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Research Article

Dynamic Properties of the Fractional-Order Logistic Equation of Complex Variables

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We study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the continuous dynamical system of the logistic equation of complex variables. The existence and uniqueness of uniformly Lyapunov stable solution will be proved.

1. Introduction

Dynamical properties and chaos synchronization of deterministic nonlinear systems have been intensively studied over the last two decades on a large number of real dynamical systems of physical nature (i.e., those that involve real variables). However, there are also many interesting cases involving complex variables. As an example, we mention here the complex Lorenz equations, complex Chen and Lü chaotic systems, and some others (see [1–8] and the references therein).

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only among mathematicians, but also among physicists and engineers (see [9–16] and references therein).

Consider the following fractional-order Logistic equation of complex variables:

$$D^{\alpha}z(t) = \rho z(t)(1 - z(t)) = \rho z(t) - \rho z^2(t), \quad t > 0, \tag{1.1}$$

$$z(0) = z_o = x_o + iy_o, \tag{1.2}$$

where

$$z(t) = x(t) + iy(t), \quad |z(t)| \le 1,$$
$$\rho = a + ib, \quad a, b > 0. \tag{1.3}$$

Here we study the dynamic properties (equilibrium points, local and global stability, chaos and bifurcation) of the continuous dynamical system of complex variables (1.1)-(1.2). The the existence of a unique uniformly stable solution and the continuous dependence of the solution on the initial data (1.2) are also proved.

Now we give the definition of fractional-order integration and fractional-order differentiation.

Definition 1.1. The fractional integral of order $\beta \in R^+$ of the function $f(t), t \in I$ is

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \tag{1.4}$$

and the Caputo's definition for the fractional order derivative of order $\alpha \in (0,1]$ of $f(t)$ is given by

$$D^\alpha f(t) = I^{1-\alpha} \frac{d}{dt} f(t). \tag{1.5}$$

2. Existence and Uniqueness

The following lemma (formulation of the problem) can be easily proved.

Lemma 2.1. *The discontinuous dynamical system* (1.1)-(1.2) *can be transformed to the system*

$$D^\alpha x(t) = ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t), \quad t > 0, \tag{2.1}$$

$$D^\alpha y(t) = bx(t) + ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t), \quad t > 0, \tag{2.2}$$

with the initial values

$$x(0) = x_o, \quad y(0) = y_o, \tag{2.3}$$

where $|x(t)| \le 1$ *and* $|y(t)| \le 1$.

Let $C[0,T]$ *be the class of continuous functions defined on* $[0,T]$.

Let Y *be the class of columns vectors* $(x(t), y(t))^\tau$, $x, y \in C[0,T]$ *with the norm*

$$\left\|(x,y)^\tau\right\|_Y = \|x\| + \|y\| = \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |y(t)|. \tag{2.4}$$

Let X be the class of columns vectors $(x(t), y(t))^{\tau},\ x, y \in C[0,T]$ *with the equivalent norm*

$$\|(x,y)^{\tau}\|_X = \|x\|^* + \|y\|^* = \sup_{t\in[0,T]} e^{-Nt}|x(t)| + \sup_{t\in[0,T]} e^{-Nt}|y(t)|, \quad N > 0. \tag{2.5}$$

Write the problem (2.1)-(2.3) *in the following matrix form:*

$$\begin{aligned} D^{\alpha}(x,y)^{\tau} = \Big(&ax(t) - by(t) - a\Big(x^2(t) - y^2(t)\Big) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\Big(x^2(t) - y^2(t)\Big)\ - 2ax(t)y(t)\Big)^{\tau}, \end{aligned} \tag{2.6}$$

and

$$(x(0), y(0))^{\tau} = (x_o, y_o)^{\tau}, \tag{2.7}$$

where τ *is the transpose of the matrix.*

Now we have the following theorem.

Theorem 2.2. *The problem* (2.6)-(2.7) *has a unique solution* $(x, y) \in X$.

Proof. Integrating (2.6) α-times we obtain

$$\begin{aligned} (x(t), y(t))^{\tau} = (x(0), y(0))^{\tau} + I^{\alpha}\Big(&ax(t) - by(t) - a\Big(x^2(t) - y^2(t)\Big) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\Big(x^2(t) - y^2(t)\Big) - 2ax(t)y(t)\Big)^{\tau}. \end{aligned} \tag{2.8}$$

Define the operator $F : X \to X$ by

$$\begin{aligned} F(x(t), y(t))^{\tau} = (x(0), y(0))^{\tau} + I^{\alpha}\Big(&ax(t) - by(t) - a\Big(x^2(t) - y^2(t)\Big) + 2bx(t)y(t),\ bx(t) \\ &+ay(t) - b\Big(x^2(t) - y^2(t)\Big) - 2ax(t)y(t)\Big)^{\tau}, \end{aligned} \tag{2.9}$$

then by direct calculations, we can get

$$\|F(x,y) - F(u,v)^{\tau}\|_X \le K\|(x,y) - (u,v)^{\tau}\|_X, \tag{2.10}$$

where

$$K = 5(a+b)\frac{1}{N^{\alpha}}. \tag{2.11}$$

Choose N large enough we find that $K < 1$ and by the contraction fixed theorem [17] the problem (2.6)-(2.7) has a unique solution $(x, y) \in X$.

From the continuity of the solution we deduce that (see [10])

$$I^{\alpha}\Big(ax(t)-by(t)-a\Big(x^2(t)-y^2(t)\Big)+2bx(t)y(t),\ bx(t) \\ +ay(t)-b\Big(x^2(t)-y^2(t)\Big)-2ax(t)y(t)\Big)^{\tau}\Big|_{t=0}=0, \tag{2.12}$$

then the solution satisfies the initial condition. Differentiating (2.8), then by the same way as in ([18, 19]), we deduce that the integral equation (2.8) satisfies the problem (2.6)-(2.7) which completes the proof. □

3. Uniform Stability

Theorem 3.1. *The solution of the problem* (2.6)–(2.7) *is uniformly stable in the sense that*

$$|x_o-x_o^*|+|y_o-y_o^*|\le\delta \Longrightarrow \|(x,y)-(x^*,y^*)\|_X\le\epsilon, \tag{3.1}$$

where $(x^*(t),y^*(t))$ *is the solution of the differential equation* (2.6) *with the initial data*

$$(x(0),y(0))^{\tau}=(x_o^*,y_o^*)^{\tau}. \tag{3.2}$$

Proof. Direct calculations give

$$\|(x,y)-(x^*,y^*)^{\tau}\|_X\le|x_o-x_o^*|+|y_o-y_o^*|+K\|(x,y)-(x^*,y^*)^{\tau}\|_X, \tag{3.3}$$

which implies that

$$\|(x,y)-((x^*,y^*)^{\tau}\|_X\le(1-K)^{-1}(|x_o-x_o^*|+|y_o-y_o^*|)\le\epsilon, \tag{3.4}$$

$$\epsilon=(1-K)^{-1}\delta. \tag{3.5}$$

□

4. Equilibrium Points and Their Asymptotic Stability

Let $\alpha\in(0,1]$ and consider the system ([9, 20–22])

$$D^{\alpha}y_1(t)=f_1(y_1,y_2), \\ D^{\alpha}y_2(t)=f_2(y_1,y_2), \tag{4.1}$$

with the initial values

$$y_1(0)=y_{o1},\ y_2(0)=y_{o2}. \tag{4.2}$$

To evaluate the equilibrium points, let

$$D^{\alpha} y_j(t) = 0 \Longrightarrow f_j\left(y_1^{\text{eq}}, y_2^{\text{eq}}\right) = 0, \quad j = 1, 2, \tag{4.3}$$

from which we can get the equilibrium points $y_1^{\text{eq}}, y_2^{\text{eq}}$.

To evaluate the asymptotic stability, let

$$y_j(t) = y_j^{\text{eq}} + \varepsilon_j(t). \tag{4.4}$$

So the the equilibrium point $(y_1^{\text{eq}}, y_2^{\text{eq}})$ is locally asymptotically stable if both the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{bmatrix} \tag{4.5}$$

evaluated at the equilibrium point satisfies $(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2)$ ([9, 20–23]).

For the fractional-order Logistic equation of complex variables consider the following:

$$\begin{aligned} D^{\alpha} x(t) &= ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t), \quad t > 0, \\ D^{\alpha} y(t) &= bx(t) + ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t), \quad t > 0. \end{aligned} \tag{4.6}$$

To evaluate the equilibrium points, let

$$\begin{aligned} D^{\alpha} x &= 0, \\ D^{\alpha} y &= 0, \end{aligned} \tag{4.7}$$

then $(x_{\text{eq}}, y_{\text{eq}}) = (0,0), (1,0)$, are the equilibrium points.

For $(x_{\text{eq}}, y_{\text{eq}}) = (0,0)$ we find that

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{4.8}$$

its eigenvalues are

$$\lambda = a \mp bi. \tag{4.9}$$

A sufficient condition for the local asymptotic stability of the equilibrium point $(0,0)$ is

$$\left|\arg(\lambda_1)\right| > \frac{\alpha\pi}{2}, \quad \left|\arg(\lambda_2)\right| > \frac{\alpha\pi}{2}, \quad 0 < \alpha < 1, \tag{4.10}$$

that is,

$$\frac{b}{a} > \tan\left(\frac{\alpha\pi}{2}\right) \tag{4.11}$$

and x_0 is small.

For $(x_{\text{eq}}, y_{\text{eq}}) = (1, 0)$ we find that

$$A = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} \tag{4.12}$$

its eigenvalues are

$$\lambda = -a \pm bi. \tag{4.13}$$

A sufficient condition for the local asymptotic stability of the equilibrium point $(1, 0)$ is $a > 0$ and x_0 is not close to zero.

5. Numerical Methods and Results

An Adams-type predictor-corrector method has been introduced and investigated further in ([24–26]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (2.1)-(2.2) by an equivalent fractional integral equations

$$\begin{aligned} x(t) &= x(0) + I^{\alpha}\left[ax(t) - by(t) - a\left(x^2(t) - y^2(t)\right) + 2bx(t)y(t)\right], \\ y(t) &= y(0) + I^{\alpha}\left[bx(t) + ay(t) - b\left(x^2(t) - y^2(t)\right) - 2ax(t)y(t)\right], \end{aligned} \tag{5.1}$$

and then apply the PECE (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions displayed in Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 for different $0 < \alpha \leq 1$. In Figures 1–4 we take $x(0) = 0.1$, $y(0) = 0.9, a = 0.1, b = 0.9$ and found that the equilibrium point $(0, 0)$ is local asymptotic stable for $\alpha = 0.8, 0.9$ because the condition $b/a > \tan(\alpha\pi/2)$ is satisfied and the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 1.0$. In Figures 5–8 we take $x(0) = 0.2$, $y(0) = 0.7, a = 0.1, b = 0.5$ and found that the equilibrium point $(0, 0)$ is local asymptotic stable for $\alpha = 0.8$ because the condition $b/a > \tan(\alpha\pi/2)$ is satisfied and the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 0.9, 1.0$. In Figures 9–12 we take $x(0) = 0.5$, $y(0) = 0.5, a = 0.1, b = 0.4$ and found that the equilibrium point $(1, 0)$ is local asymptotic stable for $\alpha = 0.8, 0.9, 1.0$.

6. Conclusions

In this paper we considered the fractional-order Logistic equations of complex variables. Here we studied the dynamic properties (equilibrium points, local and global stability, chaos

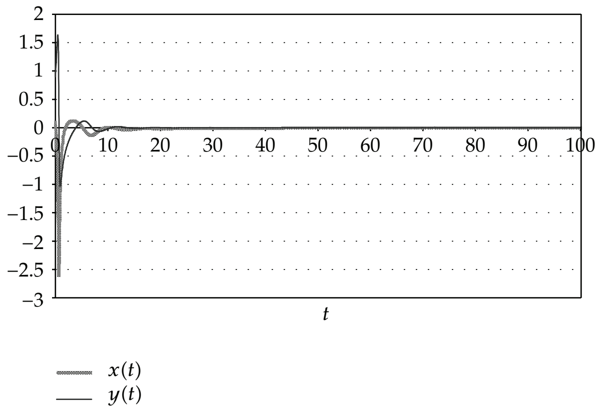

Figure 1: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 0.8.

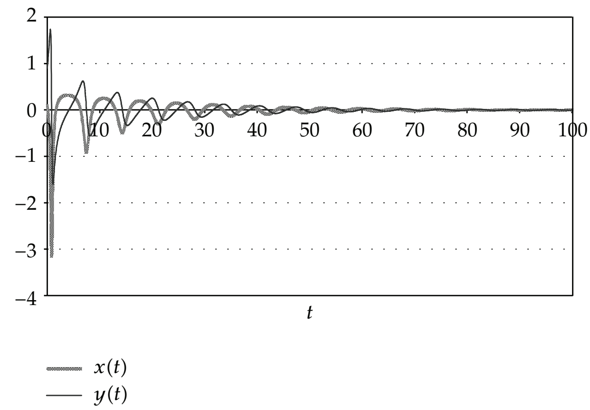

Figure 2: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 0.9.

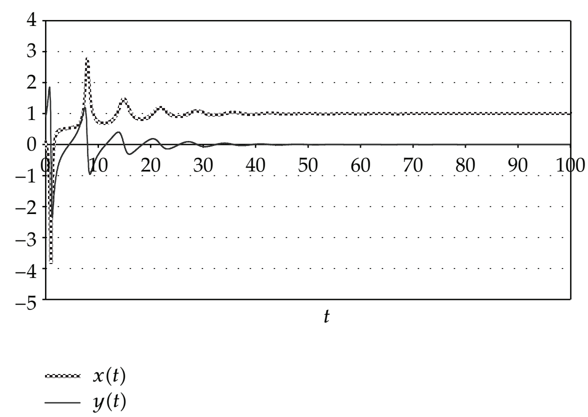

Figure 3: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$, alpha = 1.0.

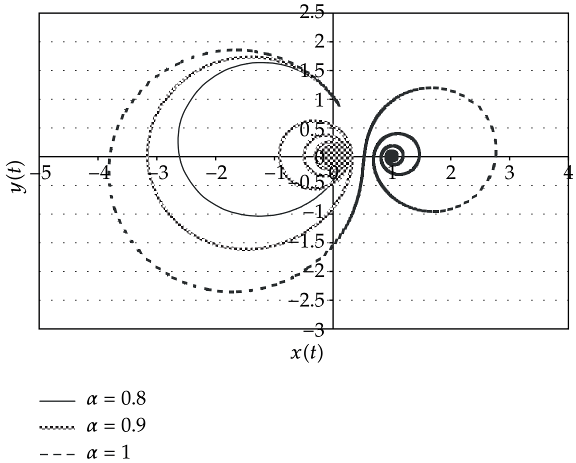

Figure 4: $x(0) = 0.1$, $y(0) = 0.9$, $a = 0.1$, $b = 0.9$.

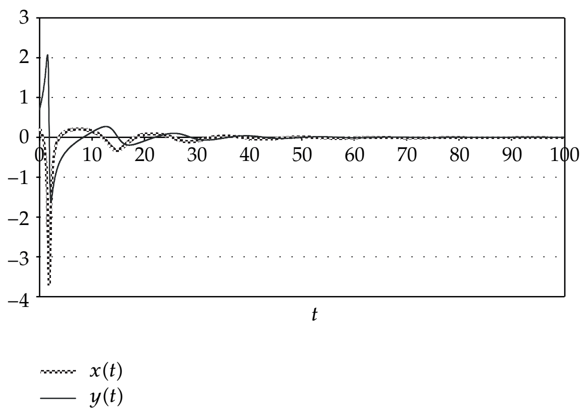

Figure 5: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 0.8.

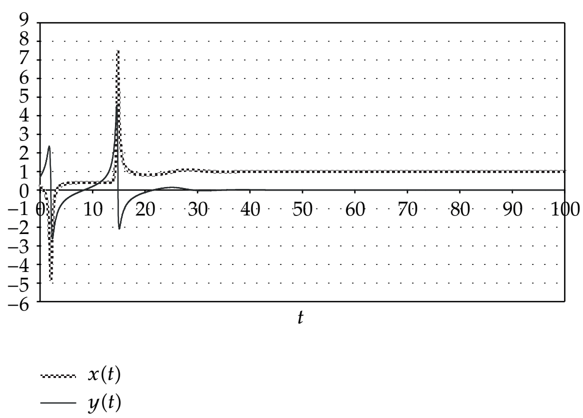

Figure 6: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 0.9.

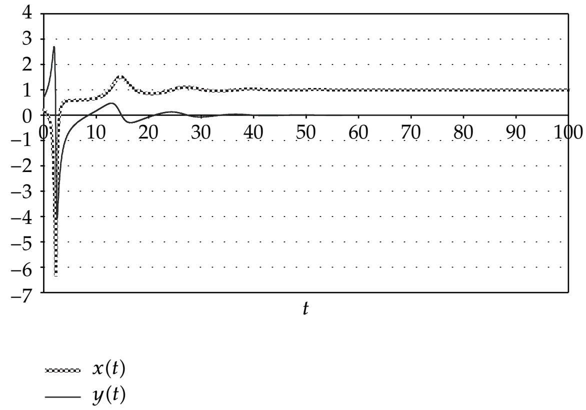

Figure 7: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$, alpha = 1.0.

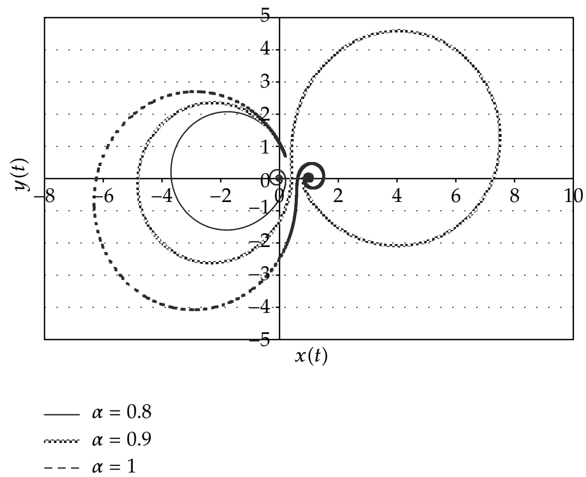

Figure 8: $x(0) = 0.2$, $y(0) = 0.7$, $a = 0.1$, $b = 0.5$.

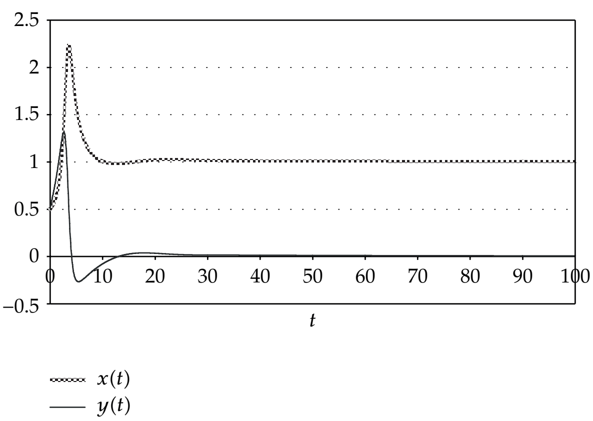

Figure 9: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 0.8.

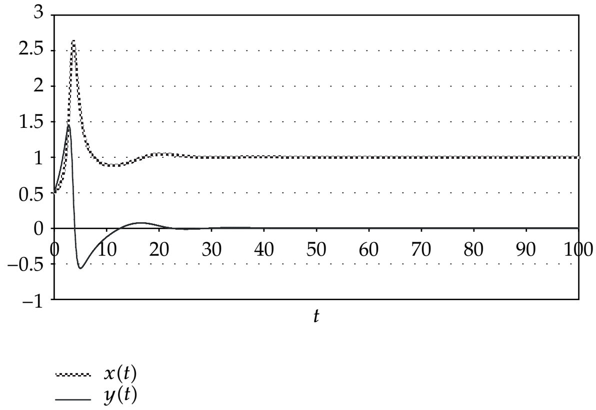

Figure 10: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 0.9.

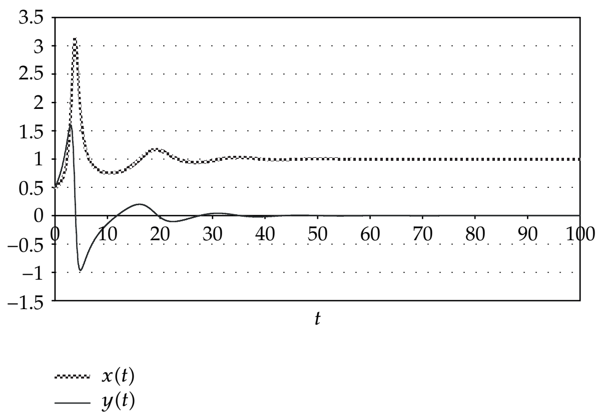

Figure 11: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$, alpha = 1.0.

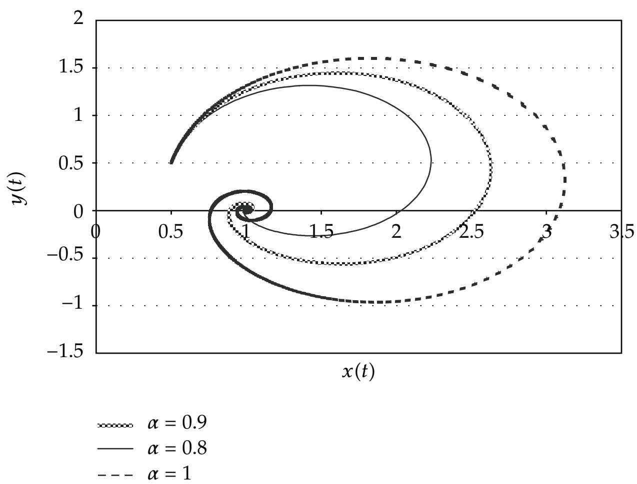

Figure 12: $x(0) = 0.5$, $y(0) = 0.5$, $a = 0.1$, $b = 0.4$.

and bifurcation). The existence of a unique uniformly stable solution and the continuous dependence of the solution on the initial data (1.2) are also proved. Also we studied the numerical solution of the system (1.1)-(1.2).

We like to argue that fractional-order equations are more suitable than integer-order ones in modeling biological, economic, and social systems (generally complex adaptive systems) where memory effects are important.

References

[1] J. D. Gibbon and M. J. McGuinness, "The real and complex Lorenz equations in rotating fluids and lasers," *Physica D*, vol. 5, no. 1, pp. 108–122, 1982.

[2] G. M. Mahmoud, S. A. Aly, and A. A. Farghaly, "On chaos synchronization of a complex two coupled dynamos system," *Chaos, Solitons and Fractals*, vol. 33, no. 1, pp. 178–187, 2007.

[3] G. M. Mahmoud, S. A. Aly, and M. A. AL-Kashif, "Dynamical properties and chaos synchronization of a new chaotic complex nonlinear system," *Nonlinear Dynamics*, vol. 51, no. 1-2, pp. 171–181, 2008.

[4] G. M. Mahmoud, T. Bountis, G. M. AbdEl-Latif, and E. E. Mahmoud, "Chaos synchronization of two different chaotic complex Chen and Lü systems," *Nonlinear Dynamics*, vol. 55, no. 1-2, pp. 43–53, 2009.

[5] C. Z. Ning and H. Haken, "Detuned lasers and the complex Lorenz equations: subcritical and supercritical Hopf bifurcations," *Physical Review A*, vol. 41, no. 7, pp. 3826–3837, 1990.

[6] A. Rauh, L. Hannibal, and N. B. Abraham, "Global stability properties of the complex Lorenz model," *Physica D*, vol. 99, no. 1, pp. 45–58, 1996.

[7] Y. Xu, W. Xu, and G.M. Mahmoud, "On a complex beam-beam interaction model with random forcing," *Physica A*, vol. 336, no. 3-4, pp. 347–360, 2004.

[8] Y. Xu, W. Xu, and G.M. Mahmoud, "On a complex Duffing system with random excitation," *Chaos, Solitons and Fractals*, vol. 35, no. 1, pp. 126–132, 2008.

[9] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, "Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 542–553, 2007.

[10] A. M. A. El-Sayed, F. M. Gaafar, and H. H. G. Hashem, "On the maximal and minimal solutions of arbitrary-orders nonlinear functional integral and differential equations," *Mathematical Sciences Research Journal*, vol. 8, no. 11, pp. 336–348, 2004.

[11] A. M. A. El-Sayed, A. E. M. El-Mesiry, and H. A. A. El-Saka, "On the fractional-order logistic equation," *Applied Mathematics Letters*, vol. 20, no. 7, pp. 817–823, 2007.

[12] R. Gorenflo and F. Mainardi, "Fractional Calculus: Integral and Differential Equations of Fractional Order," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., pp. 223–276, Springer, Wien, Austria, 1997.

[13] I. Podlubny and A. M. A. El-Sayed, On Two Definitions of Fractional Calculus, Slovak Academy of Science-Institute of Experimental Physics, 1996.

[14] I. Podlubny, *Fractional Differential Equations*, Academic Press Inc., San Diego, Calif, USA, 1999.

[15] Y. Suansook and K. Paithoonwattanakij, "Chaos in fractional order logistic model," in *Proceedings of the International Conference on Signal Processing Systems*, pp. 297–301, May 2009.

[16] I. Petras, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Nonlinear Physical Science, Springer, New York, NY, USA, 2011.

[17] R. F. Curtain and A. J. Pritchard, *Functional Analysis in Modern Applied Mathematics*, Academic Press, London, UK, 1977.

[18] A. M. A. El-Sayed and Sh. A. Abd El-Salam, "On the stability of a fractional-order differential equation with nonlocal initial condition," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 29, pp. 1–8, 2008.

[19] A. M. A. El-Sayed, "On the existence and stability of positive solution for a nonlinear fractional-order differential equation and some applications," *Alexandria Journal of Mathematics*, vol. 1, no. 1, 2010.

[20] E. Ahmed, A. M. A. El-Sayed, E. M. El-Mesiry, and H. A. A. El-Saka, "Numerical solution for the fractional replicator equation," *International Journal of Modern Physics C*, vol. 16, no. 7, pp. 1017–1025, 2005.

[21] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka, "On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems," *Physics Letters A*, vol. 358, no. 1, pp. 1–4, 2006.

[22] H. A. El-Saka, E. Ahmed, M. I. Shehata, and A. M. A. El-Sayed, "On stability, persistence, and Hopf bifurcation in fractional order dynamical systems," *Nonlinear Dynamics*, vol. 56, no. 1-2, pp. 121–126, 2009.

[23] D. Matignon, "Stability results for fractional differential equations with applications to control processing," in *Proceedings of the Computational Engineering in Systems and Application multiconference*, vol. 2, pp. 963–968, Lille, France, 1996.

[24] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application- Oriented Exposition Using Differential Operators of Caputo Type (Lecture Notes in Mathematics)*, Springer-Verlag, Berlin, Germany, 2010.

[25] A. M. A. El-Sayed, A. E. M. El-Mesiry, and H. A. A. El-Saka, "Numerical solution for multi-term fractional (arbitrary) orders differential equations," *Computational & Applied Mathematics*, vol. 23, no. 1, pp. 33–54, 2004.

[26] A. E. M. El-Mesiry, A. M. A. El-Sayed, and H. A. A. El-Saka, "Numerical methods for multi-term fractional (arbitrary) orders differential equations," *Applied Mathematics and Computation*, vol. 160, no. 3, pp. 683–699, 2005.