

Research Article

Reflective Full Subcategories of the Category of L -Posets

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This paper focuses on the relationship between L -posets and complete L -lattices from the categorical view. By considering a special class of fuzzy closure operators, we prove that the category of complete L -lattices is a reflective full subcategory of the category of L -posets with appropriate morphisms. Moreover, we characterize the Dedekind-MacNeille completions of L -posets and provide an equivalent description for them.

1. Introduction

Fuzzy order was originated from the literature of Zadeh [1] and later was broadly studied. A fuzzy ordered set (an L -poset) is the fundamental idea of fuzzy ordered structures. Some people studied it from the fuzzy set theory such as [2–8], and others considered it as a category (see [9–11]). All these works enriched the theories of fuzzy orders and fuzzy ordered structures. At present, we are about to consider fuzzy ordered sets as objects in the framework of category. As we all know that the researches on category D can be reflected on its reflective full subcategories whose structures are better than D . Searching for such subcategories provides a new approach to study primary category.

In recent years, the completion theory for fuzzy ordered sets attracts much attention. Wagner [9] introduced the enriched Dedekind-MacNeille completion for a category enriched over a commutative unital quantale. Bělohlávek [12] described the Dedekind-MacNeille completion for an L -ordered set as an application of the theory of concept lattices in the fuzzy setting. Xie et al. [13] built and characterized the Dedekind-MacNeille completions for L -posets. In [14], Wang and Zhao proposed join-completions for L -ordered sets and investigated their characterizations. It was shown that each join-completion is in bijective correspondence with a consistent fuzzy closure operator C , and it has a universal property

with respect to C -homomorphisms. It also mentioned that the relationship between the categories of L -posets and complete L -lattices will be of concern in the future. This problem is closely related to the appropriate morphisms chosen, respectively. How about when C -homomorphisms endowed on L -posets and fuzzy-join-preserving mappings are chosen for complete L -lattices? This is our first motivation for this paper. After analysis, we discover that the latter category is a reflective full subcategory of the former with respect to a special class of fuzzy closure operators, but not for any one (a counterexample is given). The special class of fuzzy closure operators is singled out in this paper. Moreover, when a completion has a universal property, we can declare that it is a join-completion. But there are so many join-completions, thus which type is it? This is the other motivation of this paper. We prove that it is exactly the Dedekind-MacNeille completion up to isomorphism.

This paper is organized as follows. In Section 2, we list some basic definitions and well-known results for fuzzy order theory. In Section 3, we discuss the relationship between C -homomorphisms and fuzzy-join-preserving mappings then single out a special class of fuzzy closure operators such that the category of complete L -lattices is a reflective full subcategory of the category of L -posets. In Section 4, we characterize the Dedekind-MacNeille completions for L -posets and give an equivalent description for them. In Section 5, we summarize all the content and reach a conclusion.

2. Preliminaries

A complete residuated lattice [15] is a structure $(L, *, \rightarrow, \vee, \wedge, 0, 1)$ such that (1) $(L, \vee, \wedge, 0, 1)$ is a complete lattice with the greatest element 1 and least element 0; (2) $(L, *, 1)$ is a commutative monoid with the identity 1, and $*$ is isotone at both arguments; (3) $(*, \rightarrow)$ is an adjoint pair, that is, $x * y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in L$. Usually, we abbreviate $(L, *, \rightarrow, \vee, \wedge, 0, 1)$ by L simply. Some basic properties of complete residuated lattices are collected here ([15–17]).

- (1) $1 \rightarrow a = a$;
- (2) $a \leq b$ if and only if $a \rightarrow b = 1$;
- (3) $(a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c$;
- (4) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c) = (a * b) \rightarrow c$;
- (5) $a \leq b \rightarrow (a * b)$;
- (6) $a * (a \rightarrow b) \leq b$;
- (7) $a * (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a * b_i)$, $a * (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} (a * b_i)$;
- (8) $a \rightarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightarrow b_i)$, $(\bigvee_{i \in I} a_i) \rightarrow b = \bigwedge_{i \in I} (a_i \rightarrow b)$;
- (9) $a \rightarrow b \geq b$;
- (10) $b \rightarrow c \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$, $b \rightarrow c \leq (c \rightarrow a) \rightarrow (b \rightarrow a)$;
- (11) $(a \rightarrow b) \rightarrow b \geq a$;
- (12) $(a \rightarrow b) * (c \rightarrow d) \leq (a * c) \rightarrow (b * d)$.

In this paper, L always denotes a complete residuated lattice unless otherwise stated, and L^X denotes the set of all L -subsets of a nonempty set X . For all $A, B \in L^X$, we define

$$\begin{aligned} (A \cap B)(x) &= A(x) \wedge B(x), & (A \cup B)(x) &= A(x) \vee B(x), \\ (A * B)(x) &= A(x) * B(x), & (A \longrightarrow B)(x) &= A(x) \longrightarrow B(x). \end{aligned} \quad (2.1)$$

Then $(L^X, *, \rightarrow, \vee, \wedge, \bar{0}, \bar{1})$ is also a complete residuated lattice. If no confusion arises, we always do not discriminate the constant value function \bar{a} with a , for example, $(a * A)(x) = a * A(x)$ and $(a \rightarrow A)(x) = a \rightarrow A(x)$ for every $x \in X$.

Fuzzy order was first introduced by Zadeh [1], from then on, different kinds of fuzzy order have been introduced and studied by different authors (the reader is referred to [3, 6, 9, 11, 18–20] for details). In this paper, we adopt the definition of fuzzy order introduced by Fan and Zhang [3, 6].

Definition 2.1. An L -partial order e (also called an L -order) on X is an L -relation satisfying:

- (1) for all $x \in X$, $e(x, x) = 1$;
- (2) for all $x, y, z \in X$, $e(x, y) * e(y, z) \leq e(x, z)$;
- (3) for all $x, y \in X$, $e(x, y) = e(y, x) = 1 \Rightarrow x = y$.

Then (X, e) is called an L -partially ordered set or an L -poset for simplicity.

It is worth noting that Bělohlávek defined another fuzzy order in [12, 16], and it was shown to be equivalent to the above definition by Yao [5].

There are some important L -posets which are mentioned in many papers, such as [3, 4, 19, 21]. For example, in an L -poset (X, e) , define $e^{op}(x, y) = e(y, x)$ for all $x, y \in X$, then (X, e^{op}) is also an L -poset. In a complete residuated lattice L , define $e_L : L \times L \rightarrow L$ by $e_L(x, y) = x \rightarrow y$ for all $x, y \in L$, then (L, e_L) is an L -poset. For all $A, B \in L^X$, define $\text{sub}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$, then (L^X, sub) is an L -poset.

In the following, some basic and very important definitions and results related to the theory of L -poset are listed. The reader is referred to [2–9, 11–13, 20] for details.

Definition 2.2. Let (X, e) be an L -poset, $A \in L^X$, $x_0 \in X$. x_0 is called a fuzzy join (resp., fuzzy meet) of A denoted by $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$) if

- (1) for all $x \in X$, $A(x) \leq e(x, x_0)$ (resp., $A(x) \leq e(x_0, x)$);
- (2) for all $y \in X$, $\bigwedge_{x \in X} (A(x) \rightarrow e(x, y)) \leq e(x_0, y)$ (resp., $\bigwedge_{x \in X} (A(x) \rightarrow e(y, x)) \leq e(y, x_0)$).

It is easy to see that the fuzzy join or the fuzzy meet is unique in an L -poset (X, e) if it exists, and $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$) if and only if $e(x_0, y) = \bigwedge_{x \in X} (A(x) \rightarrow e(x, y))$ (resp., $e(y, x_0) = \bigwedge_{x \in X} (A(x) \rightarrow e(y, x))$) for all $y \in X$. If $\sqcup A$ and $\sqcap A$ exist for all $A \in L^X$, then (X, e) is said to be a complete L -lattice.

In an L -poset (X, e) , for any $x \in X$, we usually define $\downarrow x \in L^X$ as $\downarrow x(y) = e(y, x)$ for each $y \in X$ and dually define $\uparrow x(y) = e(x, y)$. For any $A \in L^X$, it is called an L -lower set if $A(x) * e(y, x) \leq A(y)$ for all $x, y \in X$, or an L -upper set if $A(x) * e(x, y) \leq A(y)$ for all $x, y \in X$. When A is an L -lower set, we have $\text{sub}(\downarrow x, A) = A(x)$, and $\text{sub}(\uparrow x, A) = A(x)$ when A is an L -upper set.

For each map $f : X \rightarrow Y$ from a set X to an L -poset (Y, e) , there exist L -forward powerset operators $f_L^\rightarrow : L^X \rightarrow L^Y$ defined as $f_L^\rightarrow(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $y \in Y, A \in L^X$, and an L -backward powerset operator $f_L^\leftarrow : L^Y \rightarrow L^X$ defined as $f_L^\leftarrow(B) = B \circ f$ for each $B \in L^Y$.

Definition 2.3. Let $(X, e_X), (Y, e_Y)$ be L -posets and $f : X \rightarrow Y$ be a map. Then

- (1) f is said to be L -order-preserving or L -monotone if $e_X(x, y) \leq e_Y(f(x), f(y))$ for all $x, y \in X$;
- (2) f is an L -order-embedding if $e_X(x, y) = e_Y(f(x), f(y))$ for all $x, y \in X$;
- (3) f is said to be fuzzy-join-preserving if for any $A \in L^X$ such that $\sqcup A$ exists, it implies that $\sqcup f_L^\rightarrow(A)$ exists and $f(\sqcup A) = \sqcup f_L^\rightarrow(A)$;
- (4) f is said to be fuzzy-meet-preserving if for any $A \in L^X$ such that $\sqcap A$ exists, it implies that $\sqcap f_L^\rightarrow(A)$ exists and $f(\sqcap A) = \sqcap f_L^\rightarrow(A)$.

Definition 2.4. Let $(X, e_X), (Y, e_Y)$ be L -posets, $f : X \rightarrow Y, g : Y \rightarrow X$ be L -order-preserving maps. If $e_Y(f(x), y) = e_X(x, g(y))$ for all $x \in X, y \in Y$, then (f, g) is called a fuzzy Galois connection between X and Y . f is called the left adjoint of g , and dually g is the right adjoint of f .

It is worth noting that for any map $f : X \rightarrow Y$, there is a useful fuzzy Galois connection $(f_L^\rightarrow, f_L^\leftarrow)$ between (L^X, sub) and (L^Y, sub) .

Theorem 2.5. Let $(X, e_X), (Y, e_Y)$ be L -posets, $f : (X, e_X) \rightarrow (Y, e_Y)$, and let $g : (Y, e_Y) \rightarrow (X, e_X)$ be maps.

- (1) If (X, e_X) is complete, then f is L -order-preserving and has a right adjoint if and only if $f(\sqcup A) = \sqcup f_L^\rightarrow(A)$ for each $A \in L^X$;
- (2) If (Y, e_Y) is complete, then g is L -order-preserving and has a left adjoint if and only if $g(\sqcap B) = \sqcap g_L^\leftarrow(B)$ for each $B \in L^Y$.

Definition 2.6. Let (X, e_X) be an L -poset, (Y, e_Y) a complete L -lattice. If there exists an L -order-embedding $\varphi : X \rightarrow Y$, then (Y, e_Y) is said to be a completion of X via φ . Besides, if $\varphi(X)$ is join-dense in Y (refer to [13]), then we say that (Y, e_Y) is a join-completion of X via φ .

For an L -poset (X, e) and $A \in L^X$, then we have $A^u, A^l \in L^X$ which is called upper bound and lower bound of A , respectively.

$$A^u(x) = \bigwedge_{y \in X} A(y) \longrightarrow e(y, x), \quad A^l(x) = \bigwedge_{y \in X} A(y) \longrightarrow e(x, y), \quad (\forall x \in X). \quad (2.2)$$

Define $DM_L(X) = \{A \in L^X \mid A^{ul} = A\}$ and $DM_L^{op}(X) = \{A \in L^X \mid A^{lu} = A\}$. Then both $(DM_L(X), \text{sub})$ and $(DM_L^{op}(X), \text{sub}^{op})$ are join-completions of (X, e) , respectively, via L -order-embedding $\iota : X \rightarrow DM_L(X)$ given by $\iota(x) = \downarrow x$ and L -order-embedding $u : X \rightarrow DM_L^{op}(X)$ given by $u(x) = \uparrow x$ (the reader is referred to [13] for details). They are known as the Dedekind-MacNeille completions for (X, e) .

3. Reflective Full Subcategories of L -Posets

The aim of this section is to study the relationship between L -posets and complete L -lattices in categorical terms, which is closely related to the universal property of join-completions determined by a fuzzy closure operator. We refer to [22] for general category theory.

Definition 3.1. A subcategory A of a category B is said to be reflective in B if for each B -object B , there exists an A -object A and a B -morphism $r : B \rightarrow A$ such that for each B -morphism $f : B \rightarrow C$ there exists a unique A -morphism \bar{f} with $\bar{f} \circ r = f$.

Assume that the reader is familiar with the concepts of fuzzy closure operators and fuzzy closure systems. If C is a fuzzy closure operator on X , then there is an associated fuzzy closure system $\mathcal{L}_C = \{A \in L^X \mid C(A) = A\}$ and vice versa (refer to [12, 23] for details).

Definition 3.2 (see [14]). Let (X, e_X) be an L -poset. A fuzzy closure operator C on X is called a consistent fuzzy closure operator if $C(\chi\{x\}) = \downarrow x$.

As shown in the literature [14], join-completions are in bijective correspondence with consistent fuzzy closure operators.

Definition 3.3 (see [14]). Let $(X, e_X), (Y, e_Y)$ be L -posets and C a consistent fuzzy closure operator on X . A mapping $f : X \rightarrow Y$ is called a C -homomorphism if for all $y \in Y$, $f_L^-(\downarrow y) \in \mathcal{L}_C$.

Theorem 3.4 (see [14]). Let (X, e_X) be a L -poset, C a consistent fuzzy closure operator on X , and (Y, e_Y) a completion of X via φ . Then, the following are equivalent:

- (1) (Y, e_Y) is a join-completion of X via φ ;
- (2) φ is a C -homomorphism. Moreover, for each complete L -lattice (Z, e_Z) and each C -homomorphism $f : X \rightarrow Z$, there exists a unique fuzzy-join-preserving mapping $\bar{f} : Y \rightarrow Z$ such that $f = \bar{f} \circ \varphi$.

Considering Theorem 3.4, we do want to know that whether the category of complete L -lattices with fuzzy-join-preserving mappings, in symbol $L\text{-Sup}$, is a reflective full subcategories of the category whose objects are L -posets and arrows are C -homomorphisms associated with a consistent fuzzy closure operator C , which is denoted by $L\text{-POS}(C)$. In fact, the answer is positive for some special consistent fuzzy closure operators. But as in Example 3.5, when $C = \downarrow$, there exists some C -homomorphisms on complete L -lattices but not fuzzy-join-preserving. So $L\text{-Sup}$ is not a full subcategory of $L\text{-POS}(\downarrow)$. Sequentially, it is not a reflective full subcategory of $L\text{-POS}(\downarrow)$ neither.

Example 3.5. Let $X = \{0, a, b, 1\}$ with $0 \leq a, b \leq 1$ and $Y = \{0, c, 1\}$ with $0 \leq c \leq 1$, then (X, \leq) and (Y, \leq) are complete L -lattices where $L = 2$. Define $f : X \rightarrow Y$ as $f(0) = 0, f(a) = f(b) = c, f(1) = 1$. It easily checked that f is a \downarrow -homomorphism, but it is not fuzzy-join-preserving.

In the sequel, we try to seek for those special fuzzy closure operators C in order to give some reflective full subcategories for the category of L -posets.

Definition 3.6. Let (X, e) be an L -poset and C a fuzzy closure operator on X . Then, C is called an essential fuzzy closure operator if $C(A) = \downarrow \sqcup A$ for all $A \in L^X$ with $\sqcup A$ exists.

Note that an essential fuzzy closure operator is also a consistent fuzzy closure operator, not vice versa. The following proposition shows that “essential” is a sufficient and necessary condition for the equivalence of C -homomorphisms and fuzzy-join-preserving mappings on complete L -lattices.

Proposition 3.7. Let (X, e_X) , (Y, e_Y) be complete L -lattices, and let f a mapping from X to Y , C be a consistent fuzzy closure operator on X . Then

- (1) Assume that C is essential, we have f is a C -homomorphism if and only if it is fuzzy-join-preserving;
- (2) Assume that f is a C -homomorphism if and only if it is fuzzy-join-preserving, then C is essential.

Proof. (1) Suppose that C is essential. If f is an C -homomorphism, then f is L -order-preserving by Lemma 4.3 in [14]. For any $A \in L^X$, clearly $e_Y(\sqcup f_L^{\rightarrow}(A), f(\sqcup A)) = 1$. For the other direction, let $f_L^{\leftarrow}(\downarrow \sqcup f_L^{\rightarrow}(A)) = B$, then $\text{sub}(A, B) = \text{sub}(A, f_L^{\leftarrow}(\downarrow \sqcup f_L^{\rightarrow}(A))) = \text{sub}(f_L^{\rightarrow}(A), \downarrow \sqcup f_L^{\rightarrow}(A)) = 1$. By Lemma 2.19(1) in [8], we have $e_X(\sqcup A, \sqcup B) \geq \text{sub}(A, B) = 1$, and so $\downarrow \sqcup B(\sqcup A) = 1$. By assumption, $B \in \mathcal{L}_C$ and so $B = C(B) = \downarrow \sqcup B$. It implies that $f_L^{\leftarrow}(\downarrow \sqcup f_L^{\rightarrow}(A))(\sqcup A) = 1$. Thus $\downarrow \sqcup f_L^{\rightarrow}(A)(f(\sqcup A)) = 1$, that is, $e_Y(f(\sqcup A), \sqcup f_L^{\rightarrow}(A)) = 1$. Therefore, $\sqcup f_L^{\rightarrow}(A) = f(\sqcup A)$ and f is fuzzy-join-preserving. Conversely, if f is fuzzy-join-preserving. Then for any $y \in Y$, it is easy to check that $f_L^{\leftarrow}(\downarrow y) \leq \downarrow \sqcup f_L^{\leftarrow}(\downarrow y)$ at first. Furthermore,

$$\begin{aligned}
 \text{sub}(\downarrow \sqcup f_L^{\leftarrow}(\downarrow y), f_L^{\leftarrow}(\downarrow y)) &= \text{sub}(f_L^{\rightarrow}(\downarrow \sqcup f_L^{\leftarrow}(\downarrow y)), \downarrow y) \\
 &= e_Y(f(\sqcup f_L^{\leftarrow}(\downarrow y)), y) \\
 &= e_Y(\sqcup (f_L^{\rightarrow}(f_L^{\leftarrow}(\downarrow y))), y) \\
 &= 1.
 \end{aligned} \tag{3.1}$$

Hence, $f_L^{\leftarrow}(\downarrow y) = \downarrow \sqcup f_L^{\leftarrow}(\downarrow y) = C(f_L^{\leftarrow}(\downarrow y))$ since C is essential. Thus $f_L^{\leftarrow}(\downarrow y) \in \mathcal{L}_C$, it implies that f is an C -homomorphism.

(2) If C is not essential, then for C -homomorphism f , we have $B = C(B) \not\leq \downarrow \sqcup B$, where $B = f_L^{\leftarrow}(\downarrow \sqcup f_L^{\rightarrow}(A))$ for $A \in L^X$. As shown in the proof of (1), we can see that $\downarrow \sqcup B(\sqcup A) = 1$. It induces that $f_L^{\leftarrow}(\downarrow \sqcup f_L^{\rightarrow}(A))(\sqcup A) \not\leq 1$, so $e_Y(f(\sqcup A), \sqcup f_L^{\rightarrow}(A)) \not\leq 1$. Then f cannot be fuzzy-join-preserving, and it contradicts with premise. Thus C is essential. \square

Corollary 3.8. (1) If C is an essential fuzzy closure operator, then $L\text{-Sup}$ is a full subcategory of $L\text{-POS}(C)$.

(2) If $L\text{-Sup}$ is a full subcategory of $\text{POS}(C)$ associated with a consistent fuzzy closure operator, then C is essential.

Proposition 3.9. *Let (X, e) be an L -poset and C an essential fuzzy closure operator on X . Then there exists a join-completion (Y, e) of X via an C -homomorphism φ . Moreover, for each complete L -lattice (Z, e) and C -homomorphism $f : (X, e) \rightarrow (Z, e)$, there exists a unique fuzzy-join-preserving mapping $\bar{f} : (Y, e) \rightarrow (Z, e)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ \varphi \downarrow & \nearrow \bar{f} & \\ Y & & \end{array} \quad (3.2)$$

Proof. Since C is an essential fuzzy closure operator, then C is a consistent fuzzy closure operator. By the Literature [14], we can see that $(\mathcal{L}_C, \text{sub})$ is a join-completion of X via C -homomorphism ι . Thus the result follows immediately from Theorem 3.4. \square

Theorem 3.10. (1) *If C is an essential fuzzy closure operator, then $L\text{-Sup}$ is a reflective full subcategory of $L\text{-POS}(C)$.*

(2) *If $L\text{-Sup}$ is a reflective full subcategory of $L\text{-POS}(C)$ associated with a consistent fuzzy closure operator, then C is essential.*

Proof. It follows from Corollary 3.8 and Proposition 3.9. \square

Remark 3.11. For an L -poset (X, e) , let $C = {}^{ul}$. By Lemma 2.5(3) in [13], we can see that $C(A) = \downarrow \sqcup A$. So $C = {}^{ul}$ is an essential fuzzy closure operator, and its associated completion is exactly the Dedekind-MacNeille completion $(DM_L(X), \text{sub})$ via C -homomorphism ι (it is called lower-continuous mapping in Section 4). Thus $L\text{-Sup}$ is a reflective full subcategory of $L\text{-POS}({}^{ul})$.

It is easy to see that $DM_L^{op}((X, e))$ can be viewed as $DM_L((X, e^{op}))$. With the duality of the L -order, when mappings dual to C -homomorphism are chosen to serve as morphisms on L -posets, the corresponding morphisms on complete L -lattices need to be changed to fuzzy-meet-preserving mappings. Then the dual result is obtained immediately. We list it in the following and leave the proof to the reader.

Definition 3.12. Let $(X, e_X), (Y, e_Y)$ be L -posets, $f : X \rightarrow Y$ a mapping. Then f is said to be upper-continuous if $(f_L^{\leftarrow}(\uparrow y))^{lu} = f_L^{\leftarrow}(\uparrow y)$ for all $y \in Y$.

Let $L\text{-Inf}$ denote the category of complete L -lattices with fuzzy-meet-preserving mappings, and the $L\text{-Pos}$ denote the category whose objects are all L -posets and arrows are upper-continuous mappings.

Theorem 3.13. *$L\text{-Inf}$ is a reflective full subcategory of $L\text{-POS}$.*

4. The Characterizations of the Dedekind-MacNeille Completions for L -Posets

From previous works, we have known that for an L -poset, each join-completion has a universal property, and any completion with a universal property must be a join-completion.

In this section, we study the characterizations of one special join-completion—the Dedekind-MacNeille completion and obtain a stronger result that a complete L -lattice with a universal property is exactly the Dedekind-MacNeille completion up to isomorphism.

Definition 4.1. Let $(X, e_X), (Y, e_Y)$ be L -posets, $f : X \rightarrow Y$ a mapping, and $B \in L^Y$. Then f is said to be lower-continuous if $(f_L^{\leftarrow}(\downarrow y))^{\text{ul}} = f_L^{\leftarrow}(\downarrow y)$ for all $y \in Y$.

Theorem 4.2. Let $(X, e_X), (Y, e_Y)$ be L -posets, and let $f : X \rightarrow Y$ be a lower-continuous mapping. Then there exists a unique fuzzy-join-preserving mapping $F = DM_L(f) : DM_L(X) \rightarrow DM_L(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \iota_X & & \downarrow \iota_Y \\ DM_L(X) & \xrightarrow{F} & DM_L(Y) \end{array} \quad (4.1)$$

Proof. First, define $F : DM_L(X) \rightarrow DM_L(Y)$ by $F(A) = (f_L^{\rightarrow}(A))^{\text{ul}}$, then we show that F has a right adjoint. In fact, define $G : DM_L(Y) \rightarrow DM_L(X)$ by $G(A) = f_L^{\leftarrow}(A)$. Then for any $A \in DM_L(X), B \in DM_L(Y)$, $\text{sub}(F(A), B) = \text{sub}((f_L^{\rightarrow}(A))^{\text{ul}}, B) = \text{sub}(B^{\text{u}}, (f_L^{\rightarrow}(A))^{\text{u}}) = \text{sub}(f_L^{\rightarrow}(A), B^{\text{ul}}) = \text{sub}(f_L^{\rightarrow}(A), B) = \text{sub}(A, f_L^{\leftarrow}(B)) = \text{sub}(A, G(B))$, it implies that (F, G) forms a fuzzy Galois connection. By Theorem 2.5, F is fuzzy-join-preserving.

Next, to show that the diagram commutes amount to showing that $(f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}} = \downarrow_Y f(x)$ for every $x \in X$. Clearly, $(f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}} \leq \downarrow_Y f(x)$ since f is L -order-preserving. In the other direction, since $(f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}$ is an L -lower set, so for any $y \in Y$, we have $(f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}(f(x)) * e_Y(y, f(x)) \leq (f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}(y)$. It implies that $e_Y(y, f(x)) \leq (f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}(y)$, that is, $\downarrow_Y f(x)(y) \leq (f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}(y)$. Thus $\downarrow_Y f(x) \leq (f_L^{\rightarrow}(\downarrow_X x))^{\text{ul}}$, as needed.

Finally, we show that F is unique. Suppose that there is another fuzzy-join-preserving mapping F^* satisfying the diagram commutes; we show that $F(A) = F^*(A)$ for all $A \in DM_L(X)$. Since $\iota_X(X)$ is join-dense in $DM_L(X)$, it suffices to show that $F(\downarrow x) = F^*(\downarrow x)$ for all $x \in X$. That is obvious since the diagram commutes. \square

Corollary 4.3. Let $f : X \rightarrow Y$ be a lower-continuous mapping of an L -poset (X, e_X) into a complete L -lattice (Y, e_Y) . Then there exists a unique fuzzy-join-preserving mapping $\bar{f} : DM_L(X) \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \iota_X & \nearrow \bar{f} & \\ DM_L(X) & & \end{array} \quad (4.2)$$

Proof. By Theorem 4.2, there exists a commutative diagram as follows, and the mapping ι_Y is an L -order-isomorphism since Y is a complete L -lattice. Let $\bar{f} = \iota_Y^{-1} \circ F$, then \bar{f} is a unique fuzzy-join-preserving mapping such that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow \iota_X & \nearrow \bar{f} & \downarrow \iota_Y \\
 DM_L(X) & \xrightarrow{F} & DM_L(Y)
 \end{array} \quad (4.3)$$

□

Corollary 4.4. *The Dedekind-MacNeille completion operator DM_L is a covariant functor from $L\text{-POS}^{(ul)}$ to $L\text{-Sup}$.*

Corollary 4.5. *In the category $L\text{-Sup}$, $(DM_L(X), \text{sub})$ is the free object generated by an L -poset (X, e) .*

Theorem 4.6. *Let (X, e_X) be an L -poset and (Y, e_Y) a complete L -lattice. Then Y is isomorphic to the Dedekind-MacNeille completion $(DM_L(X), \text{sub})$ if and only if it satisfies the following condition.*

There exists a lower-continuous mapping $\varphi : X \rightarrow Y$ such that for every lower-continuous mapping $f : X \rightarrow Z$ of X into a complete L -lattice (Z, e_Z) , there exists a unique fuzzy-join-preserving mapping $\bar{f} : Y \rightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & Y \\
 & \searrow f & \downarrow \bar{f} \\
 & & Z
 \end{array} \quad (4.4)$$

Proof. Suppose that Y is isomorphic to $DM_L(X)$, then there exists an L -order-isomorphism $g : DM_L(X) \rightarrow Y$. We define $\varphi : X \rightarrow Y$ as $\varphi = g \circ \iota_X$, then it is easily checked that φ is a lower-continuous mapping. If $f : X \rightarrow Z$ is a lower-continuous mapping of X into complete L -lattice (Z, e_Z) . By Corollary 4.3, there exists a unique fuzzy-join-preserving mapping $h : DM_L(X) \rightarrow Z$ such that $f = h \circ \iota_X$. Let $\bar{f} = h \circ g^{-1}$, then $\bar{f} : Y \rightarrow Z$ is a unique fuzzy-join-preserving mapping such that $\bar{f} \circ \varphi = f$.

For the second part, assume that the condition holds. By Corollary 4.3, the diagrams below commute:

$$\begin{array}{ccc}
 & & DM_L(X) \\
 & \nearrow \iota_X & \downarrow \bar{\varphi} \\
 X & \xrightarrow{\varphi} & Y \\
 & \searrow \iota_X & \downarrow \bar{j}_X \\
 & & DM_L(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 & & Y \\
 & \nearrow \varphi & \downarrow \bar{i}_X \\
 X & \xrightarrow{\iota_X} & DM_L(X) \\
 & \searrow \varphi & \downarrow \bar{\varphi} \\
 & & Y
 \end{array} \quad (4.5)$$

By the uniqueness condition, $\bar{\varphi} \circ \bar{i}_X = id_{DM_L(X)}$ and $\bar{i}_X \circ \bar{\varphi} = id_Y$. Thus Y is isomorphic to $DM_L(X)$. \square

Since $DM_L^{op}((X, e))$ can be regarded as $DM_L((X, e^{op}))$, when we replace the lower-continuous mapping by upper-continuous mapping and fuzzy-join-preserving mapping by fuzzy-meet-preserving mapping, then the dual results of Theorem 4.2 and Corollary 4.3 are obtained naturally. Sequentially, the following observations are immediate, and the proofs are trivial, so we omit them.

Proposition 4.7. (1) The Dedekind-MacNeille completion operator DM_L^{op} is a covariant functor from $L\text{-Pos}$ to $L\text{-Inf}$.

(2) In the category $L\text{-Inf}$, $(DM_L^{op}(X), \text{sub}^{op})$ is the free object generated by an $L\text{-poset}$ (X, e) .

Theorem 4.8. Let (X, e_X) be an $L\text{-poset}$ and (Y, e_Y) a complete $L\text{-lattice}$. Then Y is isomorphic to the Dedekind-MacNeille completion $(DM_L^{op}(X), \text{sub}^{op})$ if and only if it satisfies the following condition.

There exists an upper-continuous mapping $\phi : X \rightarrow Y$ such that for every upper-continuous mapping $f : X \rightarrow Z$ of X into a complete $L\text{-lattice}$ (Z, e_Z) , there exists a unique fuzzy-meet-preserving mapping $\bar{f} : Y \rightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow f & \downarrow \bar{f} \\ & & Z \end{array} \quad (4.6)$$

5. Conclusion

In this paper, we consider the relationship between $L\text{-posets}$ and complete $L\text{-lattices}$ in the categorical framework. After analysis of the characterizations of the join-completions, we prove that the category of complete $L\text{-lattices}$ with fuzzy-join-preserving mappings is a reflective full subcategory of the category of $L\text{-posets}$ with C -homomorphisms related to an essential fuzzy closure operator. Furthermore, we characterize the Dedekind-MacNeille completions. It is shown that they are the unique free objects (up to isomorphism) generated by an $L\text{-poset}$ in the category of complete $L\text{-lattices}$. It offers an equivalent description for them. Maybe we will continue to focus on the completion theory for fuzzy ordered sets, we will try to propose and characterize other completions for $L\text{-posets}$ in the future.

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References

- [1] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, pp. 177–200, 1971.
- [2] R. Bělohlávek, "Fuzzy Galois connections," *Mathematical Logic Quarterly*, vol. 45, no. 4, pp. 497–504, 1999.

- [3] L. Fan, "A new approach to quantitative domain theory," *Electronic Notes in Theoretic Computer Science*, vol. 45, pp. 77–87, 2001.
- [4] W. Yao, "L-fuzzy Scott topology and Scott convergence of stratified L-filters on fuzzy dcpos," *Electronic Notes in Theoretical Computer Science*, vol. 257, pp. 135–152, 2009.
- [5] W. Yao, "Quantitative domains via fuzzy sets: part I: continuity of fuzzy directed complete posets," *Fuzzy Sets and Systems*, vol. 161, no. 7, pp. 973–987, 2010.
- [6] Q.-Y. Zhang and L. Fan, "Continuity in quantitative domains," *Fuzzy Sets and Systems*, vol. 154, no. 1, pp. 118–131, 2005.
- [7] Q.-Y. Zhang and L. Fan, "A kind of L-fuzzy complete lattice and adjoint functor theorem for LF-Poset," in *Proceedings of the 4th International Symposium on Domain Theory*, Hunan University, Changsha, China, June 2006.
- [8] Q. Zhang, W. Xie, and L. Fan, "Fuzzy complete lattices," *Fuzzy Sets and Systems*, vol. 160, no. 16, pp. 2275–2291, 2009.
- [9] K. R. Wagner, *Solving recursive domain equations with enriched categories* [Ph.D. thesis], Carnegie Mellon University, Technical Report CMU-CS-94-159, 1994.
- [10] K. R. Wagner, "Liminf convergence in Ω -categories," *Theoretical Computer Science*, vol. 184, no. 1-2, pp. 61–104, 1997.
- [11] H. Lai and D.-X. Zhang, "Complete and directed complete Ω -categories," *Theoretical Computer Science*, vol. 388, no. 1–3, pp. 1–25, 2007.
- [12] R. Bělohlávek, "Concept lattices and order in fuzzy logic," *Annals of Pure and Applied Logic*, vol. 128, no. 1–3, pp. 277–298, 2004.
- [13] W. Xie, Q. Zhang, and L. Fan, "The Dedekind-MacNeille completions for fuzzy posets," *Fuzzy Sets and Systems*, vol. 160, no. 16, pp. 2292–2316, 2009.
- [14] K. Wang and B. Zhao, "Join-completions of L-ordered sets," *Fuzzy Sets and Systems*, vol. 199, pp. 92–107, 2012.
- [15] M. Ward and R. P. Dilworth, "Residuated lattices," *Transactions of the American Mathematical Society*, vol. 45, no. 3, pp. 335–354, 1939.
- [16] R. Bělohlávek, *Fuzzy Relational Systems: Foundation and Principles*, Kluwer Academic/Plenum Publishers, New York, NY, USA, 2002.
- [17] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.
- [18] H. Lai and D. Zhang, "Fuzzy preorder and fuzzy topology," *Fuzzy Sets and Systems*, vol. 157, no. 14, pp. 1865–1885, 2006.
- [19] I. Stubbe, "Categorical structures enriched in a quantaloid: categories, distributors and functors," *Theory and Applications of Categories*, vol. 14, pp. 1–45, 2005.
- [20] I. Stubbe, "Categorical structures enriched in a quantaloid: tensored and cotensored categories," *Theory and Applications of Categories*, vol. 16, no. 14, pp. 283–306, 2006.
- [21] H.-L. Lai and D.-X. Zhang, "Many-valued complete distributivity," <http://arxiv.org/abs/math/0603590>.
- [22] S. Mac Lane, *Categories for the Working Mathematician*, Springer, New York, NY, USA, 1971.
- [23] R. Bělohlávek, "Fuzzy closure operators," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 473–489, 2001.