Research Article

Iterative Solution to a System of Matrix Equations

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An efficient iterative algorithm is presented to solve a system of linear matrix equations $A_1 X_1 B_1 + A_2 X_2 B_2 = E$, $C_1 X_1 D_1 + C_2 X_2 D_2 = F$ with real matrices X_1 and X_2. By this iterative algorithm, the solvability of the system can be determined automatically. When the system is consistent, for any initial matrices X_1^0 and X_2^0, a solution can be obtained in the absence of roundoff errors, and the least norm solution can be obtained by choosing a special kind of initial matrix. In addition, the unique optimal approximation solutions $\widehat{X}_1$ and $\widehat{X}_2$ to the given matrices $\widetilde{X}_1$ and $\widetilde{X}_2$ in Frobenius norm can be obtained by finding the least norm solution of a new pair of matrix equations $A_1 \overline{X}_1 B_1 + A_2 \overline{X}_2 B_2 = \overline{E}, C_1 \overline{X}_1 D_1 + C_2 \overline{X}_2 D_2 = \overline{F}$, where $\overline{E} = E - A_1 \widetilde{X}_1 B_1 - A_2 \widetilde{X}_2 B_2$, $\overline{F} = F - C_1 \widetilde{X}_1 D_1 - C_2 \widetilde{X}_2 D_2$. The given numerical example demonstrates that the iterative algorithm is efficient. Especially, when the numbers of the parameter matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are large, our algorithm is efficient as well.

1. Introduction

Throughout the paper, we denote the set of all $m \times n$ real matrix by $R^{m \times n}$, the transpose matrix of A by A^T, the identity matrix of order n by I_n, the Kronecker product of A and B by $A \otimes B$, the $mn \times 1$ vector formed by the vertical concatenation of the respective columns of a matrix $A \in R^{m \times n}$ by $\text{vec}(A)$, the trace of a matrix A by $\text{tr}(A)$, and the Frobenius norm of a matrix A by $\|A\|$ where $\|A\| = \sqrt{\text{tr}(A^T A)}$.

In this paper, we consider the following two problems.

Problem 1. For the given matrices $A_1 \in R^{p \times k}$, $A_2 \in R^{p \times m}$, $B_1 \in R^{r \times q}$, $B_2 \in R^{n \times q}$, $C_1 \in R^{s \times k}$, $C_2 \in R^{s \times m}$, $D_1 \in R^{r \times t}$, $D_2 \in R^{n \times t}$, $E \in R^{p \times q}$, and $F \in R^{s \times t}$, find $X_1 \in R^{k \times r}$ and $X_2 \in R^{m \times n}$ such that

$$
\begin{aligned}
A_1 X_1 B_1 + A_2 X_2 B_2 &= E, \\
C_1 X_1 D_1 + C_2 X_2 D_2 &= F.
\end{aligned}
\tag{1}
$$

Problem 2. When Problem 1 is consistent, let S denote the solution set of the pair of matrix equation (1). For the given matrices $\widetilde{X}_1 \in R^{k \times r}$, $\widetilde{X}_2 \in R^{m \times n}$, find $\{\widehat{X}_1, \widehat{X}_2\} \in S$ such that

$$
\begin{aligned}
&\left\| \widehat{X}_1 - \widetilde{X}_1 \right\|^2 + \left\| \widehat{X}_2 - \widetilde{X}_2 \right\|^2 \\
&= \min_{\{X_1, X_2\} \in S} \left(\left\| X_1 - \widetilde{X}_1 \right\|^2 + \left\| X_2 - \widetilde{X}_2 \right\|^2 \right).
\end{aligned}
\tag{2}
$$

Problem 2 is to find the optimal approximation solutions to the given matrices $\widetilde{X}_1, \widetilde{X}_2$ in the solution set of Problem 1. It occurs frequently in experiment design (see, for instance, [1]). In the recent years, the matrix optimal approximation problem has been studied extensively (e.g., [2–13]).

The research on solving matrix equation pair has been actively ongoing for the last 30 years or more. For instance, Mitra [14] gave conditions for the existence of a solution and a representation of the general common solution to $AXB = E$, $CXD = F$. Shinozaki and Sibuya [15] and van der Woude [16] discussed conditions for the existence of a common solution to $AXB = E$, $CXD = F$. Navarra et al. [5] derived sufficient and necessary conditions for the existence of a common solution to $AXB = E$, $CXD = F$. Yuan [13] obtained an analytical expression of the least-squares solutions of $AXB = E$, $CXD = F$ by using the generalized

singular value decomposition (GSVD) of matrices. Dehghan and Hajarian [17] presented some examples to show a motivation for studying the general coupled matrix equations $\sum_{j=1}^{l} A_{ij}X_{j}B_{ij} = C_{i}$, $i = 1, 2, \ldots, l$, and [18] constructed an iterative algorithm to solve the general coupled matrix equations $\sum_{j=1}^{p} A_{ij}X_{j}B_{ij} = M_{i}$, $i = 1, 2, \ldots, p$. Wang [19, 20] gave the centrosymmetric solution to the system of quaternion matrix equations $A_{1}X = C_{1}$, $A_{3}XB_{3} = C_{3}$. Wang [21] also solved a system of matrix equations over arbitrary regular rings with identity.

Recently, some finite iterative algorithms have also been developed to solve matrix equations. Ding et al. [22, 23] and Xie et al. [24, 25] studied the iterative solutions of matrix equations $AXB = F$ and $A_{i}XB_{i} = F_{i}$ and generalized Sylvester matrix equations $AXB + CXD = F$ and $AXB + CX^{T}D = F$. They presented a gradient based and a least-squares based iterative algorithms for the solution. Li et al. [26, 27] and Zhou et al. [28, 29] considered iterative method for some coupled linear matrix equations. Deng et al. [30] studied the consistent conditions and the general expressions about the Hermitian solutions of the matrix equations $(AX, XB) = (C, D)$ and designed an iterative method for its Hermitian minimum norm solutions. Li and Wu [31] gave symmetric and skew-antisymmetric solutions to certain matrix equations $A_{1}X = C_{1}$, $XB_{3} = C_{3}$ over the real quaternion algebra H. For more studies on iterative algorithms on coupled matrix equations, we refer to [3, 10–12, 17, 32–37]. Peng et al. [6] presented iterative methods to obtain the symmetric solutions of $AXB = E$, $CXD = F$. Sheng and Chen [8] presented a finite iterative method; when $AXB = E$, $CXD = F$ is consistent. Liao and Lei [38] presented an analytical expression of the least-squares solution and an algorithm for $AXB = E$, $CXD = F$ with the minimum norm. Peng et al. [7] presented an algorithm for the least-squares reflexive solution. Dehghan and Hajarian [2] presented an iterative algorithm for solving a pair of matrix equations $AXB = E$, $CXD = F$ over generalized centrosymmetric matrices. Cai and Chen [39] presented an iterative algorithm for the least-squares bisymmetric solutions of the matrix equations $AXB = E$, $CXD = F$. Yin and Huang [40] presented an iterative algorithm to solve the least squares generalized reflexive solutions of the matrix equations $AXB = E$, $CXD = F$.

However, to our knowledge, there has been little information on finding the solutions to the system (1) by iterative algorithm. In this paper, an efficient iterative algorithm is presented to solve the system (1) for any real matrices X_{1}, X_{2}. The suggested iterative algorithm, automatically determines the solvability of equations pair (1). When the pair of equations is consistent, then, for any initial matrices X_{1}^{0} and X_{2}^{0}, the solution can be obtained in the absence of round errors, and the least norm solution can be obtained by choosing a special kind of initial matrix. In addition, the unique optimal approximation solutions $\widehat{X}_{1}$ and $\widehat{X}_{2}$ to the given matrices $\widetilde{X}_{1}$ and $\widetilde{X}_{2}$ in Frobenius norm can be obtained by finding the least norm solution of a new pair of matrix equations $A_{1}\overline{X}_{1}B_{1} + A_{2}\overline{X}_{2}B_{2} = \overline{E}$, $C_{1}\overline{X}_{1}D_{1} + C_{2}\overline{X}_{2}D_{2} = \overline{F}$, where $\overline{E} = E - A_{1}\widetilde{X}_{1}B_{1} - A_{2}\widetilde{X}_{2}B_{2}$, $\overline{F} = F - C_{1}\widetilde{X}_{1}D_{1} - C_{2}\widetilde{X}_{2}D_{2}$.

The given numerical examples demonstrate that our iterative algorithm is efficient. Especially, when the numbers of the parameter matrices $A_{1}, A_{2}, B_{1}, B_{2}, C_{1}, C_{2}, D_{1}, D_{2}$ are large, our algorithm is efficient as well while the algorithm of [32] is not convergent. That is, our algorithm has merits of good numerical stability and ease to program.

The rest of this paper is outlined as follows. In Section 2, we first propose an efficient iterative algorithm for solving Problem 1; then we give some properties of this iterative algorithm. We show that the algorithm can obtain a solution group (the least Frobenius norm solution group) for any (special) initial matrix group in the absence of roundoff errors. In Section 3, a numerical example is given to illustrate that our algorithm is quite efficient.

2. Iterative Algorithm for Solving Problems 1 and 2

In this section, we present the iterative algorithm for the consistence of the system (1).

Algorithm 3. (1) Input matrices $A_{1} \in R^{p \times k}$, $A_{2} \in R^{p \times m}$, $B_{1} \in R^{r \times q}$, $B_{2} \in R^{n \times q}$, $C_{1} \in R^{s \times k}$, $C_{2} \in R^{s \times m}$, $D_{1} \in R^{r \times t}$, $D_{2} \in R^{n \times t}$, $E \in R^{p \times q}$, $F \in R^{s \times t}$, $X_{1}^{1} \in R^{k \times r}$, and $X_{2}^{1} \in R^{m \times n}$ (where X_{1}^{1}, X_{2}^{1} are any initial matrices).

(2) Calculate

$$E^{1} = E,$$

$$F^{1} = F,$$

$$P_{1}^{1} = A_{1}^{T}E^{1}B_{1}^{T} + C_{1}^{T}F^{1}D_{1}^{T},$$

$$P_{2}^{1} = A_{2}^{T}E^{1}B_{2}^{T} + C_{2}^{T}F^{1}D_{2}^{T},$$

$$\beta^{1} = \left(\text{tr}\left[\left(E^{1}\right)^{T}\left(A_{1}P_{1}^{1}B_{1} + A_{2}P_{2}^{1}B_{2}\right) \right] \right.$$
$$+ \text{tr}\left[\left(F^{1}\right)^{T}\left(C_{1}P_{1}^{1}D_{1} + C_{2}P_{2}^{1}D_{2}\right) \right] \right) \quad (3)$$
$$\times \left(\left\| A_{1}P_{1}^{1}B_{1} + A_{2}P_{2}^{1}B_{2} \right\|^{2} \right.$$
$$+ \left. \left\| C_{1}P_{1}^{1}D_{1} + C_{2}P_{2}^{1}D_{2} \right\|^{2} \right)^{-1},$$

$$\Delta X_{1}^{1} = \beta^{1}P_{1}^{1},$$

$$\Delta X_{2}^{1} = \beta^{1}P_{2}^{1},$$

$$k = 1.$$

(3) If $\Delta X = \text{diag}(\Delta X_{1}^{k}, \Delta X_{2}^{k}) = 0$ $(k = 1, 2, \ldots)$, then stop. Otherwise,

$$X_{1}^{k+1} = X_{1}^{k} + \Delta X_{1}^{k},$$
$$X_{2}^{k+1} = X_{2}^{k} + \Delta X_{2}^{k}. \quad (4)$$

(4) Calculate

$$E^{k+1} = E^k - \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2 \right),$$

$$F^{k+1} = F^k - \left(C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right),$$

$$P_1^{k+1} = A_1^T E^{k+1} B_1^T + C_1^T F^{k+1} D_1^T,$$

$$P_2^{k+1} = A_2^T E^{k+1} B_2^T + C_2^T F^{k+1} D_2^T,$$

$$\beta^{k+1} = \left(\text{tr} \left[\left(E^{k+1} \right)^T \left(A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2 \right) \right] \right.$$

$$+ \text{tr} \left[\left(F^{k+1} \right)^T \left(C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2 \right) \right] \right) \quad (5)$$

$$\times \left(\left\| A_1 P_1^{k+1} B_1 + A_2 P_2^{k+1} B_2 \right\|^2 \right.$$

$$+ \left. \left\| C_1 P_1^{k+1} D_1 + C_2 P_2^{k+1} D_2 \right\|^2 \right)^{-1},$$

$$\Delta X_1^{k+1} = \beta^{k+1} P_1^{k+1},$$

$$\Delta X_2^{k+1} = \beta^{k+1} P_2^{k+1},$$

$$k = k + 1.$$

Go to (3).

Lemma 4. *In Algorithm 3, the choice of β^k makes $\| \text{diag}(E^{k+1}, F^{k+1}) \|$ reach a minimum and $\text{diag}(E^{k+1}, F^{k+1})$ and $\text{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)$ orthogonal to each other.*

Proof. From Algorithm 3, we have

$$\left\| \text{diag} \left(E^{k+1}, F^{k+1} \right) \right\|^2$$

$$= \left\| \text{diag} \left(E^k - \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2 \right), \right. \right.$$

$$\left. \left. F^k - \left(C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right) \right) \right\|^2$$

$$= \left\| \text{diag} \left(E^k - \left(A_1 \beta^k P_1^k B_1 + A_2 \beta^k P_2^k B_2 \right), \right. \right.$$

$$\left. \left. F^k - \left(C_1 \beta^k P_1^k D_1 + C_2 \beta^k P_2^k D_2 \right) \right) \right\|^2$$

$$= \left\| E^k - \left(A_1 \beta^k P_1^k B_1 + A_2 \beta^k P_2^k B_2 \right) \right\|^2$$

$$+ \left\| F^k - \left(C_1 \beta^k P_1^k D_1 + C_2 \beta^k P_2^k D_2 \right) \right\|^2 \quad (6)$$

$$= \left\| E^k \right\|^2 + \left\| F^k \right\|^2$$

$$- 2 \left[\text{tr} \left(E^k, A_1 P_1^k B_1 + A_2 P_2^k B_2 \right) \right.$$

$$\left. + \text{tr} \left(F^k, C_1 P_1^k D_1 + C_2 P_2^k D_2 \right) \right] \beta^k$$

$$+ \left[\left\| A_1 P_1^k B_1 + A_2 P_2^k B_2 \right\|^2 \right.$$

$$\left. + \left\| C_1 P_1^k D_1 + C_2 P_2^k D_2 \right\|^2 \right] \left(\beta^k \right)^2.$$

From the above, the condition of $\| \text{diag}(E^{k+1}, F^{k+1}) \|$ reaching a minimum is

$$\beta^k = \left(\text{tr} \left(E^k, A_1 P_1^k B_1 + A_2 P_2^k B_2 \right) \right.$$

$$+ \text{tr} \left(F^k, C_1 P_1^k D_1 + C_2 P_2^k D_2 \right) \right)$$

$$\times \left(\left\| A_1 P_1^k B_1 + A_2 P_2^k B_2 \right\|^2 \right. \quad (7)$$

$$+ \left. \left\| C_1 P_1^k D_1 + C_2 P_2^k D_2 \right\|^2 \right)^{-1}.$$

On the other hand, if the choice of β^k makes $\text{diag}(E^{k+1}, F^{k+1})$ and $\text{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)$ orthogonal to each other, that is, $\text{tr}[\text{diag}(E^{k+1}, F^{k+1})^T \text{diag}(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2)] = 0$, we can have the same β^k as (7). □

Theorem 5. *Algorithm 3 is bound to be convergent.*

Proof. From Algorithm 3 and Lemma 4 we have

$$\text{diag} \left(E^k, F^k \right) = \text{diag} \left(E^{k+1}, F^{k+1} \right)$$

$$+ \text{diag} \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, \right.$$

$$\left. C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right),$$

$$\Longleftrightarrow \left\| \text{diag} \left(E^k, F^k \right) \right\|^2$$

$$= \left\| \text{diag} \left(E^{k+1}, F^{k+1} \right) + \text{diag} \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, \right. \right.$$

$$\left. \left. C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right) \right\|^2$$

$$= \left\| \text{diag} \left(E^{k+1}, F^{k+1} \right) \right\|^2$$

$$+ \left\| \text{diag} \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, \right. \right.$$

$$\left. \left. C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right) \right\|^2$$

(8)

such that

$$\left\| \text{diag} \left(E^{k+1}, F^{k+1} \right) \right\|^2$$

$$= \left\| \text{diag} \left(E^k, F^k \right) \right\|^2$$

$$- \left\| \text{diag} \left(A_1 \Delta X_1^k B_1 + A_2 \Delta X_2^k B_2, \right. \right. \quad (9)$$

$$\left. \left. C_1 \Delta X_1^k D_1 + C_2 \Delta X_2^k D_2 \right) \right\|^2$$

$$\leq \left\| \text{diag} \left(E^k, F^k \right) \right\|^2.$$

From (9), we know that Algorithm 3 is convergent. □

Lemma 6 (see [41]). *Suppose that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$; then y_0 is the least Frobenius norm solution of the system of linear equations.*

Theorem 7. *Assume that the system* (1) *is consistent. Let* $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T$ *be initial matrices where* $Y \in R^{p \times q}$, $Z \in R^{s \times t}$ *are any initial matrices, or, especially,* $X_1^1 = 0$, $X_2^1 = 0$; *then the solution generated by Algorithm 3 is the least Frobenius norm solution to* (1).

Proof. If (1) is consistent, from $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T$, using Algorithm 3, we have the iterative solution pair X_1^k, X_2^k of (1) as the following:

$$
\begin{aligned}
X_1^k &= X_1^{k-1} + \Delta X_1^{k-1} \\
&= X_1^1 + \Delta X_1^1 + \cdots + \Delta X_1^{k-1} \\
&= A_1^T Y B_1^T + C_1^T Z D_1^T \\
&\quad + A_1^T \left[\beta^1 E^1 + \cdots + \beta^{k-1} E^{k-1} \right] B_1^T \\
&\quad + C_1^T \left[\beta^1 F^1 + \cdots + \beta^{k-1} F^{k-1} \right] D_1^T \\
&= A_1^T M B_1^T + C_1^T N D_1^T, \\
X_2^k &= X_2^{k-1} + \Delta X_2^{k-1} \\
&= X_2^1 + \Delta X_2^1 + \cdots + \Delta X_1^{k-1} \\
&= A_2^T Y B_2^T + C_2^T Z D_2^T \\
&\quad + A_2^T \left[\beta^1 E^1 + \cdots + \beta^{k-1} E^{k-1} \right] B_2^T \\
&\quad + C_2^T \left[\beta^1 F^1 + \cdots + \beta^{k-1} F^{k-1} \right] D_2^T \\
&= A_2^T M B_2^T + C_2^T N D_2^T.
\end{aligned}
\tag{10}
$$

We know that (1) is equivalent to the system

$$
\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ D_1^T \otimes C_1 & D_2^T \otimes C_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{pmatrix} = \begin{pmatrix} \mathrm{vec}(E) \\ \mathrm{vec}(F) \end{pmatrix}. \tag{11}
$$

From (10) and (11) we have

$$
\begin{aligned}
& \begin{pmatrix} \mathrm{vec}\left(A_1^T M B_1^T + C_1^T N D_1^T \right) \\ \mathrm{vec}\left(A_2^T M B_2^T + C_2^T N D_2^T \right) \end{pmatrix} \\
&= \begin{pmatrix} B_1 \otimes A_1^T & D_1 \otimes C_1^T \\ B_2 \otimes A_2^T & D_2 \otimes C_2^T \end{pmatrix} \begin{pmatrix} \mathrm{vec}(M) \\ \mathrm{vec}(N) \end{pmatrix} \\
&= \begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ D_1^T \otimes C_1 & D_2^T \otimes C_2 \end{pmatrix}^T \\
&\quad \times \begin{pmatrix} \mathrm{vec}(M) \\ \mathrm{vec}(N) \end{pmatrix} \in R\left(\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 \\ D_1^T \otimes C_1 & D_2^T \otimes C_2 \end{pmatrix}^T \right),
\end{aligned}
\tag{12}
$$

where $R(*)$ is the column space of matrix $*$.

Considering Lemma 6, with the initial matrices $X_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T$, $X_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T$, where $Y \in R^{p \times q}$,

$Z \in R^{s \times t}$ are arbitrary, or, especially, $X_1^1 = 0$ and $X_2^1 = 0$, then the solution pair X_1^k, X_2^k generated by Algorithm 3 is the least Frobenius norm solution of the matrix equations (1). □

Suppose that Problem 1 is consistent. Obviously the solution set S of (1) is nonempty. For given matrices pair $\widetilde{X}_1 \in R^{k \times r}$, $\widetilde{X}_2 \in R^{m \times n}$, we can write

$$
\begin{aligned}
& A_1 X_1 B_1 + A_2 X_2 B_2 = E, \\
& C_1 X_1 D_1 + C_2 X_2 D_2 = F, \\
& \Longleftrightarrow \begin{cases} A_1 \left(X_1 - \widetilde{X}_1 \right) B_1 + A_2 \left(X_2 - \widetilde{X}_2 \right) B_2 \\ \quad = E - A_1 \widetilde{X}_1 B_1 - A_2 \widetilde{X}_2 B_2, \\ C_1 \left(X_1 - \widetilde{X}_1 \right) D_1 + C_2 \left(X_2 - \widetilde{X}_2 \right) D_2 \\ \quad = F - C_1 \widetilde{X}_1 D_1 - C_2 \widetilde{X}_2 D_2. \end{cases}
\end{aligned}
\tag{13}
$$

Let $\overline{X}_1 = X_1 - \widetilde{X}_1$, $\overline{X}_2 = X_2 - \widetilde{X}_2$, $\overline{E} = E - A_1 \widetilde{X}_1 B_1 - A_2 \widetilde{X}_2 B_2$, and $\overline{F} = F - C_1 \widetilde{X}_1 D_1 - C_2 \widetilde{X}_2 D_2$. Then Problem 2 is equivalent to find the least Frobenius norm solution pair of the system

$$
\begin{aligned}
& A_1 \overline{X}_1 B_1 + A_2 \overline{X}_2 B_2 = \overline{E}, \\
& C_1 \overline{X}_1 D_1 + C_2 \overline{X}_2 D_2 = \overline{F},
\end{aligned}
\tag{14}
$$

which can be obtained using Algorithm 3 with the initial matrix pair $\overline{X}_1^1 = A_1^T Y B_1^T + C_1^T Z D_1^T$, $\overline{X}_2^1 = A_2^T Y B_2^T + C_2^T Z D_2^T$ where $Y \in R^{p \times q}$ and $Z \in R^{s \times t}$ are arbitrary, or especially, $\overline{X}_1^1 = 0$, $\overline{X}_2^1 = 0$, and the solution of the matrix optimal approximation Problem 2 can be represented as $\widehat{X}_1 = \overline{X}_1^k + \widetilde{X}_1$, $\widehat{X}_2 = \overline{X}_2^k + \widetilde{X}_2$.

3. An Example

In this section, we show a numerical example to illustrate the efficiency of Algorithm 3. All computations are performed by MATLAB 7. For the influence of the error of calculation, we consider the matrix R as a zero matrix if $\|R\| < 10^{-10}$.

Example 1. Consider the solution of the linear matrix equations

$$
\begin{aligned}
& A_1 X_1 B_1 + A_2 X_2 B_2 = E, \\
& C_1 X_1 D_1 + C_2 X_2 D_2 = F,
\end{aligned}
\tag{15}
$$

where

$$
A_1 = \begin{pmatrix} 139 & 105 & 54 \\ 124 & 176 & 50 \\ 159 & 35 & 175 \\ 191 & 196 & 147 \end{pmatrix},
$$

$$
B_1 = \begin{pmatrix} 13 & 117 & 103 & 87 & 116 \\ 198 & 85 & 67 & 45 & 152 \end{pmatrix},
$$

$$
A_2 = \begin{pmatrix} 27 & 60 \\ 2 & 132 \\ 179 & 57 \\ 40 & 94 \end{pmatrix},
$$

$$B_2 = \begin{pmatrix} 106 & 76 & 92 & 12 & 83 \\ 128 & 157 & 114 & 121 & 61 \\ 42 & 136 & 159 & 10 & 175 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 3 & 88 & 192 \\ 154 & 100 & 145 \\ 194 & 43 & 82 \\ 198 & 129 & 149 \\ 158 & 64 & 54 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 88 & 31 & 140 \\ 71 & 135 & 146 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 4 & 63 \\ 115 & 3 \\ 90 & 77 \\ 9 & 137 \\ 5 & 19 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 96 & 90 & 55 \\ 111 & 143 & 51 \\ 24 & 179 & 173 \end{pmatrix},$$

$$E = \begin{pmatrix} 8433077 & 7598166 & 7137224 & 3600960 & 9942247 \\ 11267058 & 9622068 & 9126164 & 4374104 & 12443246 \\ 14871294 & 16123511 & 15980470 & 6538271 & 20550138 \\ 16234322 & 14317981 & 13168541 & 7214805 & 18956522 \end{pmatrix},$$

$$F = \begin{pmatrix} 8104691 & 10054438 & 13979223 \\ 8672308 & 12546367 & 16384832 \\ 7666584 & 11382258 & 13330665 \\ 10613517 & 13831240 & 17573398 \\ 4111503 & 5032847 & 7285952 \end{pmatrix}.$$

$$(16)$$

In this example, the numbers of the parameter matrices $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are larger than the numbers of the parameter matrices in the example of [32]. It can be verified that these matrix equations are consistent and have the solution as

$$X_1 = \begin{pmatrix} 53 & 48 \\ 32 & 129 \\ 175 & 193 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 133 & 2 & 164 \\ 174 & 27 & 86 \end{pmatrix}. \qquad (17)$$

Let

$$X_1^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (18)$$

(1) Using Algorithm 3 and iterate 10309 steps, we obtain the least Frobenius norm solution pair of the matrix equation in Example 1 as follows:

$$X_1 = \begin{pmatrix} 53.0000 & 48.0000 \\ 32.0000 & 129.0000 \\ 175.0000 & 193.0000 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 133.0000 & 2.0000 & 164.0000 \\ 174.0000 & 27.0000 & 86.0000 \end{pmatrix}. \qquad (19)$$

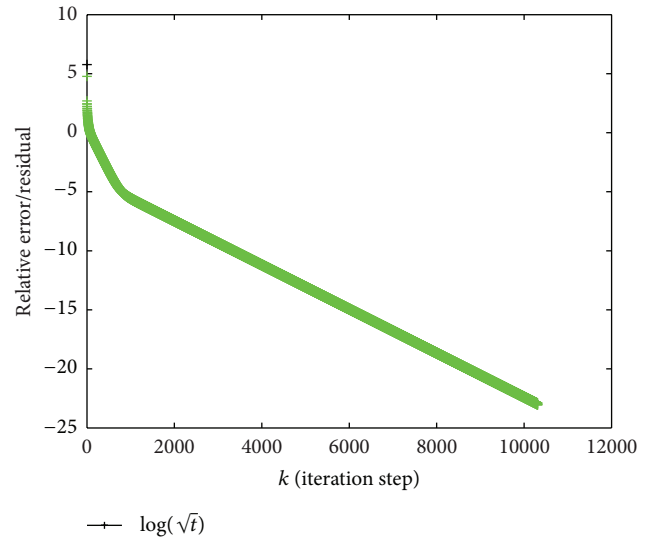The obtained sequence $\|\Delta X\|$ are presented in Figure 1.

FIGURE 1: The obtained sequence $\|\Delta X\|$ by Algorithm 3 for Example 1.
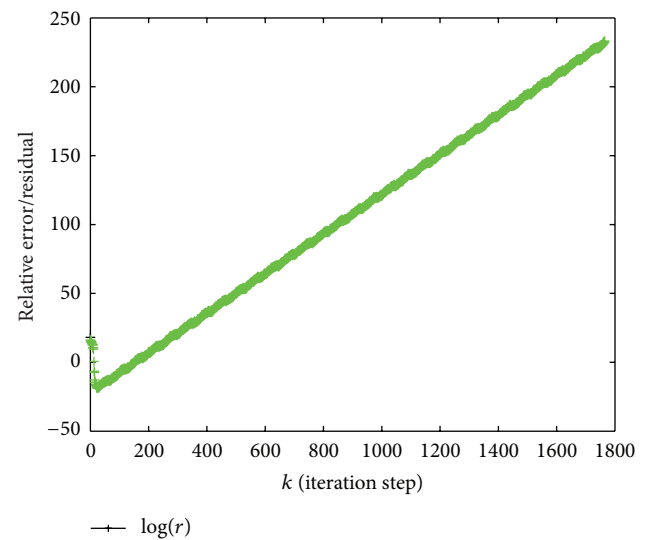

FIGURE 2: The relative error of the solution and the residual by the algorithm of [32] for Example 1.

(2) Using the algorithm of [32], to this example, the iteration is not convergent. The obtained result is presented in Figure 2.

This numerical example demonstrates that our algorithm has merits of good numerical stability and ease to program.

Acknowledgments

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