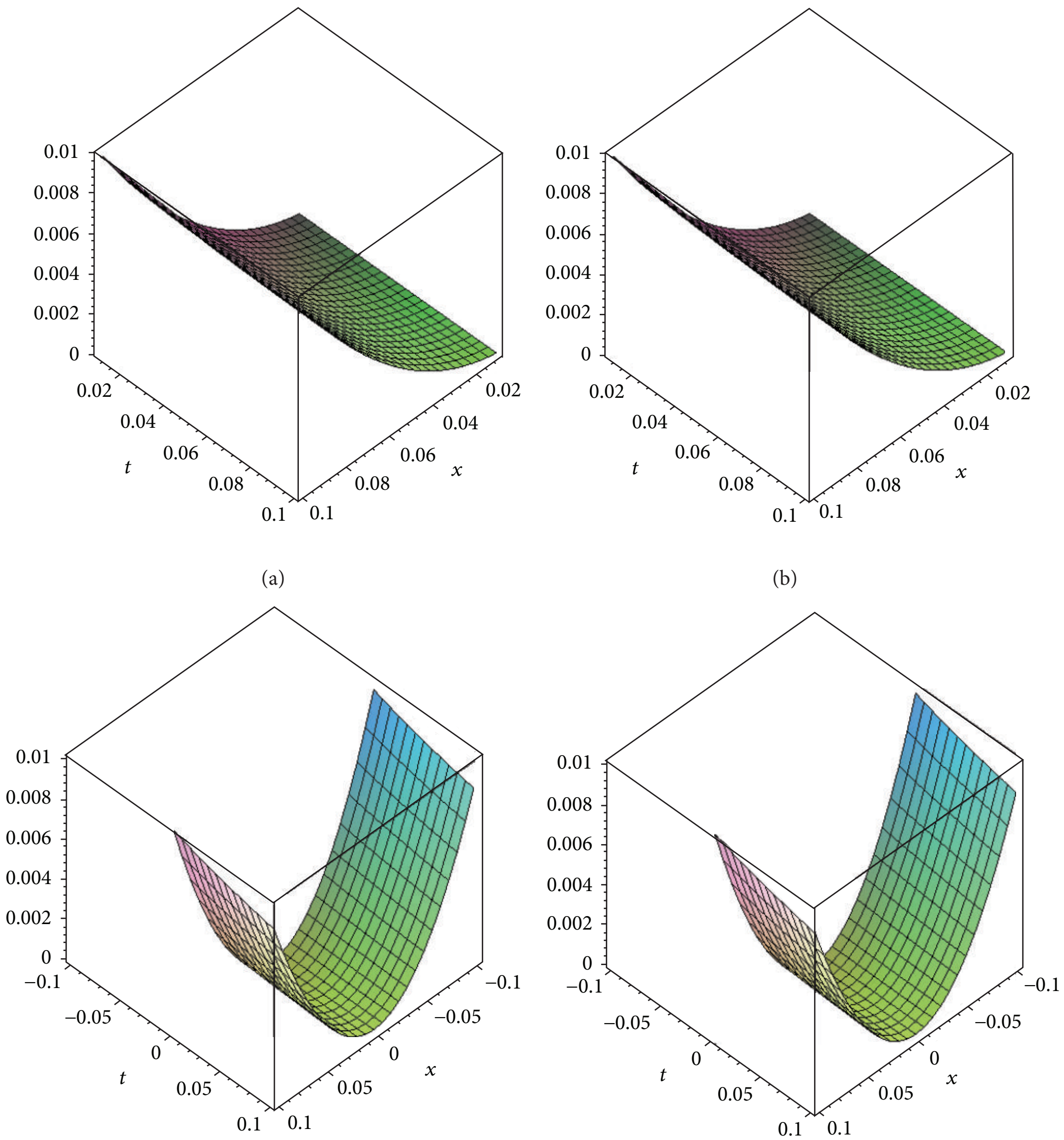
FIGURE 3: (a) Exact solution of Example 2 for $\alpha = 2.0$ (b) ADM solution of Example 2 for $\alpha = 2.0$ (c) Multivariate Padé approximation of ADM solution for $\alpha = 2.0$ in Example 2.

obtain multivariate Padé equations of (33) for $m = 4$ and $n = 2$, we use (10). By using (10), we obtain

$$\begin{aligned}
&p(x,t) \\
&= \begin{vmatrix} x^2(1-2t) + 3.000000000x^2t^2 & x^2(1-2t) & x^2 \\ -4.000000001x^2t^3 & 3.000000000x^2t^2 & -2x^2t \\ 5.000000000x^2t^4 & -4.000000001x^2t^3 & 3.000000000x^2t^2 \end{vmatrix} \\
&= -20.00000000t^4\left(0.28\times10^{-9}t^2 - 0.34\times10^{-9}t\ (-0.04999999986\right)x^6, \\
&q(x,t) \\
&= \begin{vmatrix} 1 & 1 & 1 \\ -4.000000001x^2t^3 & 3.000000000x^2t^2 & -2x^2t \\ 5.000000000x^2t^4 & -4.000000001x^2t^3 & 3.000000000x^2t^2 \end{vmatrix} \\
&= 20.00000000t^4\left(0.0499999999 + 0.1000000001t + 0.0500000004t^2\right)x^4.
\end{aligned} \tag{34}$$

So, the multivariate Padé approximation of order $(4, 2)$ for (33), that is,

$$\begin{aligned}
[4,2]_{(x,t)} = &-1.000000000\left(0.28\times10^{-9}t^2 - 0.34\times10^{-9}t\right. \\
&\left.-0.04999999986\right)x^2 \\
&\times\left(0.0499999999 + 0.1000000001t\right. \\
&\left.+0.0500000004t^2\right)^{-1}.
\end{aligned} \tag{35}$$



(a) (b) (c) (d)

Figure 4: (a) ADM solution of Example 2 for $\alpha = 1.5$ (b) Multivariate Padé approximation of ADM solution for $\alpha = 1.5$ in Example 2 (c) ADM solution of Example 2 for $\alpha = 1.75$ (d) Multivariate Padé approximation of ADM solution for $\alpha = 1.75$ in Example 2.

For $\alpha = 1.5$ (32) is

$$\begin{aligned} u(x,t) = {} & x^2(1-2t) + 6x^2 \\ & \times \left(0.7522527782t^{1.5} - 1.203604445t^{2.5} \right. \\ & \left. +0.6877739683t^{3.5}\right) + 12.00000000x^2t^{3.0}. \end{aligned} \tag{36}$$

For simplicity, let $t^{1/2} = a$; then

$$\begin{aligned} u(x,a) = {} & x^2\left(1-2a^2\right) + 6x^2 \\ & \times \left(0.7522527782a^3 \right. \\ & \left. -1.203604445a^5 + 0.6877739683t^7\right) \\ & + 12.00000000x^2a^6. \\ = {} & x^2 - x^2 2a^2 + 4.513516669x^2a^3 \\ & - 7.221626670x^2a^5 + 4.126643810x^2a^7 \\ & + 12.00000000x^2a^6. \end{aligned} \tag{37}$$

Using (10) to calculate multivariate Padé equations of (37) for $m = 7$ and $n = 2$, we use (10). By using (10), we obtain

Table 1: Numerical values when $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1.0$ for (13).

x	t	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		
		u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{Exact}
0.01	0.01	0.001118860666	0.001118859998	$0.7125363089 \times 10^{-21}$	$0.7125363089 \times 10^{-21}$	0.00009999999987	0.0001000000000	0.0001
0.02	0.02	0.003138022015	0.003138007574	$0.2579676158 \times 10^{-18}$	$0.2579676148 \times 10^{-18}$	0.0003999999915	0.0004000000000	0.0004
0.03	0.03	0.005716640278	0.005716554826	$0.8097412617 \times 10^{-17}$	$0.8097412644 \times 10^{-17}$	0.0008999999028	0.0009000000000	0.0009
0.04	0.04	0.008727882362	0.008727584792	$0.9339702880 \times 10^{-16}$	$0.9339702881 \times 10^{-16}$	0.001599999454	0.001600000000	0.0016
0.05	0.05	0.01209607907	0.01209530419	$0.6224118481 \times 10^{-15}$	$0.6224118464 \times 10^{-15}$	0.002499997917	0.002500000000	0.0025
0.06	0.06	0.01576873408	0.01576705530	$0.2931772549 \times 10^{-14}$	$0.2931772552 \times 10^{-14}$	0.003599993779	0.003600000000	0.0036
0.07	0.07	0.01970633078	0.01970312778	$0.1086905954 \times 10^{-13}$	$0.1086905954 \times 10^{-13}$	0.004899984313	0.004900000000	0.0049
0.08	0.08	0.02387754051	0.02387197301	$0.3381735603 \times 10^{-13}$	$0.3381735608 \times 10^{-13}$	0.006399965047	0.006400000000	0.0064
0.09	0.09	0.02825661342	0.02824760190	$0.9203528751 \times 10^{-13}$	$0.9203528751 \times 10^{-13}$	0.008099929141	0.008100000000	0.0081
0.1	0.1	0.03282181204	0.03280802591	$0.2253790147 \times 10^{-12}$	$0.2253790145 \times 10^{-12}$	0.009999866667	0.010000000000	0.01

Table 2: Numerical values when $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2.0$ for (28).

x	t	$\alpha = 1.5$		$\alpha = 1.75$		$\alpha = 2.0$		
		u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{ADM}	u_{MPA}	u_{Exact}
0.01	0.01	0.00009844537131	0.00009844533110	0.00009811632420	0.00009811625316	0.00009802960500	0.00009802960486	0.00009802960494
0.02	0.02	0.0003889833219	0.0003889809228	0.0003855443171	0.0003855410232	0.0003844675200	0.0003844675123	0.0003844675125
0.03	0.03	0.0008664034309	0.0008663775704	0.0008529749012	0.0008529437792	0.0008483364450	0.0008483363176	0.0008483363182
0.04	0.04	0.001527388854	0.001527250241	0.001492265504	0.001492112228	0.001479290880	0.001479289940	0.001479289941
0.05	0.05	0.002370102454	0.002369595331	0.002296248891	0.002295720729	0.002267578126	0.002267573695	0.002267573696
0.06	0.06	0.003393997434	0.003392539471	0.003258613123	0.003257161294	0.003204002880	0.003203987182	0.003203987184
0.07	0.07	0.004599726730	0.004596176215	0.004373819216	0.004370404791	0.004279895445	0.004279849765	0.004279849769
0.08	0.08	0.005989109024	0.005981449991	0.005637043845	0.005629880400	0.005487083520	0.005486968446	0.005486968450
0.09	0.09	0.007565131844	0.007550068937	0.007044140677	0.007030367394	0.006817867605	0.006817607941	0.006817607945
0.1	0.1	0.009331981000	0.009304436794	0.008591616573	0.008566895894	0.008265000000	0.008264462804	0.008264462810

$$p(x,a) = \begin{vmatrix} x^2 - x^2 2a^2 + 4.513516669x^2a^3 - 7.221626670x^2a^5 & x^2 - x^2 2a^2 + 4.513516669x^2a^3 & x^2 - x^2 2a^2 + 4.513516669x^2a^3 \\ 12.00000000x^2t^6 & -7.221626670x^2a^5 & 0 \\ 4.126643810x^2a^7 & 12.00000000x^2a^6 & -7.221626670x^2a \end{vmatrix}$$

$$= 49.51972572\left(8.235760151a^5 + 0.879185531a^4 + 1.253427636a^3 + 1.403426542a^2 + 1.750000001a + 1.053153890\right)x^6a^{10},$$

$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 12.00000000x^2a^6 & -7.221626670x^2a^5 & 0 \\ 4.126643810x^2a^7 & 12.00000000x^2a^6 & -7.221626670x^2a \end{vmatrix}$$

$$= 49.51972572\left(1.053153890 + 1.750000001a + 3.509734322a^2\right)x^4a^{10}, \tag{38}$$

recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(7, 2)$ for (36), that is,

$$\begin{aligned}[7,2]_{(x,t)} = &\left(8.235760151t^{5/2} + 0.879185531t^2 \right.\\ &+ 1.253427636t^{3/2} + 1.403426542t \\ &\left. +1.750000001\sqrt{t} + 1.053153890\right)x^2 \\ &\times\left(1.053153890 + 1.750000001\sqrt{t}\right.\\ &\left. +3.509734322t\right)^{-1}.\end{aligned} \tag{39}$$

For $\alpha = 1.75$ (32) is

$$\begin{aligned}u(x,t) = {}& x^2(1-2t) \\ &+ 6x^2\left(0.6217515726t^{1.75} - 0.9043659240t^{2.75}\right.\\ &\left. +0.4823284927t^{3.75}\right) \\ &+ 6.189965715x^2t^{3.5}.\end{aligned} \tag{40}$$

For simplicity, let $t^{1/4} = a$; then

$$\begin{aligned}u(x,a) = {}& x^2\left(1-2a^4\right) \\ &+ 6x^2\left(0.6217515726a^7 - 0.9043659240a^{11}\right.\\ &\left. +0.4823284927a^{15}\right) + 6.189965715x^2a^{14} \\ = {}& x^2 - 2x^2a^4 + 3.730509436x^2a^7 \\ &- 5.426195544x^2a^{11} + 2.893970956x^2a^{15} \\ &+ 6.189965715x^2a^{14}.\end{aligned} \tag{41}$$

Using (10) to calculate multivariate Padé equations of (41) for $m = 15$ and $n = 2$, we use (10). By using (10), we obtain

$$p(x,a) = \begin{vmatrix} x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} & x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} & x^2 - 2x^2a^4 + 3.730509436x^2a^7 - 5.426195544x^2a^{11} \\ 6.189965715x^2a^{14} & 0 & 0 \\ 2.893970956x^2a^{15} & 6.189965715x^2a^{14} & 0 \end{vmatrix}$$

$$= -38.31567556x^6a^{28}\left(-1 + 2a^4 - 3.730509436a^7 + 5.426195544a^{11}\right),$$

$$q(x,a) = \begin{vmatrix} 1 & 1 & 1 \\ 6.189965715x^2a^{14} & 0 & 0 \\ 2.893970956x^2a^{15} & 6.189965715x^2a^{14} & 0 \end{vmatrix} = 38.31567556x^4a^{28}, \tag{42}$$

recalling that $t^{1/4} = a$, we get multivariate Padé approximation of order $(15, 2)$ for (40), that is,

$$\begin{aligned}[15,2]_{(x,t)} = {}& -38.31567556x^6t^7 \\ &\times\left(-1 + 2t - 3.730509436t^{7/4}\right.\\ &\left. +5.426195544t^{11/4}\right) \\ &\times\left(38.31567556x^4t^7\right)^{-1}.\end{aligned} \tag{43}$$

Table 2, Figures 3(a), 3(b), 3(c), 4(a), 4(b), 4(c), and 4(d) show the approximate solutions for (28) obtained for different values of α using the decomposition method (ADM) and the multivariate Padé approximation (MPA). The value of $\alpha = 2$ is for the exact solution $u(x,t) = (x/t+1)^2$ [24].

6. Concluding Remarks

The fundamental goal of this paper has been to construct an approximate solution of nonlinear partial differential equations of fractional order by using multivariate Padé approximation. The goal has been achieved by using the multivariate Padé approximation and comparing with the Adomian decomposition method. The present work shows the validity and great potential of the multivariate Padé approximation for solving nonlinear partial differential equations of fractional order from the numerical results. Numerical results obtained using the multivariate Padé approximation and the Adomian decomposition method are in agreement with exact solutions.

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