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Research Article

Approximate Euler-Lagrange Quadratic Mappings in Fuzzy Banach Spaces

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We consider general solution and the generalized Hyers-Ulam stability of an Euler-Lagrange quadratic functional equation $f(rx + sy) + rsf(x - y) = (r + s)[rf(x) + sf(y)]$ in fuzzy Banach spaces, where r, s are nonzero rational numbers with $r^2 + rs + s^2 - 1 \neq 0$, $r + s \neq 0$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for additive mappings on Banach spaces. Hyers's theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. Cholewa [6] noticed that the theorem of F. Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation. In particular, Rassias investigated the Hyers-Ulam stability for the relative Euler-Lagrange functional equation

$$f(ax+by) + f(bx-ay) = \left(a^2+b^2\right)[f(x)+f(y)] \tag{2}$$

in [8–10]. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [11–14]).

The theory of fuzzy space has much progressed as the theory of randomness has developed. Some mathematicians have defined fuzzy norms on a vector space from various points of view [15–19]. Following Cheng and Mordeson [20] and Bag and Samanta [15] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [21] and investigated some properties of fuzzy normed spaces [22].

We use the definition of fuzzy normed spaces given [15, 18, 23].

Definition 1 (see [15, 18, 23]). Let X be a real vector space. A function $N : X \times \mathbf{R} \to [0,1]$ is said to be a fuzzy norm on X if, for all $x, y \in X$ and all $s, t \in \mathbf{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N_3) $N(cx,t) = N(x, t/|c|)$ for $c \neq 0$;

(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;

(N_5) $N(x,\cdot)$ is a nondecreasing function on $\mathbf{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbf{R}$.

The pair (X, N) is called a fuzzy normed vector space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [18, 24].

Definition 2 (see [15, 18, 23]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converges to x if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$, and one denotes it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 3 (see [15, 18, 23]). Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbf{N}$ such that, for all $n \geq n_0$ and all $p > 0$, one has $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well known that every convergent sequence in a fuzzy normed space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete, and the fuzzy normed vector space is called a fuzzy Banach space.

It is said that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [22]).

We recall the fixed point theorem from [25], which is needed in Section 4.

Theorem 4 (see [25, 26]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{3}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$, for all $y \in Y$.*

In 1996, Isac and Rassias [27] were the first to provide new application of fixed point theorems to the proof of stability theory of functional equations. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [28–30] and references therein).

Recently, Kim et al. [31] investigated the solution and the stability of the Euler-Lagrange quadratic functional equation

$$\begin{aligned} f(kx+ly) + f(kx-ly) &= kl\left[f(x+y) + f(x-y)\right] \\ &\quad + 2(k-l)\left[kf(x) - lf(y)\right], \end{aligned} \tag{4}$$

where k, l are non-zero rational numbers with $k \neq l$.

Najati and Jung [32] have observed the Hyers-Ulam stability of the generalized quadratic functional equation

$$f(rx+sy) + rsf(x-y) = rf(x) + sf(y), \tag{5}$$

where r, s are non-zero rational numbers with $r + s = 1$.

In this paper, we generalize the above quadratic functional equation (5) to investigate the generalized Hyers-Ulam stability of an Euler-Lagrange quadratic functional equation

$$f(rx+sy) + rsf(x-y) = (r+s)\left[rf(x) + sf(y)\right] \tag{6}$$

in fuzzy Banach spaces, where r, s are non-zero rational numbers with $r^2 + rs + s^2 - 1 \neq 0$, $r + s \neq 0$. In particular, if $r + s = 1$ in the functional equation (6), then $r^2 + rs + s^2 - 1 \neq 0$ is trivial and so (6) reduces to (5).

2. General Solution of (6)

Lemma 5 (see [31]). *A mapping $f : X \to Y$ between linear spaces satisfies the functional equation*

$$\begin{aligned} f(kx+ly) + f(kx-ly) &= kl\left[f(x+y) + f(x-y)\right] \\ &\quad + 2(k-l)\left[kf(x) - lf(y)\right], \end{aligned} \tag{7}$$

where k, l are non-zero rational numbers with $k \neq l$ if and only if f is quadratic.

Lemma 6. *Let X and Y be vector spaces and $f : X \to Y$ an odd function satisfying (6). Then $f \equiv 0$.*

Proof. Putting $x = 0$ (resp., $y = 0$) in (6), we get

$$f(sy) = s(s+2r) f(y), \qquad f(rx) = r^2 f(x) \tag{8}$$

for all $x, y \in X$. Replacing y by $-y$ in (6) and adding the obtained functional equation to (6), we get

$$\begin{aligned} f(rx+sy) + f(rx-sy) &= 2r(r+s) f(x) \\ &\quad - rs\left[f(x+y) + f(x-y)\right] \end{aligned} \tag{9}$$

for all $x, y \in X$. Replacing y by ry in (9) and using (8), we get

$$\begin{aligned} rf(x+sy) + rf(x-sy) &= 2(r+s) f(x) \\ &\quad - s\left[f(x+ry) + f(x-ry)\right] \end{aligned} \tag{10}$$

for all $x, y \in X$. Again if we replace x by sx in (10) and use (8), we get

$$\begin{aligned} &r(2r+s)\left[f(x+y) + f(x-y)\right] \\ &= 2(r+s)(2r+s) f(x) - \left[f(sx+ry) + f(sx-ry)\right] \end{aligned} \tag{11}$$

for all $x, y \in X$. Exchanging x for y in (6) and using the oddness of f, we have

$$f(sx+ry) = (r+s)\left[rf(y) + sf(x)\right] + rsf(x-y) \tag{12}$$

for all $x, y \in X$. Replacing y by $-y$ in (12) and adding the obtained functional equation to (12), we get

$$f(sx+ry)+f(sx-ry)=2s(r+s)f(x)+rs[f(x+y)+f(x-y)] \tag{13}$$

for all $x, y \in X$. So it follows from (11) and (13) that

$$f(x+y)+f(x-y)=2f(x) \tag{14}$$

for all $x, y \in X$. It easily follows from (14) that f is additive; that is, $f(x+y)=f(x)+f(y)$ for all $x, y \in X$. Since r is a rational number, $f(rx)=rf(x)$ for all $x \in X$. Therefore, it follows from (8) that $r(r-1)f(x)=0$ for all $x \in X$. Since r, s are nonzero, we infer that $f \equiv 0$ if $r \neq 1$.

If $r=1$, then $s \neq 0, -1$, and thus we see easily that $f \equiv 0$ by the similar argument above. □

Lemma 7. *Let X and Y be vector spaces and $f: X \to Y$ an even function satisfying* (6). *Then f is quadratic.*

Proof. Putting $x=y=0$ in (6), we get $f(0)=0$ since $r^2+rs+s^2-1 \neq 0$. Replacing x by $x+y$ in (6), we obtain

$$f(rx+(r+s)y)=(r+s)[rf(x+y)+sf(y)]-rsf(x) \tag{15}$$

for all $x, y \in X$. Replacing y by $-y$ in (15) and using the evenness of f, we get

$$f(rx-(r+s)y)=(r+s)[rf(x-y)+sf(y)]-rsf(x) \tag{16}$$

for all $x, y \in X$. Adding (15) and (16), we get

$$\begin{aligned} f(rx+(r+s)y)&+f(rx-(r+s)y)\\ &=r(r+s)[f(x+y)+f(x-y)]\\ &\quad-2s[rf(x)-(r+s)f(y)] \end{aligned} \tag{17}$$

for all $x, y \in X$. Thus (17) can be rewritten by

$$f(kx+ly)+f(kx-ly)=kl[f(x+y)+f(x-y)]+2(k-l)[kf(x)-lf(y)], \tag{18}$$

where $k:=r, l:=r+s$ for all $x, y \in X$. Therefore, it follows from Lemma 5 that f is quadratic. □

Theorem 8. *Let $f: X \to Y$ be a function between vector spaces X and Y. Then f satisfies* (6) *if and only if f is quadratic.*

Proof. Let f_o and f_e be the odd and the even parts of f. Suppose that f satisfies (6). It is clear that f_o and f_e satisfy (6). By Lemmas 6 and 7, $f_o \equiv 0$ and f_e is quadratic. Since $f=f_o+f_e$, we conclude that f is quadratic.

Conversely, if a mapping f is quadratic, then it is easy to see that f satisfies (6). □

3. Stability of (6) by Direct Method

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

For notational convenience, given a mapping $f: X \to Y$, we define a difference operator $D_{rs}f: X^2 \to Y$ of (6) by

$$\begin{aligned} D_{rs}f(x,y) :=\, & f(rx+sy)+rsf(x-y)\\ &-(r+s)[rf(x)+sf(y)] \end{aligned} \tag{19}$$

for all $x, y \in X$.

Theorem 9. *Assume that a mapping $f: X \to Y$ with $f(0)=0$ satisfies the inequality*

$$N(D_{rs}f(x,y),t) \geq N'(\varphi(x,y),t), \tag{20}$$

and $\varphi: X^2 \to Z$ is a mapping for which there is a constant $c \in \mathbf{R}$ satisfying $0<|c|<(r+s)^2$ such that

$$N'(\varphi((r+s)x,(r+s)y),t) \geq N'(c\varphi(x,y),t) \tag{21}$$

for all $x \in X$ and all $t>0$. Then one can find a unique Euler-Lagrange quadratic mapping $Q: X \to Y$ satisfying the equation $D_{rs}Q(x,y)=0$ and the inequality

$$N(f(x)-Q(x),t) \geq N'\left(\frac{\varphi(x,x)}{(r+s)^2-|c|},t\right), \quad t>0, \tag{22}$$

for all $x \in X$.

Proof. We observe from (21) that

$$\begin{aligned} N'(\varphi&((r+s)^n x,(r+s)^n y),t)\\ &\geq N'(c^n\varphi(x,y),t)\\ &=N'\left(\varphi(x,y),\frac{t}{|c|^n}\right), \quad t>0,\\ N'(\varphi&((r+s)^n x,(r+s)^n y),|c|^n t)\\ &\geq N'(\varphi(x,y),t), \quad t>0, \end{aligned} \tag{23}$$

for all $x, y \in X$. Putting $y:=x$ in (20), we obtain

$$\begin{gathered} N\left(f((r+s)x)-(r+s)^2 f(x),t\right) \geq N'(\varphi(x,x),t),\\ \text{or}\quad N\left(f(x)-\frac{f((r+s)x)}{(r+s)^2},\frac{t}{(r+s)^2}\right) \geq N'(\varphi(x,x),t) \end{gathered} \tag{24}$$

for all $x \in X$. Therefore it follows from (23), (24) that

$$\begin{aligned} N&\left(\frac{f((r+s)^n x)}{(r+s)^{2n}}-\frac{f\left((r+s)^{n+1}x\right)}{(r+s)^{2(n+1)}},\frac{|c|^n t}{(r+s)^{2(n+1)}}\right)\\ &\geq N'(\varphi((r+s)^n x,(r+s)^n x),|c|^n t)\\ &\geq N'(\varphi(x,x),t) \end{aligned} \tag{25}$$

for all $x \in X$ and any integer $n \geq 0$. So

$$\begin{aligned} &N\left(f(x) - \frac{f((r+s)^n x)}{(r+s)^{2n}}, \sum_{i=0}^{n-1} \frac{|c|^i t}{(r+s)^{2(i+1)}}\right) \\ &= N\left(\sum_{i=0}^{n-1}\left(\frac{f((r+s)^i x)}{(r+s)^{2i}} - \frac{f((r+s)^{i+1} x)}{(r+s)^{2(i+1)}}\right), \right. \\ &\qquad \left. \sum_{i=0}^{n-1} \frac{|c|^i t}{(r+s)^{2(i+1)}}\right) \\ &\geq \min_{0\leq i\leq n-1}\left\{N\left(\frac{f((r+s)^i x)}{(r+s)^{2i}} - \frac{f((r+s)^{i+1} x)}{(r+s)^{2(i+1)}}, \right.\right. \\ &\qquad \left.\left. \frac{|c|^i t}{(r+s)^{2(i+1)}}\right)\right\} \\ &\geq N'(\varphi(x,x), t), \quad t > 0, \end{aligned} \tag{26}$$

which yields

$$\begin{aligned} &N\left(\frac{f((r+s)^m x)}{(r+s)^{2m}} - \frac{f((r+s)^{m+p} x)}{(r+s)^{2(m+p)}}, \sum_{i=m}^{m+p-1} \frac{|c|^i t}{(r+s)^{2(i+1)}}\right) \\ &= N\left(\sum_{i=m}^{m+p-1}\left(\frac{f((r+s)^i x)}{(r+s)^{2i}} - \frac{f((r+s)^{i+1} x)}{(r+s)^{2(i+1)}}\right), \right. \\ &\qquad \left. \sum_{i=m}^{m+p-1} \frac{|c|^i t}{(r+s)^{2(i+1)}}\right) \\ &\geq \min_{m\leq i\leq m+p-1}\left\{N\left(\frac{f((r+s)^i x)}{(r+s)^{2i}} - \frac{f((r+s)^{i+1} x)}{(r+s)^{2(i+1)}}, \right.\right. \\ &\qquad \left.\left. \frac{|c|^i t}{(r+s)^{2(i+1)}}\right)\right\} \\ &\geq N'(\varphi(x,x), t), \quad t > 0, \end{aligned} \tag{27}$$

for all $x \in X$ and any integers $p > 0$, $m \geq 0$. Hence one obtains

$$\begin{aligned} &N\left(\frac{f((r+s)^m x)}{(r+s)^{2m}} - \frac{f((r+s)^{m+p} x)}{(r+s)^{2(m+p)}}, t\right) \\ &\quad \geq N'\left(\varphi(x,x), \frac{t}{\sum_{i=m}^{m+p-1}\left(|c|^i/(r+s)^{2(i+1)}\right)}\right) \end{aligned} \tag{28}$$

for all $x \in X$ and any integers $p > 0$, $m \geq 0$, $t > 0$. Since $\sum_{i=m}^{m+p-1}(|c|^i/(r+s)^{2i})$ is convergent series, we see by taking the limit $m \to \infty$ in the last inequality that a sequence $\{f((r+s)^n x)/(r+s)^{2n}\}$ is Cauchy in the fuzzy Banach space (Y, N) and so it converges in Y. Therefore a mapping $Q : X \to Y$ defined by

$$Q(x) := N - \lim_{n\to\infty} \frac{f((r+s)^n x)}{(r+s)^{2n}} \tag{29}$$

is well defined for all $x \in X$. It means that $\lim_{n\to\infty} N(f((r+s)^n x)/(r+s)^{2n} - Q(x), t) = 1$, $t > 0$, for all $x \in X$. In addition, we see from (26) that

$$\begin{aligned} &N\left(f(x) - \frac{f((r+s)^n x)}{(r+s)^{2n}}, t\right) \\ &\quad \geq N'\left(\varphi(x,x), \frac{t}{\sum_{i=0}^{n-1}\left(|c|^i/(r+s)^{2(i+1)}\right)}\right) \end{aligned} \tag{30}$$

and so, for any $\varepsilon > 0$,

$$\begin{aligned} &N(f(x) - Q(x), t) \\ &\geq \min\left\{N\left(f(x) - \frac{f((r+s)^n x)}{(r+s)^{2n}}, (1-\varepsilon)t\right), \right. \\ &\qquad \left. N\left(\frac{f((r+s)^n x)}{(r+s)^{2n}} - Q(x), \varepsilon t\right)\right\} \\ &\geq N'\left(\varphi(x,x), \frac{(1-\varepsilon)t}{\sum_{i=0}^{n-1}\left(|c|^i/(r+s)^{2(i+1)}\right)}\right) \\ &\geq N'\left(\varphi(x,x), (1-\varepsilon)\left((r+s)^2 - |c|\right)t\right), \\ &\qquad 0 < \varepsilon < 1, \end{aligned} \tag{31}$$

for sufficiently large n and for all $x \in X$ and all $t > 0$. Since ε is arbitrary and N' is left continuous, we obtain

$$\begin{aligned} &N(f(x) - Q(x), t) \geq N'\left(\varphi(x,x), \left((r+s)^2 - |c|\right)t\right), \\ &\qquad t > 0, \end{aligned} \tag{32}$$

for all $x \in X$, which yields the approximation (22).

In addition, it is clear from (20) and (N_5) that the following relation

$$\begin{aligned} &N\left(\frac{D_{rs}f((r+s)^n x, (r+s)^n y)}{(r+s)^{2n}}, t\right) \\ &\geq N'\left(\varphi((r+s)^n x, (r+s)^n y), (r+s)^{2n} t\right) \\ &\geq N'\left(\varphi(x,y), \frac{(r+s)^{2n}}{|c|^n} t\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{33}$$

holds for all $x, y \in X$ and all $t > 0$. Therefore, we obtain by use of

$$\lim_{n\to\infty} N\left(\frac{f((r+s)^n x)}{(r+s)^{2n}} - Q(x), t\right) = 1 \quad (t > 0) \tag{34}$$

that

$$\begin{aligned}
&N\left(D_{rs}Q\left(x,y\right),t\right)\\
&\geq \min\left\{N\left(D_{rs}Q\left(x,y\right)-\frac{D_{rs}f\left((r+s)^{n}x,(r+s)^{n}y\right)}{(r+s)^{2n}},\frac{t}{2}\right),\right.\\
&\qquad\left.N\left(\frac{D_{rs}f\left((r+s)^{n}x,(r+s)^{n}y\right)}{(r+s)^{2n}},\frac{t}{2}\right)\right\}\\
&=N\left(\frac{D_{rs}f\left((r+s)^{n}x,(r+s)^{n}y\right)}{r^{2n}},\frac{t}{2}\right),\\
&\qquad\qquad(\text{for sufficiently large } n)\\
&\geq N'\left(\varphi\left(x,y\right),\frac{(r+s)^{2n}}{2|c|^{n}}t\right),\quad t>0\\
&\longrightarrow 1\quad \text{as } n\longrightarrow\infty
\end{aligned}\tag{35}$$

which implies $D_{rs}Q(x,y)=0$ by (N_2). Thus we find that Q is an Euler-Lagrange quadratic mapping satisfying (6) and (22) near the approximate quadratic mapping $f:X\rightarrow Y$.

To prove the aforementioned uniqueness, we assume now that there is another quadratic mapping $Q':X\rightarrow Y$ which satisfies (22). Then one establishes by using the equality $Q'((r+s)^{n}x)=(r+s)^{2n}Q(x)$ and (22) that

$$\begin{aligned}
&N\left(Q\left(x\right)-Q'\left(x\right),t\right)\\
&=N\left(\frac{Q\left((r+s)^{n}x\right)}{(r+s)^{2n}}-\frac{Q'\left((r+s)^{n}x\right)}{(r+s)^{2n}},t\right)\\
&\geq \min\left\{N\left(\frac{Q\left((r+s)^{n}x\right)}{(r+s)^{2n}}-\frac{f\left((r+s)^{n}x\right)}{(r+s)^{2n}},\frac{t}{2}\right),\right.\\
&\qquad\left.N\left(\frac{f\left((r+s)^{n}x\right)}{(r+s)^{2n}}-\frac{Q'\left((r+s)^{n}x\right)}{(r+s)^{2n}},\frac{t}{2}\right)\right\}\\
&\geq N'\left(\varphi\left((r+s)^{n}x,(r+s)^{n}x\right),\frac{\left((r+s)^{2}-|c|\right)(r+s)^{2n}t}{2}\right)\\
&\geq N'\left(\varphi\left(x,x\right),\frac{\left((r+s)^{2}-|c|\right)(r+s)^{2n}t}{2|c|^{n}}\right),\quad t>0,\ \forall n\in\mathbb{N},
\end{aligned}\tag{36}$$

which tends to 1 as $n\rightarrow\infty$ by (N_5). Therefore one obtains $Q(x)=Q'(x)$ for all $x\in X$, completing the proof of uniqueness. □

We remark that, if $r+s=1$ in Theorem 9, then $N'(\varphi(x,y),t)\geq N'(\varphi(x,y),t/|c|^{n})\rightarrow 1$ as $n\rightarrow\infty$, and so $\varphi(x,y)=0$ for all $x,y\in X$. Hence $D_{rs}f(x,y)=0$ for all $x,y\in X$ and f is itself a quadratic mapping.

Theorem 10. *Assume that a mapping $f:X\rightarrow Y$ with $f(0)=0$ satisfies the inequality*

$$N\left(D_{rs}f\left(x,y\right),t\right)\geq N'\left(\varphi\left(x,y\right),t\right)\tag{37}$$

and $\varphi:X^{2}\rightarrow Z$ is a mapping for which there is a constant $c\in\mathbb{R}$ satisfying $|c|>(r+s)^{2}$ such that

$$N'\left(\varphi\left(\frac{x}{(r+s)},\frac{y}{(r+s)}\right),t\right)\geq N'\left(\frac{1}{c}\varphi\left(x,y\right),t\right),\quad t>0,\tag{38}$$

for all $x\in X$ and all $t>0$. Then one can find a unique Euler-Lagrange quadratic mapping $Q:X\rightarrow Y$ satisfying the equation $D_{rs}Q(x,y)=0$ and the inequality

$$N\left(f\left(x\right)-Q\left(x\right),t\right)\geq N'\left(\frac{\varphi\left(x,x\right)}{|c|-(r+s)^{2}},t\right),\quad t>0,\tag{39}$$

for all $x\in X$.

Proof. It follows from (24) and (38) that

$$N\left(f\left(x\right)-(r+s)^{2}f\left(\frac{x}{(r+s)}\right),\frac{t}{|c|}\right)\geq N'\left(\varphi\left(x,x\right),t\right),\quad t>0\tag{40}$$

for all $x\in X$. Therefore it follows that

$$\begin{aligned}
&N\left(f\left(x\right)-(r+s)^{2n}f\left(\frac{x}{(r+s)^{n}}\right),\sum_{i=0}^{n-1}\frac{(r+s)^{2i}}{|c|^{i+1}}t\right)\\
&\qquad\geq N'\left(\varphi\left(x,x\right),t\right),\quad t>0,
\end{aligned}\tag{41}$$

for all $x\in X$ and any integer $n>0$. Thus we see from the last inequality that

$$\begin{aligned}
&N\left(f\left(x\right)-(r+s)^{2n}f\left(\frac{x}{(r+s)^{n}}\right),t\right)\\
&\geq N'\left(\varphi\left(x,x\right),\frac{t}{\sum_{i=0}^{n-1}\left((r+s)^{2i}/|c|^{i+1}\right)}\right)\\
&\geq N'\left(\varphi\left(x,x\right),\left(|c|-(r+s)^{2}\right)t\right),\quad t>0.
\end{aligned}\tag{42}$$

The remaining assertion goes through by the similar way to the corresponding part of Theorem 9. □

We also observe that, if $r+s=1$ in Theorem 10, then $N'(\varphi(x,y),t)\geq N'(\varphi(x,y),|c|^{n}t)\rightarrow 1$ as $n\rightarrow\infty$, and so $\varphi(x,y)=0$ for all $x,y\in X$. Hence $D_{rs}f=0$ and f is itself a quadratic mapping.

Corollary 11. *Let X be a normed space and $(\mathbf{R},N')$ a fuzzy normed space. Assume that there exist real numbers $\theta_1,\theta_2\geq 0$ and p is real number such that either $p<2$ or $p>2$. If a mapping $f:X\rightarrow Y$ with $f(0)=0$ satisfies the inequality*

$$N\left(D_{rs}f\left(x,y\right),t\right)\geq N'\left(\theta_1\|x\|^{p}+\theta_2\|y\|^{p},t\right)\tag{43}$$

for all $x, y \in X$ and all $t > 0$. Then one can find a unique Euler-Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{rs}Q(x, y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \le \begin{cases} N'\left(\dfrac{(\theta_1+\theta_2)\|x\|^p}{(r+s)^2 - |r+s|^p}, t\right), & \text{if } p<2,\ |r+s|>1, \\ & (p>2,\ |r+s|<1) \\ N'\left(\dfrac{(\theta_1+\theta_2)\|x\|^p}{|r+s|^p - (r+s)^2}, t\right), & \text{if } p>2,\ |r+s|>1, \\ & (p<2,\ |r+s|<1) \end{cases} \tag{44}$$

for all $x \in X$ and all $t > 0$.

Proof. Taking $\varphi(x, y) = \theta_1\|x\|^p + \theta_2\|y\|^p$ and applying Theorems 9 and 10, we obtain the desired approximation, respectively. □

Corollary 12. *Assume that, for $r + s \neq 1$, there exists a real number $\theta \ge 0$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(D_{rs}f(x, y), t) \ge N'(\theta, t) \tag{45}$$

for all $x, y \in X$ and all $t > 0$. Then one can find a unique Euler-Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{rs}Q(x, y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \ge N'\left(\frac{\theta}{\left|(r+s)^2 - 1\right|}, t\right) \tag{46}$$

for all $x \in X$ and all $t > 0$.

We remark that, if $\theta = 0$, then $N(D_{rs}f(x, y), t) \ge N'(0, t) = 1$, and so $D_{rs}f(x, y) = 0$. Thus we get that $f = Q$ is itself a quadratic mapping.

4. Stability of (6) by Fixed Point Method

Now, in the next theorem, we are going to consider a stability problem concerning the stability of (6) by using a fixed point theorem of the alternative for contraction mappings on generalized complete metric spaces due to Margolis and Diaz [25].

Theorem 13. *Assume that there exists constant $c \in \mathbf{R}$ with $|c| \neq 1$ and $q > 0$ satisfying $0 < |c|^{1/q} < (r + s)^2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(D_{rs}f(x, y), t_1 + t_2) \ge \min\left\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q)\right\} \tag{47}$$

for all $x, y \in X$, $t_i > 0$ $(i = 1, 2)$, and $\varphi : X \to Z$ is a mapping satisfying

$$N'(\varphi((r+s)x), t) \ge N'(c\varphi(x), t) \tag{48}$$

for all $x \in X$ and all $t > 0$. Then there exists a unique Euler-Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{rs}Q(x, y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \ge N'\left(\frac{2^q\varphi(x)}{\left((r+s)^2 - |c|^{1/q}\right)^q}, t^q\right) \tag{49}$$

for all $x \in X$ and all $t > 0$.

Proof. We consider the set of functions

$$\Omega := \{g : X \longrightarrow Y \mid g(0) = 0\} \tag{50}$$

and define a generalized metric on Ω as follows:

$$d_\Omega(g, h) := \inf\left\{K \in (0, \infty) : N(g(x) - h(x), Kt) \ge N'(\varphi(x), t^q),\ \forall x \in X,\ \forall t > 0\right\}. \tag{51}$$

Then one can easily see that (Ω, d_Ω) is a complete generalized metric space [33, 34].

Now, we define an operator $J : \Omega \to \Omega$ as

$$Jg(x) = \frac{g((r+s)x)}{(r+s)^2} \tag{52}$$

for all $g \in \Omega$, $x \in X$.

We first prove that J is strictly contractive on Ω. For any $g, h \in \Omega$, let $\varepsilon \in [0, \infty)$ be any constant with $d_\Omega(g, h) \le \varepsilon$. Then we deduce from the use of (48) and the definition of $d_\Omega(g, h)$ that

$$\begin{aligned} & N(g(x) - h(x), \varepsilon t) \ge N'(\varphi(x), t^q), \quad \forall x \in X,\ t > 0 \\ & \Longrightarrow N\left(\frac{g((r+s)x)}{(r+s)^2} - \frac{h((r+s)x)}{(r+s)^2}, \frac{|c|^{1/q}\varepsilon t}{(r+s)^2}\right) \\ & \qquad \ge N'(\varphi((r+s)x), |c|\, t^q) \\ & \Longrightarrow N\left(Jg(x) - Jh(x), \frac{|c|^{1/q}\varepsilon t}{(r+s)^2}\right) \\ & \qquad \ge N'(\varphi(x), t^q), \quad \forall x \in X,\ t > 0, \\ & \Longrightarrow d_\Omega(Jg, Jh) \le \frac{|c|^{1/q}\varepsilon}{(r+s)^2}. \end{aligned} \tag{53}$$

Since ε is arbitrary constant with $d_\Omega(g, h) \le \varepsilon$, we see that, for any $g, h \in \Omega$,

$$d_\Omega(Jg, Jh) \le \frac{|c|^{1/q}}{(r+s)^2} d_\Omega(g, h), \tag{54}$$

which implies J is strictly contractive with constant $|c|^{1/q}/(r + s)^2 < 1$ on Ω.

We now want to show that $d(f, Jf) < \infty$. If we put $y := x$, $t_i := t$ $(i = 1, 2)$ in (47), then we arrive at

$$N\left(f(x) - \frac{f((r+s)x)}{(r+s)^2}, \frac{2t}{(r+s)^2}\right) \ge N'(\varphi(x), t^q), \tag{55}$$

which yields $d_\Omega(f, Jf) \leq 2/(r+s)^2$ and so $d_\Omega(J^n f, J^{n+1} f) \leq d_\Omega(f, Jf) \leq 2/(r+s)^2$ for all $n \in \mathbf{N}$.

Using the fixed point theorem of the alternative for contractions on generalized complete metric spaces due to Margolis and Diaz [25], we see the following (i), (ii), and (iii).

(i) There is a mapping $Q : X \to Y$ with $Q(0) = 0$ such that

$$d_\Omega(f, Q) \leq \frac{1}{1 - \left(|c|^{1/q}/(r+s)^2\right)} d_\Omega(f, Jf) \leq \frac{2}{(r+s)^2 - |c|^{1/q}} \tag{56}$$

and Q is a fixed point of the operator J; that is, $(1/(r+s)^2)Q((r+s)x) = JQ(x) = Q(x)$ for all $x \in X$. Thus we can get

$$\begin{gathered} N\left(f(x) - Q(x), \frac{2t}{(r+s)^2 - |c|^{1/q}}\right) \geq N'\left(\varphi(x), t^q\right), \\ N\left(f(x) - Q(x), t\right) \geq N'\left(\varphi(x), \frac{\left((r+s)^2 - |c|^{1/q}\right)^q}{2^q} t^q\right) \end{gathered} \tag{57}$$

for all $t > 0$ and all $x \in X$.

(ii) Consider $d_\Omega(J^n f, Q) \to 0$ as $n \to \infty$. Thus we obtain

$$\begin{aligned} &N\left(\frac{f((r+s)^n x)}{(r+s)^{2n}} - Q(x), t\right) \\ &= N\left(f((r+s)^n x) - Q((r+s)^n x), (r+s)^{2n} t\right) \\ &\geq N'\left(\frac{2^q \varphi((r+s)^n x)}{\left((r+s)^2 - |c|^{1/q}\right)^q}, (r+s)^{2nq} t^q\right) \\ &= N'\left(\frac{2^q \varphi(x)}{\left((r+s)^2 - |c|^{1/q}\right)^q}, \left(\frac{(r+s)^{2q}}{|c|}\right)^n t^q\right) \\ &\longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \ \left(\frac{(r+s)^{2q}}{|c|} > 1\right) \end{aligned} \tag{58}$$

for all $t > 0$ and all $x \in X$, that is; the mapping $Q : X \to Y$ given by

$$N - \lim_{n \to \infty} \frac{f((r+s)^n x)}{(r+s)^{2n}} = Q(x) \tag{59}$$

is welldefined for all $x \in X$. In addition, it follows from conditions (47), (48), and (N_4) that

$$\begin{aligned} &N\left(\frac{D_{rs} f((r+s)^n x, (r+s)^n y)}{(r+s)^{2n}}, t\right) \\ &\geq N'\left(\varphi((r+s)^n x), \frac{(r+s)^{2nq} t^q}{2^q}\right) \\ &= N'\left(|c|^n \varphi(x), \frac{(r+s)^{2nq} t^q}{2^q}\right) \\ &= N'\left(\varphi(x), \left(\frac{(r+s)^{2q}}{|c|}\right)^n \frac{t^q}{2^q}\right) \\ &\longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \ t > 0, \end{aligned} \tag{60}$$

for all $x \in X$. Therefore we obtain by use of (N_4), (59), and (60)

$$\begin{aligned} &N\left(D_{rs} Q(x, y), t\right) \\ &\geq \min\left\{N\left(D_{rs} Q(x, y) - \frac{D_{rs} f((r+s)^n x, (r+s)^n y)}{(r+s)^{2n}}, \frac{t}{2}\right),\right. \\ &\qquad \left. N\left(\frac{D_{rs} f((r+s)^n x, (r+s)^n y)}{(r+s)^{2n}}, \frac{t}{2}\right)\right\} \\ &= N\left(\frac{D_{rs} f((r+s)^n x, (r+s)^n y)}{(r+s)^{2n}}, \frac{t}{2}\right) \\ &\qquad (\text{for sufficiently large } n) \\ &\geq \min\left\{N'\left(\varphi(x), \left(\frac{(r+s)^{2q}}{|c|}\right)^n \frac{t^q}{4^q}\right),\right. \\ &\qquad \left. N'\left(\varphi(y), \left(\frac{(r+s)^{2q}}{|c|}\right)^n \frac{t^q}{4^q}\right)\right\} \\ &\longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \ t > 0, \end{aligned} \tag{61}$$

which implies $D_{rs} Q(x, y) = 0$ by (N_2), and so the mapping Q is quadratic satisfying (6).

(iii) The mapping Q is a unique fixed point of the operator J in the set $\Delta = \{g \in \Omega \mid d_\Omega(f, g) < \infty\}$. Thus if we assume that there exists another Euler-Lagrange type quadratic mapping $Q' : X \to Y$ satisfying (49), then

$$\begin{gathered} Q'(x) = \frac{Q'((r+s)x)}{(r+s)^2} = JQ'(x), \\ d_\Omega(f, Q') \leq \frac{2}{\left((r+s)^2 - |c|^{1/q}\right)} < \infty, \end{gathered} \tag{62}$$

and so Q' is a fixed point of the operator J and $Q' \in \Delta = \{g \in \Omega \mid d_\Omega(f, g) < \infty\}$. By the uniqueness of the fixed point of J in Δ, we find that $Q = Q'$, which proves the uniqueness of Q satisfying (49). This ends the proof of the theorem. □

Theorem 14. *Assume that there exists constant $c \in \mathbf{R}$ with $|c| \neq 1$ and $q > 0$ satisfying $|c|^{1/q} > (r+s)^2$ such that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(D_{rs}f(x,y), t_1 + t_2) \geq \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q)\} \tag{63}$$

for all $x, y \in X$, $t_i > 0$ $(i = 1, 2)$, and $\varphi : X \to Z$ is a mapping satisfying

$$N'\left(\varphi\left(\frac{x}{(r+s)}\right), t\right) \geq N'\left(\frac{1}{c}\varphi(x), t\right) \tag{64}$$

for all $x \in X$. Then there exists a unique Euler-Lagrange quadratic mapping $Q : X \to Y$ satisfying the equation $D_{rs}Q(x,y) = 0$ and the inequality

$$N(f(x) - Q(x), t) \geq N'\left(\frac{2^q\varphi(x)}{\left(|c|^{1/q} - (r+s)^2\right)^q}, t^q\right), \quad t > 0, \tag{65}$$

for all $x \in X$.

Proof. The proof of this theorem is similar to that of Theorem 13. □

Remark 15. In a real space with a fuzzy norm $N(x,t) = N'(x,t) = t/(t + \|x\|)$, the stability result obtained by the direct method is somewhat different from the stability result obtained by the fixed point method as follows. Let X be a normed space and Y a Banach space. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfy the inequality

$$\|D_{rs}f(x,y)\| \leq \theta_1\|x\|^{p_1} + \theta_2\|y\|^{p_2} \tag{66}$$

for all $x, y \in X$ and $X \setminus \{0\}$ if $p_1, p_2 < 0$. Assume that there exist real numbers $\theta_1, \theta_2 \geq 0$ and p_1, p_2 such that either $p_1, p_2 < 2$, $|r+s| > 1$ ($p_1, p_2 > 2$, $|r+s| < 1$, resp.) or $p_1, p_2 > 2$, $|r+s| > 1$ ($p_1, p_2 < 2$, $|r+s| < 1$, resp.). Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality:

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\theta_1\|x\|^{p_1}}{(r+s)^2 - |r+s|^{p_1}} + \dfrac{\theta_2\|x\|^{p_2}}{(r+s)^2 - |r+s|^{p_2}}, & \text{if } p_1, p_2 < 2,\ |r+s| > 1, \\ & (p_1, p_2 > 2,\ |r+s| < 1,\ \text{resp.}), \\ \dfrac{\theta_1\|x\|^{p_1}}{|r+s|^{p_1} - (r+s)^2} + \dfrac{\theta_2\|x\|^{p_2}}{|r+s|^{p_2} - (r+s)^2}, & \text{if } p_1, p_2 > 2,\ |r+s| > 1 \\ & (p_1, p_2 < 2,\ |r+s| < 1,\ \text{resp.}) \end{cases} \tag{67}$$

for all $x \in X$ and $X\backslash\{0\}$ if $p_1, p_2 < 0$, which is verified by using the direct method together with the following inequality

$$\begin{aligned} &\left\|f(x) - \frac{f((r+s)^n x)}{(r+s)^n}\right\| \\ &\leq \frac{1}{(r+s)^2}\sum_{i=0}^{n-1}\left(\frac{\theta_1|r+s|^{p_1 i}\|x\|^{p_1}}{|r+s|^{2i}} + \frac{\theta_2|r+s|^{p_2 i}\|x\|^{p_2}}{|r+s|^{2i}}\right), \\ &\left\|f(x) - (r+s)^{2n} f\left(\frac{x}{(r+s)^n}\right)\right\| \\ &\leq \frac{1}{(r+s)^2}\sum_{i=1}^{n}\left(\frac{\theta_1|r+s|^{2i}\|x\|^{p_1}}{|r+s|^{p_1 i}} + \frac{\theta_2|r+s|^{2i}\|x\|^{p_2}}{|r+s|^{p_2 i}}\right), \end{aligned} \tag{68}$$

for all $x \in X$.

On the other hand, assume that there exist real numbers $\theta_1, \theta_2 \geq 0$ and p_1, p_2 such that either $\max\{p_1, p_2\} < 2$, $|r+s| > 1$ ($\min\{p_1, p_2\} > 2$, $|r+s| < 1$, resp.) or $\min\{p_1, p_2\} > 2$, $|r+s| > 1$ ($\max\{p_1, p_2\} < 2$, $|r+s| < 1$, resp.). Then there exists a unique quadratic function $Q : X \to Y$ which satisfies the inequality

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\theta_1\|x\|^{p_1} + \theta_2\|x\|^{p_2}}{(r+s)^2 - |r+s|^{\max\{p_1,p_2\}}}, & \text{if } \max\{p_1, p_2\} < 2,\ |r+s| > 1, \\ \dfrac{\theta_1\|x\|^{p_1} + \theta_2\|x\|^{p_2}}{(r+s)^2 - |r+s|^{\min\{p_1,p_2\}}}, & \text{if } \min\{p_1, p_2\} > 2,\ |r+s| < 1, \\ \dfrac{\theta_1\|x\|^{p_1} + \theta_2\|x\|^{p_2}}{|r+s|^{\min\{p_1,p_2\}} - (r+s)^2}, & \text{if } \min\{p_1, p_2\} > 2,\ |r+s| > 1, \\ \dfrac{\theta_1\|x\|^{p_1} + \theta_2\|x\|^{p_2}}{|r+s|^{\max\{p_1,p_2\}} - (r+s)^2}, & \text{if } \max\{p_1, p_2\} < 2,\ |r+s| < 1 \end{cases} \tag{69}$$

for all $x \in X$ and $X \setminus \{0\}$ if $p_1, p_2 < 0$, which is established by using the fixed point method together with

$$c = \begin{cases} |r+s|^{\max\{p_1,p_2\}}, & \text{if } \max\{p_1, p_2\} < 2,\ |r+s| > 1, \\ |r+s|^{\min\{p_1,p_2\}}, & \text{if } \min\{p_1, p_2\} > 2,\ |r+s| < 1, \\ |r+s|^{\min\{p_1,p_2\}}, & \text{if } \min\{p_1, p_2\} > 2,\ |r+s| > 1, \\ |r+s|^{\max\{p_1,p_2\}}, & \text{if } \max\{p_1, p_2\} < 2,\ |r+s| < 1. \end{cases} \tag{70}$$

Therefore, we observe that the corresponding subsequential four stability results by the direct method are sharper than the corresponding subsequential four stability results obtained by the fixed point method.

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