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Research Article

The Zeros of Orthogonal Polynomials for Jacobi-Exponential Weights

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This paper gives the estimates of the zeros of orthogonal polynomials for Jacobi-exponential weights.

1. Introduction and Results

This paper deals with the zeros of orthogonal polynomials for Jacobi-exponential weights. Let w be a weight in $\mathbf{I} := (a,b)$, $-\infty \le a < 0 < b \le \infty$, for which the moment problem possesses a unique solution. Denote by $\mathbf{N}$ the set of positive integers. $\mathbf{P}_n$ stands for the set of polynomials of degree at most n.

Assume that $W = e^{-Q}$ where $Q : \mathbf{I} \to [0,\infty)$ is continuous. Also, let $0 < p < \infty$,

$$\begin{gathered} a \le t_r < t_{r-1} < \cdots < t_2 < t_1 \le b, \\ p_i > \frac{-1}{p}, \quad i = 1,2,\ldots,r, \\ U(x) = \prod_{i=1}^{r} |x - t_i|^{p_i}. \end{gathered} \tag{1.1}$$

The letters $c, C_0, C_1, \ldots$ stand for positive constants independent of variables and indices, unless otherwise indicated, and their values may be different at different occurrences, even in subsequent formulas. Moreover, $C_n \sim D_n$ means that there are two constants c_1 and

c_2 such that $c_1 \le C_n/D_n \le c_2$ for the relevant range of n. We write $c = c(\lambda)$ or $c \neq c(\lambda)$ to indicate dependence on or independence of a parameter λ.

Definition 1.1 (see [1, Definition 1.7, page 14]). Given $c, t \ge 0$ and a nonnegative Borel measure ν with compact support in $\mathbf{C}$ and total mass $\le t$, one says that

$$P(z) := c \exp\left(\int \ln|z-t| d\nu(t)\right) \tag{1.2}$$

is an exponential of a potential of mass $\le t$. One denotes the set of all such P by $\mathcal{P}_t$.

One notes that, for $P \in \mathbf{P}_n$,

$$|P| \in \mathcal{P}_t, \quad t \ge n. \tag{1.3}$$

Definition 1.2 (see [1, page 19]). Let w be a weight in $\mathbf{I}$. For $0 < p < \infty$, generalized Christoffel functions with respect to w for $z \in \mathbf{C}$ are defined by

$$\lambda_{p,n}(w; z) = \inf_{P \in \mathbf{P}_n} \left(\frac{\|Pw\|_{L_p(\mathbf{I})}}{|P(z)|}\right)^p. \tag{1.4}$$

For $p = \infty$, generalized Christoffel functions with respect to w for $z \in \mathbf{C}$ are defined by

$$\lambda_{\infty,n}(w; z) = \inf_{P \in \mathbf{P}_n} \frac{\|Pw\|_{L_\infty(\mathbf{I})}}{|P(z)|}. \tag{1.5}$$

Obviously, for the classical Christoffel function $\lambda_n(w^2; x)$ with respect to w^2, we have

$$\lambda_n\left(w^2; x\right) = \lambda_{2,n-1}(w; x). \tag{1.6}$$

A function $f : (c,d) \to (0,\infty)$ is said to be *quasi-increasing* (or *quasi-decreasing*) if there exists $C > 0$ such that

$$f(x) \le (\text{or } \ \ge) C f(y), \quad c < x \le y < d. \tag{1.7}$$

Definition 1.3 (see [1, pages 10–12]). Let $a < 0 < b$. Assume that $W = e^{-Q}$ where $Q : \mathbf{I} \to [0,\infty)$ satisfies the following properties

(a) $Q' \in C(\mathbf{I})$ and $Q(0) = 0$.

(b) Q' is nondecreasing in $\mathbf{I}$.

(c) We have

$$\lim_{t \to a+} Q(t) = \lim_{t \to b-} Q(t) = \infty. \tag{1.8}$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0, \tag{1.9}$$

is quasi-decreasing in $(a, 0)$ and quasi-increasing in $(0, b)$, respectively. Moreover

$$T(t) \geq \Lambda > 1, \quad t \in \mathbf{I} \setminus \{0\}. \tag{1.10}$$

(e) There exists $\epsilon_0 \in (0,1)$ such that, for $y \in \mathbf{I} \setminus \{0\}$,

$$T(y) \sim T\left(y\left[1 - \frac{\epsilon_0}{T(y)}\right]\right). \tag{1.11}$$

Then we write $W \in \mathcal{F}$.

(f) In addition, assume that there exist $C, \epsilon_1 > 0$ such that, for all $x \in \mathbf{I} \setminus \{0\}$,

$$\int_{x-\epsilon_1|x|/T(x)}^{x} \frac{|Q'(t) - Q'(x)|}{|t-x|^{3/2}} dt \leq C\left|Q'(x)\right| \left[\frac{T(x)}{|x|}\right]^{1/2}. \tag{1.12}$$

Then we write $W \in \mathcal{F}(\mathrm{Lip}(1/2))$.

For $W \in \mathcal{F}$ and $t > 0$, the Mhaskar-Rahmanov-Saff numbers $a_{-t} := a_{-t}(Q) < 0 < a_t := a_t(Q)$ are defined by the equations

$$\begin{aligned} t &= \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{xQ'(x)}{[(x-a_{-t})(a_t - x)]^{1/2}} dx, \\ 0 &= \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{Q'(x)}{[(x-a_{-t})(a_t - x)]^{1/2}} dx. \end{aligned} \tag{1.13}$$

Put for $t > 0$,

$$\Delta_t := \Delta_t(Q) := [a_{-t}, a_t],$$

$$\delta_t := \delta_t(Q) := \frac{1}{2}(a_t + |a_{-t}|), \qquad \eta_{\pm t} := \eta_{\pm t}(Q) := \left[tT(a_{\pm t})\sqrt{\frac{|a_{\pm t}|}{\delta_t}} \right]^{-2/3},$$

$$\varphi_t(x) := \varphi_t(Q; x) := \begin{cases} \dfrac{|x - a_{-2t}||x - a_{2t}|}{t\sqrt{[|x - a_{-t}| + |a_{-t}|\eta_{-t}][|x - a_t| + a_t\eta_t]}}, & x \in [a_{-t}, a_t], \\ \varphi_t(a_t), & x \in (a_t, b), \\ \varphi_t(a_{-t}), & x \in (a, a_{-t}), \end{cases} \tag{1.14}$$

$$\mathbf{J}_{L,t} := \mathbf{J}_{L,t}(Q) := \left[a_{-t}(1 + L\eta_{-t}), a_t(1 + L\eta_t)\right], \quad L > 0,$$

$$\mathbf{K}_{L,t} := \mathbf{K}_{L,t}(Q) := [-1 + L(1 + a_{-t}), 1 - L(1 - a_t)], \quad L > 1.$$

Let

$$U_t(x) := \prod_{i=1}^{r} \left(|x - t_i| + \frac{\delta_t}{t} \right)^{p_i}, \qquad \rho := \rho(U) := \sum_{i=1}^{r} \max\{p_i, 0\}. \tag{1.15}$$

In 1994 and 2001, Levin and Lubinsky [1, 2] published their monographs on orthogonal polynomials for exponential weights W^2. Then they [3, 4] discussed orthogonal polynomials for exponential weights $x^{2\alpha}W(x)^2, \alpha > -1/2$, in $[0, b)$, since the results of [1, 2] cannot be applied to such weights. Kasuga and Sakai [5] considered generalized Freud weights $|x|^{2\alpha}W(x)^2$ in $(-\infty, \infty)$. Recently the second author [6] obtained the L_p Christoffel functions for Jacobi-exponential weights UW, which are the combination of the two best important weights: Jacobi weight and the exponential weight, and restricted range inequalities.

Theorem 1.4 (see [6, Theorem 1.1]). *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$, $L > 0$, and $0 < p < \infty$. Assume that*

$$\lim_{t \to \infty} \frac{|a_{-t}|}{a_t} = \gamma, \quad 0 < \gamma < \infty. \tag{1.16}$$

Then there exists $n_0 > 0$ such that, for $n \ge n_0$ and $x \in \mathbf{J}_{L,n}$, the relation

$$\lambda_{p,n}(UW; x) \sim \varphi_n(x) U_n(x)^p W(x)^p \tag{1.17}$$

uniformly holds.

Theorem 1.5 (see [6, Theorem 1.2]). *Let $W = e^{-Q(x)}$, where $Q : \mathbf{I} \to [0, \infty)$ is convex with $Q(a+) = Q(b-) = \infty$ and $Q(x) > Q(0) = 0$, $x \in \mathbf{I} \setminus \{0\}$. Let $0 < p \le \infty$. Assume that relation* (1.16) *is valid. Then there exist $C, t_0 > 0$ such that, for $t \ge t_0$ and $P \in \mathcal{P}_{t-\rho-2/p}$,*

$$\begin{aligned} \|PUW\|_{L_p(\mathbf{I})} &\le C\|PUW\|_{L_p(\Delta_t)}, \\ \|PU_tW\|_{L_p(\mathbf{I})} &\le C\|PU_tW\|_{L_p(\Delta_t)}. \end{aligned} \tag{1.18}$$

Theorem 1.6 (see [6, Theorem 1.3]). *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$, $L > 0$, and $0 < p < \infty$. Assume that relation* (1.16) *is valid. Then there exist $C, t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_t$,*

$$\|PUW\|_{L_p(\mathbf{I})} \leq C\|PUW\|_{L_p([a_{-t}(1-L\eta_{-t}),a_t(1-L\eta_t)])}. \tag{1.19}$$

In this paper we discuss the zeros of orthogonal polynomials for Jacobi-exponential weights UW and restricted range inequalities.

Theorem 1.7. *Let $W \in \mathcal{F}(\mathrm{Lip}\,(1/2))$. Assume that* (1.16) *is valid, and*

$$a < t_r < \cdots < t_1 < b, \tag{1.20}$$

$$\varphi_t(x) = O(1), \quad t \longrightarrow \infty. \tag{1.21}$$

Then

$$x_{kn} - x_{k+1,n} \leq c\varphi_n(x_{kn}), \quad k = 1, 2, \ldots, n-1. \tag{1.22}$$

Theorem 1.8. *Let $W = e^{-Q(x)}$, where $Q : \mathbf{I} \to [0, \infty)$ is convex with $Q(a+) = Q(b-) = \infty$ and $Q(x) > Q(0) = 0$, $x \in \mathbf{I} \setminus \{0\}$. Let $0 < p \leq \infty$. Assume that all p_i are positive and relation* (1.16) *is valid. Then there exist $t_0 > 0$ such that, for $t \geq t_0$ and $P \in \mathcal{P}_{t-\rho-2/p}$,*

$$\|PUW\|_{L_p(\mathbf{I}\setminus\Delta_t)} \leq \|PUW\|_{L_p(\Delta_t)}. \tag{1.23}$$

Theorem 1.9. *Let the assumptions of Theorem 1.8 prevail. Then*

$$x_{1n} < a_{n+\rho+1/2}, \tag{1.24}$$

$$x_{nn} > a_{-n-\rho-1/2}. \tag{1.25}$$

Theorem 1.10. *Let $W \in \mathcal{F}(\mathrm{Lip}(1/2))$. Then*

$$x_{1n} \geq a_n(1 - c\eta_n), \tag{1.26}$$

$$x_{nn} \leq a_{-n}(1 - c\eta_{-n}). \tag{1.27}$$

If all $p_i \geq 0$, then

$$1 - \frac{x_{1n}}{a_n} \sim \eta_n, \tag{1.28}$$

$$1 - \frac{x_{nn}}{a_{-n}} \sim \eta_{-n}. \tag{1.29}$$

Here we should point out that our main result (Theorem 1.7) cannot follow from [7] given by Mastroianni and Totik, because in general Jacobi-exponential weights UW are not doubling weights, although Jacobi weights U are doubling weights. A doubling weight means that the measure of a twice enlarged interval is less than a constant times the measure of the original interval. For example, for $W(t) = \exp(-t^2)$, by L'Hospital rule

$$\begin{aligned}\lim_{d\to\infty}\frac{\int_{d/2}^{5d/2}\exp(-t^2)dt}{\int_{d}^{2d}\exp(-t^2)dt} &= \lim_{d\to\infty}\frac{\exp\left(-(5d/2)^2\right)-\exp\left(-(d/2)^2\right)}{\exp\left(-(2d)^2\right)-\exp(-d^2)}\\ &= \lim_{d\to\infty}\frac{\exp(3d^2/4)-\exp(-21d^2/4)}{1-\exp(-3d^2)} = \infty,\end{aligned} \tag{1.30}$$

which implies that $W(t) = \exp(-t^2)$ is not a doubling weight.

We will give some auxiliary lemmas in Section 2 and the proofs of Theorems 1.7–1.10 in Section 3, respectively.

2. Auxiliary Lemmas

Lemma 2.1 (Levin and Lubinsky [1, Lemma 3.5, pages 71-72]). *Let $W \in \mathcal{F}$. Then for fixed $L > 1$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t. \tag{2.1}$$

Moreover, there exists $\tau_0 > 0$ such that, for $t \geq \tau \geq \tau_0$, the inequalities

$$1 \leq \frac{\delta_t}{\delta_\tau} \leq c\left(\frac{t}{\tau}\right)^{1/\Lambda} \tag{2.2}$$

hold.

Lemma 2.2 (Shi [6]). *Let $W \in \mathcal{F}$. Then, for large enough t,*

$$a_{2t} \geq a_t(1+\eta_t). \tag{2.3}$$

Lemma 2.3. *Let $\mathbf{I} = (-1,1)$, $W \in \mathcal{F}$, and $L > 1$. Then, for $x \in \mathbf{K}_{L,t}$,*

$$\varphi_t(x) \sim \frac{1}{t}\left[(a_{2t}-x)(x-a_{-2t})\right]^{1/2}, \tag{2.4}$$

$$\varphi_t(x) \leq \frac{c\delta_t}{t}. \tag{2.5}$$

Proof. By the same argument as that of [8, (2.25)] we can prove (2.4). By (2.4) and (2.1) for $x \in \mathbf{K}_{L,t}$,

$$\varphi_t(x) \leq \frac{c}{t}\cdot\frac{1}{2}(a_{2t}-a_{-2t}) \leq \frac{c\delta_t}{t}. \tag{2.6}$$

□

Lemma 2.4. *Let $W \in \mathcal{F}$. Then, for $x \in \mathbf{I}$,*

$$\varphi_t(x) \leq \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. \tag{2.7}$$

Proof. By the definition of φ_t it is enough to prove (2.7) for $x \in \Delta_t$. Without loss of generality we can assume that $0 \leq x \leq a_t$. By Lemma 3.11(b) in [1, page 81] for $t > 0$,

$$\left|\frac{a_{2t}}{a_t} - 1\right| \sim \frac{1}{T(a_t)}. \tag{2.8}$$

By Lemma 2.12 in [8], (2.3), (2.1), and (2.8),

$$\begin{aligned} S(x) &= \frac{a_{2t} - x}{a_t(1+\eta_t) - x} \cdot \frac{a_{-2t} - x}{a_{-t}(1+\eta_{-t}) - x} \\ &\leq \frac{a_{2t} - a_t}{a_t\eta_t} \cdot \frac{a_{-2t}}{a_{-t}(1+\eta_{-t})} \leq c\frac{a_{2t}/a_t - 1}{\eta_t} \leq \frac{c}{\eta_t T(a_t)}. \end{aligned} \tag{2.9}$$

By (1.63) in [1, page 15],

$$\eta_t T(a_t) \geq t^{-2/3} T(a_t)^{1/3} \tag{2.10}$$

and hence

$$S(x) \leq c t^{2/3} T(a_t)^{-1/3}. \tag{2.11}$$

Thus

$$\begin{aligned} \varphi_t(x) &= \frac{[(a_{2t} - x)(x - a_{-2t})]^{1/2}}{t} S(x)^{1/2} \\ &\leq \frac{c\delta_t}{t}\left[t^{2/3}T(a_t)^{-1/3}\right]^{1/2} = \frac{c\delta_t}{t^{2/3}T(a_t)^{1/6}}. \end{aligned} \tag{2.12}$$

□

Let $\mathbf{I}_k = [x_{k+1,n}, x_{kn}]$, $d_k = x_{kn} - x_{k+1,n}$, $k = 1, 2, \ldots, n-1$. Let, for $n \geq n_0$ and $d := \min_{1 \leq i \leq r-1}(t_i - t_{i+1})$,

$$\max_{1 \leq k \leq n-1} d_k \leq \frac{d}{4}. \tag{2.13}$$

Lemma 2.5. *For fixed index k, $1 \leq k \leq n-1$, let j, $1 \leq j \leq r$, satisfy*

$$\min_{x \in \mathbf{I}_k} \left| x - t_j \right| = \min_{1 \leq i \leq r} \min_{x \in \mathbf{I}_k} |x - t_i|. \tag{2.14}$$

Then

$$\prod_{i\neq j}|x_{\kappa n}-t_i|^{p_i}\sim\prod_{i\neq j}\left(|x_{\kappa n}-t_i|+\frac{\delta_n}{n}\right)^{p_i}\sim\prod_{i\neq j}|x-t_i|^{p_i},\quad x\in\mathbf{I}_k,\ \kappa=k,k+1. \tag{2.15}$$

Proof. We give the proof of (2.15) for $\kappa=k$ only, the proof of (2.15) for $\kappa=k+1$ being similar.

We claim that, for $i\neq j$,

$$|x_{kn}-t_i|\geq\frac{3}{8}d. \tag{2.16}$$

In fact, suppose without loss of generality that $x_{kn}\geq t_j$. It is enough to show (2.16) for $i=j-1$. Because $|x_{kn}-t_{j+1}|\geq t_j-t_{j+1}\geq d$.

If $t_j\in\mathbf{I}_k$ then by (2.13)

$$|x_{kn}-x_{k+1,n}|\leq\frac{d}{4}\leq t_{j-1}-t_j\leq\left|t_{j-1}-x_{kn}\right|+\left|x_{kn}-t_j\right| \tag{2.17}$$

and hence

$$\left|x_{k+1,n}-t_j\right|\leq\left|x_{kn}-t_{j-1}\right|; \tag{2.18}$$

if $t_j\notin\mathbf{I}_k$ then by (2.14)

$$\left|x_{k+1,n}-t_j\right|=\min_{x\in\mathbf{I}_k}\left|x-t_j\right|\leq\min_{x\in\mathbf{I}_k}\left|x-t_{j-1}\right|=t_{j-1}-x_{kn}, \tag{2.19}$$

which again implies (2.18). Then by (2.18)

$$\begin{aligned}d\leq\left|t_{j-1}-t_j\right|&\leq\left|t_{j-1}-x_{kn}\right|+|x_{kn}-x_{k+1,n}|+\left|x_{k+1,n}-t_j\right|\\&\leq2\left|x_{kn}-t_{j-1}\right|+d_k\\&\leq2\left|x_{kn}-t_{j-1}\right|+\frac{1}{4}d\end{aligned} \tag{2.20}$$

and hence $|x_{kn}-t_{j-1}|\geq3d/8$. This proves (2.16).

With the help of (2.16) for $x\in\mathbf{I}_k$ and $i\neq j$,

$$\begin{aligned}|x-t_i|&\leq|x_{kn}-t_i|+|x-x_{kn}|\leq|x_{kn}-t_i|+\frac{d}{4}\leq\frac{5}{3}|x_{kn}-t_i|,\\|x-t_i|&\geq|x_{kn}-t_i|-|x-x_{kn}|\geq|x_{kn}-t_i|-\frac{d}{4}\geq\frac{1}{3}|x_{kn}-t_i|.\end{aligned} \tag{2.21}$$

Hence

$$|x-t_i|\sim|x_{kn}-t_i|. \tag{2.22}$$

Furthermore, by (2.2) with $\tau = 1$

$$\frac{\delta_t}{t} \le c\delta_1 t^{1/\Lambda - 1} = o(1), \quad t \longrightarrow \infty. \tag{2.23}$$

So for $i \neq j$,

$$|x_{kn} - t_i| \sim |x_{kn} - t_i| + \frac{\delta_n}{n}. \tag{2.24}$$

This proves (2.15). □

By the same argument as that of Lemma 7.2.7 in [9, page 157] replacing $1/n$ by C_n, we can get its extension.

Lemma 2.6. *Let* $p \ge 0$, $B_n \ge A_n \ge 0$, $C_n \ge 0$, $\sigma = \pm 1$, *and*

$$B_n^{p+1} + \sigma A_n^{p+1} \le C C_n \left[(B_n + C_n)^p + (A_n + C_n)^p \right]. \tag{2.25}$$

Then

$$B_n + \sigma A_n \le c C_n. \tag{2.26}$$

Lemma 2.7. *Let* $W \in \mathcal{F}$. *Let* (1.16), (1.20), *and* (1.21) *prevail. Then there exists* $t_0 > 0$ *such that, for* $t \ge t_0$ *and for each index* j, $1 \le j \le r$,

$$\left|x - t_j\right| + \frac{\delta_t}{t} \sim \left|x - t_j\right| + \varphi_t(x) \tag{2.27}$$

holds uniformly for $x \in \mathbf{I}$.

Proof. Let $0 < \epsilon < \min\{b - t_1, t_r - a\}$ and $\Delta = [t_r - \epsilon, t_1 + \epsilon]$. We separate two cases.
Case 1 ($x \in \Delta$). In this case by (2.3) and (2.1),

$$\varphi_t(x) \ge \frac{1}{t} \left[(a_{2t} - x)(x - a_{-2t}) \right]^{1/2} \ge \frac{1}{t} \left[(a_{2t} - t_1 - \epsilon)(t_r - \epsilon - a_{-2t}) \right]^{1/2} \ge \frac{c\delta_t}{t} \tag{2.28}$$

which coupled with (2.5) gives

$$\varphi_t(x) \sim \frac{\delta_t}{t}. \tag{2.29}$$

Hence (2.27) follows.
Case 2 ($x \notin \Delta$). In this case by (2.23),

$$\left|x - t_j\right| \ge \epsilon \ge \frac{c\delta_t}{t} \tag{2.30}$$

and by (1.21)

$$\left|x - t_j\right| \geq \epsilon \geq c\varphi_t(x). \tag{2.31}$$

Again (2.27) follows. □

Corollary 2.8. *Let $W \in \mathcal{F}$. Let* (1.16) *and* (1.20) *prevail. If*

$$\frac{\delta_t}{t^{2/3}T(a_t)^{1/6}} = O(1), \quad t \longrightarrow \infty \tag{2.32}$$

then (2.27) *holds.*

In particular, if $\Lambda \geq 3/2$ then (2.32), (1.21), *and* (2.27) *hold.*

Proof. By (2.7) relation (2.32) implies (1.21). Then by Lemma 2.7 relation (2.27) is valid.

In particular, if $\Lambda \geq 3/2$ then by (2.2) with $\tau = \tau_0$ relation (2.32) is valid and hence (1.21) and (2.27) hold. □

3. Proof of Theorems

3.1. Proof of Theorem 1.7

Denote by ℓ_{kn}'s the fundamental polynomials based on the zeros x_{kn}'s. By Theorem 1.4 and Lemma 11.8 in [8, pages 320-321]

$$\begin{aligned}
&\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2} \\
&\quad = \int_I \left[\ell_{kn}(t)^2 W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2 W(x_{k+1,n})^{-2}\right] W(t)^2 U(t)^2 dt \\
&\quad \geq \int_{x_{k+1,n}}^{x_{kn}} \left[\ell_{kn}(t)^2 W(x_{kn})^{-2} + \ell_{k+1,n}(t)^2 W(x_{k+1,n})^{-2}\right] W(t)^2 U(t)^2 dt \\
&\quad \geq \frac{1}{2}\int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt.
\end{aligned} \tag{3.1}$$

On the other hand, by Theorem 1.4,

$$\begin{aligned}
&\lambda_n(WU; x_{kn})W(x_{kn})^{-2} + \lambda_n(WU; x_{k+1,n})W(x_{k+1,n})^{-2} \\
&\quad \leq c\left[\varphi_n(x_{kn})U_n(x_{kn})^2 + \varphi_n(x_{k+1,n})U_n(x_{k+1,n})^2\right].
\end{aligned} \tag{3.2}$$

Then for $\overline{\varphi}_n(x_{kn}) := \max\{\varphi_n(x_{kn}), \varphi_n(x_{k+1,n})\}$,

$$\int_{x_{k+1,n}}^{x_{kn}} U(t)^2 dt \leq c\overline{\varphi}_n(x_{kn})\left[U_n(x_{kn})^2 + U_n(x_{k+1,n})^2\right]. \tag{3.3}$$

Let j be defined by (2.14). Using Lemma 2.5 it follows from (3.3) that

$$\int_{x_{k+1,n}}^{x_{kn}} \left|t-t_j\right|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn})\left[\left(\left|x_{kn}-t_j\right|+\frac{\delta_n}{n}\right)^{2p_j}+\left(\left|x_{k+1,n}-t_j\right|+\frac{\delta_n}{n}\right)^{2p_j}\right]. \tag{3.4}$$

Further, by (2.27),

$$\int_{x_{k+1,n}}^{x_{kn}} \left|t-t_j\right|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn})\left\{\left[\left|x_{kn}-t_j\right|+\overline{\varphi}_n(x_{kn})\right]^{2p_j}+\left[\left|x_{k+1,n}-t_j\right|+\overline{\varphi}_n(x_{kn})\right]^{2p_j}\right\}. \tag{3.5}$$

By calculation from (3.5) we get

$$\begin{aligned}&\frac{1}{2p_j+1}\left[\left|x_{kn}-t_j\right|^{2p_j+1}+\sigma\left|x_{k+1,n}-t_j\right|^{2p_j+1}\right]\\&\quad=\int_{x_{k+1,n}}^{x_{kn}} \left|t-t_j\right|^{2p_j} dt \le c\overline{\varphi}_n(x_{kn})\left\{\left[\left|x_{kn}-t_j\right|+\overline{\varphi}_n(x_{kn})\right]^{2p_j}+\left[\left|x_{k+1,n}-t_j\right|+\overline{\varphi}_n(x_{kn})\right]^{2p_j}\right\},\end{aligned} \tag{3.6}$$

where

$$\sigma=\begin{cases}1, & t_j\in \mathbf{I}_k,\\ -1, & t_j\notin \mathbf{I}_k.\end{cases} \tag{3.7}$$

We separate two cases.

Case 1 ($p_j \ge 0$). Using Lemma 2.6 it follows from (3.6) that

$$x_{kn}-x_{k+1,n}\le c\overline{\varphi}_n(x_{kn}). \tag{3.8}$$

Case 2 ($p_j < 0$). Suppose without loss of generality that $x_{k+1,n} > t_j$ for the case when $t_j \notin \mathbf{I}_k$. By (3.6),

$$\begin{aligned}&\frac{1}{2p_j+1}\left[\left|x_{kn}-t_j\right|^{2p_j+1}+\sigma\left|x_{k+1,n}-t_j\right|^{2p_j+1}\right]\\&\quad=\int_{x_{k+1,n}}^{x_{kn}} \left|t-t_j\right|^{2p_j} dt \le c_0\overline{\varphi}_n(x_{kn})\min\left\{\overline{\varphi}_n(x_{kn})^{2p_j},\left|x_{k+1,n}-t_j\right|^{2p_j}\right\}.\end{aligned} \tag{3.9}$$

Subcase 2.1 ($t_j \in \mathbf{I}_k$). Inequality (3.9) gives

$$\left|x_{\kappa n}-t_j\right|^{2p_j+1}\le c\overline{\varphi}_n(x_{kn})^{2p_j+1},\quad \kappa=k,k+1 \tag{3.10}$$

which yields (3.8).

Subcase 2.2 ($t_j \notin \mathbf{I}_k$). In this case we distinguish two subcases.

(1) $|x_{k+1,n} - t_j| \geq 2c_0\overline{\varphi}_n(x_{kn})$, where c_0 is given by (3.9). In this case

$$\begin{aligned}\int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j} dt &= \int_{x_{k+1,n}}^{x_{kn}} (t-t_j)(t-t_j)^{2p_j-1} dt \\ &\geq (x_{k+1,n}-t_j)\int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j-1} dt \\ &= (x_{k+1,n}-t_j)\frac{1}{2|p_j|}\left[(x_{k+1,n}-t_j)^{2p_j} - (x_{kn}-t_j)^{2p_j}\right] \\ &\geq \frac{c_0\overline{\varphi}_n(x_{kn})}{|p_j|}\left[(x_{k+1,n}-t_j)^{2p_j} - (x_{kn}-t_j)^{2p_j}\right],\end{aligned} \tag{3.11}$$

which by (3.9) gives

$$(x_{k+1,n}-t_j)^{2p_j} \leq (1-|p_j|)^{-1}(x_{kn}-t_j)^{2p_j} \leq 2(x_{kn}-t_j)^{2p_j}. \tag{3.12}$$

On the other hand, by (3.9) and (3.12),

$$\begin{aligned}c_0\overline{\varphi}_n(x_{kn})(x_{k+1,n}-t_j)^{2p_j} &\geq \int_{x_{k+1,n}}^{x_{kn}} (t-t_j)^{2p_j} dt \geq (x_{kn}-t_j)^{2p_j}(x_{kn}-x_{k+1,n}) \\ &\geq \frac{1}{2}(x_{k+1,n}-t_j)^{2p_j}(x_{kn}-x_{k+1,n})\end{aligned} \tag{3.13}$$

and hence (3.8) follows.

(2) $|x_{k+1,n} - t_j| < 2c_0\overline{\varphi}_n(x_{kn})$. By (3.9),

$$\begin{aligned}c_0\overline{\varphi}_n(x_{kn})^{2p_j+1} &\geq \frac{1}{2p_j+1}\left[(x_{kn}-t_j)^{2p_j+1} - (x_{k+1,n}-t_j)^{2p_j+1}\right] \\ &\geq \frac{1}{2p_j+1}\left[(x_{kn}-t_j)^{2p_j+1} - (2c_0\overline{\varphi}_n(x_{kn}))^{2p_j+1}\right].\end{aligned} \tag{3.14}$$

So $x_{kn} - t_j \leq c\overline{\varphi}_n(x_{kn})$ and (3.8) follows.

Finally, applying Theorem 5.7(b) in [1, page 125] we conclude $\overline{\varphi}_n(x_{kn}) \sim \varphi_n(x_{kn})$ and hence (1.22) follows from (3.8).

3.2. *Proof of Theorem 1.8*

For $P \in \mathcal{P}_{t-\rho-2/p}$, we have $PU \in \mathcal{P}_{t-2/p}$ and hence apply Theorem 1.8 in [1, page 15] to obtain (1.23).

3.3. Proof of Theorem 1.9

Use the same argument as that of Theorem 11.1 in [1, page 313].

3.4. Proof of Theorem 1.10

We give the proofs of (1.26) and (1.28) only, the proofs of (1.27) and (1.29) being similar.

First let us prove (1.26). Choose $\alpha, \beta > 1$ so that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad 2\beta p_i > -1, \quad i = 1, \ldots, r. \tag{3.15}$$

Let L_n denote the linear map of Δ_n onto $[-1, 1]$. By Lemma 11.7 in [1, page 318] there exists $y_n \in \Delta_n$ such that

$$L_n(y_n) = \cos\frac{2\pi}{m}, \quad m = m(n), \tag{3.16}$$

and for large enough n and $R_n \in \mathbf{P}_{n-2m}$ such that

$$R_n(x)W(x)^{1/\alpha} \geq C_1, \quad x \in [0, y_n], \tag{3.17}$$

$$\left\| R_n W^{1/\alpha} \right\|_{L_\infty(\mathbf{I})} \leq C_2. \tag{3.18}$$

Using (11.7) in [1, page 318] in the form

$$1 - \frac{x_{1n}}{a_n} = \min_{P \in \mathbf{P}_{n-1}} \frac{\int_{\mathbf{I}} (1 - x/a_n)(PUW)^2(x)dx}{\int_{\mathbf{I}} (PUW)^2(x)dx}. \tag{3.19}$$

Again choose [1, page 319]

$$P(x) = R_n(x) V_{m,\cos(2\pi/m)}(L_n(x))^2 \in \mathbf{P}_{n-2}. \tag{3.20}$$

Applying Theorem 1.5 and (3.18), and using the same argument as that in [1, pages 319-320], we can get

$$
\begin{aligned}
&\int_{\mathbf{I}}\left(1-\frac{x}{a_n}\right)(PUW)^2(x)dx\\
&\quad\le c\int_{\Delta_n}\left(1-\frac{x}{a_n}\right)(PUW)^2(x)dx\\
&\quad= c\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]\left[\left(U(x)W(x)^{1/\beta}\right)^2\right]dx\\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]^{\alpha}dx\right\}^{1/\alpha}\left\{\int_{\Delta_n}\left[\left(U(x)W(x)^{1/\beta}\right)^2\right]^{\beta}dx\right\}^{1/\beta}\\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)\left(P(x)W(x)^{1/\alpha}\right)^2\right]^{\alpha}dx\right\}^{1/\alpha}\\
&\quad\le c\left\{\int_{\Delta_n}\left[\left(1-\frac{x}{a_n}\right)V_{m,\cos(2\pi/m)}(L_n(x))^4\right]^{\alpha}dx\right\}^{1/\alpha}\\
&\quad=\frac{c\delta_n}{a_n}\left\{\int_{\Delta_n}\left[(1-L_n(x))V_{m,\cos(2\pi/m)}(L_n(x))^4\right]^{\alpha}dx\right\}^{1/\alpha}\\
&\quad=\frac{c\delta_n^2}{a_n}\left\{\int_{-1}^{1}\left[(1-t)V_{m,\cos(2\pi/m)}(t)^4\right]^{\alpha}dt\right\}^{1/\alpha}\\
&\quad\le\frac{c\delta_n^2}{a_nm^4}\left\{\int_{-\infty}^{\infty}\left[(1+|v|)\min\left\{1,\frac{c}{|v|}\right\}^4\right]^{\alpha}dv\right\}^{1/\alpha}\\
&\quad\le ca_n\eta_n^2.
\end{aligned}
\tag{3.21}
$$

On the other hand, by (3.17),

$$
\begin{aligned}
\int_{\mathbf{I}}(PUW)^2(x)dx &\ge \int_{y_n(1-C_1\eta_n)}^{y_n}(PUW)^2(x)dx\\
&\ge \int_{y_n(1-C_1\eta_n)}^{y_n}V_{m,\cos(2\pi/m)}(L_n(x))^4U(x)^2dx.
\end{aligned}
\tag{3.22}
$$

By (1.20) for large enough n, we have

$$
U(x)\ge c>0,\quad x\in\left[y_n(1-C_1\eta_n),y_n\right].
\tag{3.23}
$$

Hence (3.22) implies

$$\begin{aligned}\int_{\mathbf{I}}(PUW)^2(x)dx &\geq c\int_{y_n(1-C_1\eta_n)}^{y_n} V_{m,\cos(2\pi/m)}(L_n(x))^4dx\\ &= c\delta_n\int_{\cos(2\pi/m)-C_1y_n\eta_n/\delta_n}^{\cos(2\pi/m)} V_{m,\cos(2\pi/m)}(t)^4dt.\end{aligned} \tag{3.24}$$

But in [1, page 320] the following estimate is given:

$$\delta_n\int_{\cos(2\pi/m)-C_1y_n\eta_n/\delta_n}^{\cos(2\pi/m)} V_{m,\cos(2\pi/m)}(t)^4dt \geq ca_n\eta_n. \tag{3.25}$$

Substituting this estimate into (3.24) gives

$$\int_{\mathbf{I}}(PUW)^2(x)dx \geq ca_n\eta_n \tag{3.26}$$

which coupled with (3.21) yields (1.26).

Next let us prove (1.28). We already know that

$$a_n(1-c\eta_n) \leq x_{1n} < a_{n+\rho+1/2} = a_n(1+o(\eta_n)), \tag{3.27}$$

by (1.26) and (1.24). We must prove that, for some $c_1 > 0$, and n large enough, we have

$$x_{1n} < a_n(1-c_1\eta_n). \tag{3.28}$$

We use the idea for the proof of Corollary 13.4(b) in [1, pages 380-381] with modification. By the same argument as that proof with $A = a_{n+\rho+1/2}(1-\varepsilon\eta_n)$ instead, applying Theorem 1.8 we obtain

$$1-\frac{x_{1n}}{A} = \lambda_n\left((UW)^2, x_{1n}\right)^{-1}\int_{\mathbf{I}}\left(1-\frac{x}{A}\right)(\ell_{1n}UW)(x)^2dx, \tag{3.29}$$

$$\begin{aligned}&\int_{\mathbf{I}}\left(1-\frac{x}{A}\right)(\ell_{1n}UW)(x)^2dx\\ &\quad= \int_a^A\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx - \int_A^b\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\quad\geq \int_a^A\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\qquad - \int_A^{a_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx - \int_{\Delta_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx\\ &\quad\geq -2\int_A^{a_{n+\rho+1/2}}\left|1-\frac{x}{A}\right|(\ell_{1n}UW)(x)^2dx,\end{aligned} \tag{3.30}$$

where ℓ_{1n} denotes the fundamental polynomial of Lagrange interpolation based on the zeros of the nth orthogonal polynomial with respect to the weight $(UW)^2$.

But

$$\begin{aligned}
&\int_A^{a_{n+\rho+1/2}} \left|1 - \frac{x}{A}\right| (\ell_{1n}UW)(x)^2 dx \\
&\quad \le \left(\frac{a_{n+\rho+1/2}}{A} - 1\right) \int_{\mathbf{I}} (\ell_{1n}UW)(x)^2 dx = \left(\frac{a_{n+\rho+1/2}}{A} - 1\right) \lambda_n\left((UW)^2, x_{1n}\right)
\end{aligned} \tag{3.31}$$

which, coupled with (3.30) and (3.29), gives

$$1 - \frac{x_{1n}}{A} \ge -c\varepsilon\eta_n. \tag{3.32}$$

Thus

$$\begin{aligned}
\frac{x_{1n}}{a_n} &= \frac{x_{1n}}{A} \frac{A}{a_{n+\rho+1/2}} \frac{a_{n+\rho+1/2}}{a_n} \\
&\le (1 + c\varepsilon\eta_n)(1 - \varepsilon\eta_n)(1 + o(\eta_n)) \\
&< 1 - c_1\eta_n,
\end{aligned} \tag{3.33}$$

for n large enough, provided $\varepsilon > 0$ is small enough.

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References

[1] A. L. Levin and D. S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, vol. 4 of *CMS Books in Mathematics*, Springer, New York, NY, USA, 2001.

[2] A. L. Levin and D. S. Lubinsky, "Christoffel functions and orthogonal polynomials for exponential weights on $[-1, 1]$," *Memoirs of the American Mathematical Society*, no. 535, 1994.

[3] A. L. Levin and D. S. Lubinsky, "Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 199–256, 2005.

[4] A. L. Levin and D. S. Lubinsky, "Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$. II," *Journal of Approximation Theory*, vol. 139, no. 1-2, pp. 107–143, 2006.

[5] T. Kasuga and R. Sakai, "Orthonormal polynomials with generalized Freud-type weights," *Journal of Approximation Theory*, vol. 121, no. 1, pp. 13–53, 2003.

[6] Y. G. Shi, "On generalized Christoffel functions," *Acta Mathematica Hungarica*, vol. 135, no. 3, pp. 213–228, 2012.

[7] G. Mastroianni and V. Totik, "Uniform spacing of zeros of orthogonal polynomials," *Constructive Approximation*, vol. 32, no. 2, pp. 181–192, 2010.

[8] Y. G. Shi and R. Liu, "Generalized Christoffel functions for Jacobi-exponential weights on $[-1, 1]$," submitted to *Acta Mathematica Scientia*.

[9] Y. G. Shi, *Power Orthogonal Polynomials*, Nova Science Publishers, New York, NY, USA, 2006.