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Research Article

Generalized Difference Spaces of Non-Absolute Type of Convergent and Null Sequences

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The aim of the present paper is to introduce the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of generalized difference sequences which generalize the paper due to Mursaleen and Noman (2010). These spaces are the BK-spaces of non-absolute type and norm isomorphic to the spaces c_0 and c, respectively. Furthermore, we derive some inclusion relations determine the α-, β-, and γ-duals of those spaces, and construct their Schauder bases. Finally, we characterize some matrix classes from the spaces $c_0^\lambda(B)$, and $c^\lambda(B)$ to the spaces ℓ_p, c_0, and c.

1. Introduction

By ω, we denote the space of all complex valued sequences. Any vector subspace of ω is called a *sequence space*. A sequence space E with a linear topology is called a K-space provided each of the maps $p_i : E \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space is called an FK-space provided E is a complete linear metric space. An FK-space whose topology is normable is called a BK-space (see [1, pages 272-273]) which contains ϕ, the set of all finitely nonzero sequences. We write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent, and null sequences, respectively, which are BK-spaces with the usual sup-norm defined by $\|x\|_\infty = \sup|x_k|$, where, here and in the sequel, the supremum is taken over all $k \in \mathbb{N}$. Also by ℓ_1 and ℓ_p, we denote the spaces of all absolutely and p-absolutely convergent series, respectively, which are BK-spaces with the usual norm defined by $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$, where $1 \leq p < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively.

Let X and Y be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}$, A-transform of x, exists and is in Y, where

$$A_n(x) := \sum_k a_{nk} x_k, \quad \forall n \in \mathbb{N}. \tag{1.1}$$

By $(X : Y)$, we denote the class of all infinite matrices $A = (a_{nk})$ such that $A : X \to Y$. Thus $A \in (X : Y)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$. A sequence $x \in \omega$ is said to be A-summable to l if Ax converges to l, which is called the A-limit of x.

The *domain* X_A of an infinite matrix A in a sequence space X is defined by

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}. \tag{1.2}$$

We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. Also, we write $e^{(k)}$ for the sequence whose only nonzero term is a 1 in the kth place for each $k \in \mathbb{N}$.

The approach of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, for example, [2–14]. They introduced the sequence spaces $(\ell_\infty)_{N_q}$ and c_{N_q} in [14], $(\ell_p)_{C_1} = X_p$ and $(\ell_\infty)_{C_1} = X_\infty$ in [10], $\mu_G = Z(u, v; \mu)$ in [9], $(\ell_\infty)_{R^t} = r_\infty^t$, $c_{R^t} = r_c^t$ and $(c_0)_{R^t} = r_0^t$ in [8], $(\ell_p)_{R^t} = r_p^t$ in [2], $(c_0)_{E^t} = e_0^r$ and $c_{E^r} = e_c^r$ in [3], $(\ell_p)_{E^r} = e_p^r$ and $(\ell_\infty)_{E^r} = e_\infty^r$ in [4], $(c_0)_{A^r} = a_0^r$ and $c_{A^r} = a_c^r$ in [5], $[c_0(u, p)]_{A^r} = a_0^r(u, p)$ and $[c(u, p)]_{A^r} = a_c^r(u, p)$ in [6], $(\ell_p)_{A^r} = a_p^r$ and $(\ell_\infty)_{A^r} = a_\infty^r$ in [7] and $(c_0)_{C_1} = \widetilde{c}_0$ and $c_{C_1} = \widetilde{c}$ in [11], $\nu_{B(r,s,t)} = \nu(B)$ in [12], and $f_{B(r,s,t)} = f(B)$ in [13]; where, N_q, C_1, R^t, and E^r denote the Nörlund, Cesáro, Riesz, and Euler means, respectively, A^r, G, and $B(r, s, t)$ are, respectively, defined in [5, 9, 12], $\mu \in \{c_0, c, \ell_p\}$, $\nu \in \{\ell_\infty, c, c_0, \ell_p\}$ and $1 \leq p < \infty$. Also $c_0(u, p)$ and $c(u, p)$ denote the sequence spaces generated from the Maddox's spaces $c_0(p)$ and $c(p)$ by Başarir [15]. In the present paper, following [2–14], we introduce the difference sequence spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of non-absolute type and derive some related results. We also establish some inclusion relations. Furthermore, we determine the α-, β-, and γ-duals of those spaces and construct their bases. Finally, we characterize some classes of infinite matrices concerning the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$.

The rest of this paper is organized, as follows.

In Section 2, the BK-spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of generalized difference sequences are introduced. Section 3 is devoted to inclusion relations concerning with the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$. In Sections 4 and 5, the Schauder bases of the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ are given and the α-, β-, and γ-duals of the generalized difference sequence spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of non-absolute type are determined, respectively. In Section 6, the classes $(c^\lambda(B) : \ell_p)$, $(c_0^\lambda(B) : \ell_p)$, $(c^\lambda(B) : c)$, $(c^\lambda(B) : c_0)$, $(c_0^\lambda(B) : c)$, and $(c_0^\lambda(B) : c_0)$ of matrix transformations are characterized, where $1 \leq p \leq \infty$. Also, by means of a given basic lemma, the characterizations of some other classes involving the Euler, difference, Riesz, and Cesàro sequence spaces are derived. In the final section of the paper, we note the significance of the present results in the literature related with difference sequence spaces and record some further suggestions.

2. The Difference Sequence Spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of Non-Absolute Type

The difference sequence spaces have been studied by several authors in different ways (see e.g. [12, 16–21]). In the present section, we introduce the spaces $c_0^\lambda(\Delta)$, and $c^\lambda(\Delta)$, and show that these spaces are BK-spaces of non-absolute type which are norm isomorphic to the spaces c_0 and c, respectively.

We assume throughout that $\lambda = (\lambda_k)_{k=0}^\infty$ is a strictly increasing sequence of positive reals tending to ∞, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots, \qquad \lim_{k\to\infty} \lambda_k = \infty. \tag{2.1}$$

Recently, Mursaleen and Noman [22] studied the sequence spaces c_0^λ and c^λ of non-absolute type, and later they introduced the difference sequence spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ in [21] of non-absolute type as follows:

$$\begin{gathered} c_0^\lambda(\Delta) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) = 0 \right\}, \\ c^\lambda(\Delta) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(x_k - x_{k-1}) \text{ exists} \right\}. \end{gathered} \tag{2.2}$$

Here and after, we use the convention that any term with a negative subscript is equal to zero, for example, $\lambda_{-1} = 0$ and $x_{-1} = 0$. With the notation of (1.2) we can redefine the spaces $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ by

$$c_0^\lambda(\Delta) = \left(c_0^\lambda\right)_\Delta, \qquad c^\lambda(\Delta) = \left(c^\lambda\right)_\Delta, \tag{2.3}$$

where Δ denotes the band matrix representing the difference operator, that is, $\Delta x = (x_k - x_{k-1}) \in \omega$ for $x = (x_k) \in \omega$.

Let r and s be nonzero real numbers and define the generalized difference matrix $B(r,s) = \{b_{nk}(r,s)\}$ by

$$b_{nk}(r,s) := \begin{cases} r, & k = n, \\ s, & k = n-1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

for all $k, n \in \mathbb{N}$. The $B(r,s)$-transform of a sequence $x = (x_k)$ is

$$B(r,s)_k(x) = r x_k + s x_{k-1}, \quad \forall k \in \mathbb{N}. \tag{2.5}$$

We note that the matrix $B(r,s)$ can be reduced to the difference matrices Δ in case $r = 1$ and $s = -1$. So, the results related to the matrix domain of the matrix $B(r,s)$ are more general and more comprehensive than the consequences of the matrices domain of Δ and include them.

Now, following Başar and Altay [18] and Aydın and Başar [17], we proceed slightly differently to Kızmaz [19] and the other authors following him and employ a technique of obtaining a new sequence space by means of the matrix domain of a triangle limitation method.

We thus introduce the difference sequence spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$, which are the generalization of the spaces $c_0^{\lambda}(\Delta)$ and $c^{\lambda}(\Delta)$ introduced by Mursaleen and Noman [21], as follows:

$$c_0^{\lambda}(B) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1}) = 0 \right\},$$
$$c^{\lambda}(B) = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1}) \text{ exists} \right\}. \tag{2.6}$$

With the notation of (1.2), we can redefine the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ as

$$c_0^{\lambda}(B) = \left(c_0^{\lambda}\right)_B, \qquad c^{\lambda}(B) = \left(c^{\lambda}\right)_B, \tag{2.7}$$

where B denotes the generalized difference matrix $B(r,s) = \{b_{nk}(r,s)\}$ defined by (2.4).

It is immediate by (2.7) that the sets $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are linear spaces with coordinatewise addition and scalar multiplication, that is, $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are the sequence spaces of generalized differences.

On the other hand, we define the triangle matrix $\widehat{\Lambda} = (\widehat{\lambda}_{nk})$ by

$$\widehat{\lambda}_{nk} := \begin{cases} \dfrac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k)}{\lambda_n}, & k < n, \\ r\dfrac{(\lambda_n - \lambda_{n-1})}{\lambda_n}, & k = n, \\ 0, & k > n \end{cases} \tag{2.8}$$

for all $n, k \in \mathbb{N}$. With a direct calculation we derive the equality

$$\widehat{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1}), \quad \forall n \in \mathbb{N} \tag{2.9}$$

and every $x = (x_k) \in \omega$ which leads us together with (1.2) to the fact that

$$c_0^{\lambda}(B) = (c_0)_{\widehat{\Lambda}}, \qquad c^{\lambda}(B) = c_{\widehat{\Lambda}}. \tag{2.10}$$

Further, for any sequence $x = (x_k)$ we define the sequence $y(\lambda) = \{y_k(\lambda)\}$ which will be frequently used as the $\widehat{\Lambda}$-transform of x, that is, $y(\lambda) = \widehat{\Lambda}(x)$ and so we have

$$y_k(\lambda) = \sum_{j=0}^{k-1} \frac{r(\lambda_j - \lambda_{j-1}) + s(\lambda_{j+1} - \lambda j)}{\lambda_k} x_j + r\frac{\lambda_k - \lambda_{k-1}}{\lambda_k} x_k, \quad \forall k \in \mathbb{N}, \tag{2.11}$$

Where, here and in what follows, the summation running from 0 to $k-1$ is equal to zero when $k=0$.

Moreover, it is clear by (2.9) that the relation (2.11) can be written as follows:

$$y_k(\lambda) = \frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})(rx_j + sx_{j-1}), \quad \forall k \in \mathbb{N}. \tag{2.12}$$

We assume throughout that the sequences $x=(x_k)$ and $y=(y_k)$ are connected by the relation (2.11).

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *The difference sequence spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are BK-spaces with the norm $\|x\|_{c_0^{\lambda}(B)} = \|x\|_{c^{\lambda}(B)} = \|\widehat{\Lambda}(x)\|_{\infty}$, that is,*

$$\|x\|_{c_0^{\lambda}(B)} = \|x\|_{c^{\lambda}(B)} = \sup_{n\in\mathbb{N}}\left|\widehat{\Lambda}_n(x)\right|. \tag{2.13}$$

Proof. Since (2.10) holds and c_0 and c are BK-spaces with respect to their natural norms (see [23, pages 217-218]) and the matrix $\widehat{\Lambda}$ is a triangle, Theorem 4.3.12 of Wilansky [24, page 63] gives the fact that $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are BK-spaces with the given norms. This completes the proof. □

Remark 2.2. One can easily check that the absolute property does not hold on the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$, that is, $\|x\|_{c_0^{\lambda}(B)} \neq \||x|\|_{c_0^{\lambda}(B)}$ and $\|x\|_{c^{\lambda}(B)} \neq \||x|\|_{c^{\lambda}(B)}$ for at least one sequence in the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$, and this shows that $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$.

Now, we give the final theorem of this section.

Theorem 2.3. *The sequence spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ of non-absolute type are norm isomorphic to the spaces c_0 and c, respectively, that is, $c_0^{\lambda}(B) \cong c_0$ and $c^{\lambda}(B) \cong c$.*

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0^{\lambda}(B)$ and c_0. Consider the transformation T defined, with the notation of (2.11), from $c_0^{\lambda}(B)$ to c_0 by $x \mapsto y(\lambda)$. Then, $Tx = y(\lambda) = \widehat{\Lambda}(x) \in c_0$ for every $x \in c_0^{\lambda}(B)$ and the linearity of T is clear. Further, it is trivial that $x=\theta$ whenever $Tx=\theta$ and hence T is injective.

Furthermore, let $y=(y_k)\in c_0$ and define the sequence $x=\{x_k(\lambda)\}$ by

$$x_k(\lambda) := \frac{1}{r}\sum_{j=0}^{k}\left(\frac{-s}{r}\right)^{k-j}\sum_{i=j-1}^{j}(-1)^{j-i}\frac{\lambda_i}{\lambda_j - \lambda_{j-1}}y_i, \quad \forall k\in\mathbb{N}. \tag{2.14}$$

Then, we obtain

$$rx_k(\lambda) + sx_{k-1}(\lambda) = \sum_{i=k-1}^{k}(-1)^{k-i}\frac{\lambda_i}{\lambda_k - \lambda_{k-1}}y_i, \quad \forall k\in\mathbb{N}. \tag{2.15}$$

Hence, for every $n \in \mathbb{N}$, we get by (2.9)

$$\begin{aligned}\widehat{\Lambda}_n(x) &= \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1}) \\ &= \frac{1}{\lambda_n}\sum_{k=0}^{n}\sum_{i=k-1}^{k}(-1)^{k-i}\lambda_i y_i \\ &= y_n.\end{aligned} \tag{2.16}$$

This shows that $\widehat{\Lambda}(x) = y$ and since $y \in c_0$, we conclude that $\widehat{\Lambda}(x) \in c_0$. Thus, we deduce that $x \in c_0^{\lambda}(B)$ and $Tx = y$. Hence T is surjective.

Moreover, one can easily see for every $x \in c_0^{\lambda}(B)$ that

$$\|Tx\|_{\infty} = \|y(\lambda)\|_{\infty} = \left\|\widehat{\Lambda}(x)\right\|_{\infty} = \|x\|_{c_0^{\lambda}(B)} \tag{2.17}$$

which means that T is norm preserving. Consequently T is a linear bijection which show that the spaces $c_0^{\lambda}(B)$ and c_0 are linearly isomorphic.

It is clear that if the spaces $c_0^{\lambda}(B)$ and c_0 are replaced by the spaces $c^{\lambda}(B)$ and c, respectively, then we obtain the fact that $c^{\lambda}(B) \cong c$. This completes the proof. □

3. The Inclusion Relations

In the present section, we establish some inclusion relations concerning with the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$. We may begin with the following theorem.

Theorem 3.1. *The inclusion $c_0^{\lambda}(B) \subset c^{\lambda}(B)$ strictly holds.*

Proof. It is obvious that the inclusion $c_0^{\lambda}(B) \subset c^{\lambda}(B)$ holds. Further to show that this inclusion is strict, consider the sequence $x = (x_k)$ defined by $x_k = \sum_{j=0}^{k}(-s/r)^j/r$ for all $k \in \mathbb{N}$. Then, we obtain by (2.9) that

$$\widehat{\Lambda}_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1}) \quad \forall n \in \mathbb{N} \tag{3.1}$$

which shows that $\widehat{\Lambda}(x) = e$ and hence $\widehat{\Lambda}(x) \in c \setminus c_0$, where $e = (1, 1, 1, \ldots)$. Thus, the sequence x is in $c^{\lambda}(B)$ but not in $c_0^{\lambda}(B)$. Hence, the inclusion $c_0^{\lambda}(B) \subset c^{\lambda}(B)$ is strict and this completes the proof. □

Theorem 3.2. *If $s + r = 0$, then the inclusion $c \subset c_0^{\lambda}(B)$ strictly holds.*

Proof. Suppose that $s + r = 0$ and $x \in c$. Then $B(r, s)x = (rx_k + sx_{k-1}) \in c_0$ and hence $B(r, s)x \in c_0^{\lambda}$, since the inclusion $c_0 \subset c_0^{\lambda}$. This shows that $x \in c_0^{\lambda}(B)$. Consequently, the inclusion $c \subset c_0^{\lambda}(B)$ holds. Further consider the sequence $y = (y_k)$ defined by $y_k = \sqrt{k+1}$ for all $k \in \mathbb{N}$. Then, it is trivial that $y \notin c$. On the other hand, it can easily seen that $B(r, s)y \in c_0$. Hence,

$B(r,s)y \in c_0^{\lambda}$ which means that $y \in c_0^{\lambda}(B)$. Thus, the sequence y is in $c_0^{\lambda}(B)$ but not in c. We therefore deduce that the inclusion $c \subset c_0^{\lambda}(B)$ is strict. This completes the proof. □

On the other hand, we recall that if $A \in (c : c)$ and $B \in (c : c)$, then $AB \in (c : c)$, namely, $\widehat{\Lambda} = (\widehat{\lambda}_{nk})$ is stronger than the ordinary convergence, hence we have the following

Corollary 3.3. *The inclusions $c_0 \subset c_0^{\lambda}(B)$ and $c \subset c^{\lambda}(B)$ strictly hold.*

Further, it is obvious that the sequence y, defined in the proof of Theorem 3.2, is in $c_0^{\lambda}(B)$ but not in ℓ_∞. This leads us to the following result.

Corollary 3.4. *Although the spaces ℓ_∞ and $c_0^{\lambda}(B)$ overlap, the space ℓ_∞ does not include the space $c_0^{\lambda}(B)$.*

Now, to prove the next theorem, we need the following lemma [24, page 4].

Lemma 3.5. *$A \in (\ell_\infty : c_0)$ if and only if $\lim_{n\to\infty} \sum_k |a_{nk}| = 0$.*

Theorem 3.6. *The inclusion $\ell_\infty \subset c_0^{\lambda}(B)$ strictly holds if and only if $z \in c_0^{\lambda}$, where the sequence $z = (z_k)$ is defined by*

$$z_k = \left| \frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k)}{\lambda_k - \lambda_{k-1}} \right|, \quad \forall k \in \mathbb{N}. \tag{3.2}$$

Proof. Suppose that the inclusion $\ell_\infty \subset c_0^{\lambda}(B)$ holds. Then we obtain that $\widehat{\Lambda}(x) \in c_0$ for every $x \in \ell_\infty$ and hence the matrix $\widehat{\Lambda} = (\widehat{\lambda}_{nk})$ is in the class $(\ell_\infty : c_0)$. Thus it follows by Lemma 3.5 that

$$\lim_{n\to\infty} \sum_k \left| \widehat{\lambda}_{nk} \right| = 0. \tag{3.3}$$

Now, by taking into account the definition of the matrix $\widehat{\Lambda} = (\widehat{\lambda}_{nk})$ given by (2.8), we have for every $n \in \mathbb{N}$ that

$$\sum_k \left| \widehat{\lambda}_{nk} \right| = \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k)| + |r| \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}. \tag{3.4}$$

Thus, the condition (3.3) implies both

$$\lim_{n\to\infty} |r| \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = 0, \tag{3.5}$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k)| = 0. \tag{3.6}$$

Now we have for every $n \geq 1$ that

$$\frac{1}{\lambda_n}\sum_{k=0}^{n-1}|r(\lambda_k-\lambda_{k-1})+s(\lambda_{k+1}-\lambda_k)| = \frac{\lambda_{n-1}}{\lambda_n}\left[\frac{1}{\lambda_{n-1}}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k\right] \tag{3.7}$$

and since $\lim_{n\to\infty}(\lambda_{n-1}/\lambda_n)=1$ by (3.5), we obtain by (3.6) that

$$\lim_{n\to\infty}\frac{1}{\lambda_{n-1}}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k = 0 \tag{3.8}$$

which shows that $z=(z_k)\in c_0^{\lambda}$.

Conversely, suppose that $z=(z_k)\in c_0^{\lambda}$. Then we have (3.8). Further, for every $n\geq 1$, we derive that

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k=0}^{n-1}|r(\lambda_k-\lambda_{k-1})+s(\lambda_{k+1}-\lambda_k)| &= \frac{1}{\lambda_n}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k\\ &\leq \frac{1}{\lambda_{n-1}}\sum_{k=0}^{n-1}(\lambda_k-\lambda_{k-1})z_k.\end{aligned} \tag{3.9}$$

Then, (3.9) and (3.8) together imply that (3.6) holds. On the other hand, we have for every $n\geq 1$ that

$$\begin{aligned}\left|\frac{r\lambda_{n-1}+s(\lambda_n-\lambda_0)}{\lambda_n}\right| &= \left|\frac{1}{\lambda_n}\sum_{k=0}^{n-1}r(\lambda_k-\lambda_{k-1})+s(\lambda_{k+1}-\lambda_k)\right|\\ &\leq \frac{1}{\lambda_n}\sum_{k=0}^{n-1}|r(\lambda_k-\lambda_{k-1})+s(\lambda_{k+1}-\lambda_k)|.\end{aligned} \tag{3.10}$$

Therefore, it follows by (3.6) that $\lim_{n\to\infty}[r\lambda_{n-1}+s(\lambda_n-\lambda_0)]/\lambda_n=0$. Particularly, if we take $r=1$ and $s=-1$, then we have $\lim_{n\to\infty}[\lambda_n-\lambda_{n-1}-\lambda_0]/\lambda_n=0$ which shows that (3.5) holds. Thus, we deduce by the relation (3.4) that (3.3) holds. This leads us with Lemma 3.5 to the consequence that $\widehat{\Lambda}\in(\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty\subset c_0^{\lambda}(B)$ holds and is strict by Corollary 3.4. This completes the proof. □

4. The Bases for the Spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$

In the present section, we give two sequences of the points of the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ which form the bases for those spaces.

If a normed sequence space X contains a sequence (b_n) with the property that for every $x\in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty}\|x-(\alpha_0b_0+\alpha_1b_1+\cdots+\alpha_nb_n)\|=0 \tag{4.1}$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for X. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and is written as $x = \sum_k \alpha_k b_k$.

Now, since the transformation T defined from $c_0^\lambda(B)$ to c_0 in the proof of Theorem 2.3 is an isomorphism, the inverse image of the basis $\{e^{(k)}\}_{k=0}^{\infty}$ of the space c_0 is the basis for the new space $c_0^\lambda(B)$. Therefore, we have the following.

Theorem 4.1. *Let $\alpha_k(\lambda) = \widehat{\Lambda}_k(x)$ for all $k \in \mathbb{N}$ and $l = \lim_{k\to\infty}\widehat{\Lambda}_k(x)$. Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{k=0}^{\infty}$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(\lambda) := \begin{cases} \left(\dfrac{-s}{r}\right)^{n-k}\left[\dfrac{\lambda_k}{r(\lambda_k - \lambda_{k-1})} + \dfrac{\lambda_k}{s(\lambda_{k+1} - \lambda_k)}\right], & k < n, \\ \dfrac{1}{r}\dfrac{\lambda_k}{(\lambda_k - \lambda_{k-1})}, & k = n, \\ 0, & k > n. \end{cases} \tag{4.2}$$

Then, the following statements hold.

(a) *The sequence $\{b^{(k)}(\lambda)\}_{k=0}^{\infty}$ is a basis for the space $c_0^\lambda(B)$ and any $x \in c_0^\lambda(B)$ has a unique representation of the form $x = \sum_k \alpha_k(\lambda) b^{(k)}(\lambda)$.*

(b) *The sequence $\{b, b^{(0)}(\lambda), b^{(1)}(\lambda), \ldots\}$ is a basis for the space $c^\lambda(B)$ and any $x \in c^\lambda(B)$ has a unique representation of the form $x = lb + \sum_k[\alpha_k(\lambda) - l]b^{(k)}(\lambda)$, where $b = (b_k) = \{\sum_{j=0}^{k}(-s/r)^j/r\}_{k=0}^{\infty}$.*

Finally, it easily follows from Theorem 2.1 that $c_0^\lambda(B)$ and $c^\lambda(B)$ are the Banach spaces with their natural norms. Then by Theorem 4.1 we obtain the following.

Corollary 4.2. *The difference sequence spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ are seprable.*

5. The α-, β-, and γ-Duals of the Spaces $c_0^\lambda(B)$ and $c^\lambda(B)$

In this section, we state and prove the theorems determining the α-, β-, and γ-duals of the generalized difference sequence spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ of non-absolute type.

For arbitrary sequence spaces X and Y, the set $M(X, Y)$ defined by

$$M(X, Y) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in Y \ \forall x = (x_k) \in X\} \tag{5.1}$$

is called the *multiplier space* of X and Y. One can easily observe for a sequence space Z with $Y \subset Z$ and $Z \subset X$ that the inclusions $M(X, Y) \subset M(X, Z)$ and $M(X, Y) \subset M(Z, Y)$ hold, respectively.

With the notation of (5.1), the α-, β-, and γ-duals of a sequence space X, which are respectively, denoted by X^α, X^β, and X^γ, are defined by

$$X^\alpha = M(X, \ell_1), \qquad X^\beta = M(X, cs), \qquad X^\gamma = M(X, bs). \tag{5.2}$$

It is clear that $X^\alpha \subset X^\beta \subset X^\gamma$. Also it can be obviously seen that the inclusions $X^\alpha \subset Y^\alpha$, $X^\beta \subset Y^\beta$, and $X^\gamma \subset Y^\gamma$ hold whenever $Y \subset X$.

Now, we may begin with quoting the following lemmas (see [25]) which are needed to prove Theorems 5.5 to 5.8.

Lemma 5.1. *$A = (a_{nk}) \in (c_0 : \ell_1) = (c : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty. \tag{5.3}$$

Lemma 5.2. *$A = (a_{nk}) \in (c_0 : c)$ if and only if*

$$\lim_{n \to \infty} a_{nk} = \alpha_k \quad \text{for each fixed } k \in \mathbb{N}, \tag{5.4}$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{5.5}$$

Lemma 5.3. *$A = (a_{nk}) \in (c : c)$ if and only if (5.4) and (5.5) hold, and*

$$\lim_{n \to \infty} \sum_k a_{nk} \text{ exists}. \tag{5.6}$$

Lemma 5.4. *$A = (a_{nk}) \in (c : \ell_\infty) = (c_0 : \ell_\infty)$ if and only if (5.5) holds.*

Now, we prove the following result.

Theorem 5.5. *The α-dual of the spaces $c_0^\lambda(B)$ and $c^\lambda(B)$ is the set*

$$h_1^\lambda = \left\{ a = (a_n) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} h_{nk}^\lambda \right| < \infty \right\}, \tag{5.7}$$

where the matrix $H^\lambda = (h_{nk}^\lambda)$ is defined via the sequence $a = (a_n) \in \omega$ by

$$h_{nk}^{(\lambda)} = \begin{cases} \left(\dfrac{-s}{r} \right)^{n-k} \left[\dfrac{\lambda_k}{r(\lambda_k - \lambda_{k-1})} + \dfrac{\lambda_k}{s(\lambda_{k+1} - \lambda_k)} \right] a_n, & k < n, \\ \dfrac{\lambda_n}{r(\lambda_n - \lambda_{n-1})} a_n, & k = n, \\ 0, & k > n \end{cases} \tag{5.8}$$

for all $n, k \in \mathbb{N}$.

Proof. Let $a = (a_n) \in \omega$. Then, by bearing in mind the relations (2.11) and (2.14), it is immediate that the equality

$$a_n x_n = \sum_{k=0}^{n} \frac{1}{r} \left(\frac{-s}{r} \right)^{n-k} \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} a_n y_i = H_n^\lambda(y) \tag{5.9}$$

holds for all $n \in \mathbb{N}$. Thus, we observe by (5.9) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0^{\lambda}(B)$ or $c^{\lambda}(B)$ if and only if $H^{\lambda} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_n) \in \{c_0^{\lambda}(B)\}^{\alpha}$ or $a = (a_n) \in \{c^{\lambda}(B)\}^{\alpha}$ if and only if $H^{\lambda} \in (c_0 : \ell_1) = (c : \ell_1)$. We therefore obtain by Lemma 5.4 with H^{λ} instead of A that $a = (a_n) \in \{c_0^{\lambda}(B)\}^{\alpha} = \{c^{\lambda}(B)\}^{\alpha}$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K} h_{nk}^{\lambda} \right| < \infty \tag{5.10}$$

which leads us to the consequence that $\{c_0^{\lambda}(B)\}^{\alpha} = \{c^{\lambda}(B)\}^{\alpha} = h_1^{\lambda}$. This completes the proof. □

Theorem 5.6. *Define the sets h_2^{λ}, h_3^{λ}, h_4^{λ}, and h_5^{λ}, as follows:*

$$\begin{aligned}
h_2^{\lambda} &= \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} \left(\frac{-s}{r} \right)^{n-j} a_j \text{ exists for each } \mathrm{k} \in \mathbb{N} \right\}, \\
h_3^{\lambda} &= \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\widehat{a}_k(n)| < \infty \right\}, \\
h_4^{\lambda} &= \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{r} \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})} a_k \right| < \infty \right\}, \\
h_5^{\lambda} &= \left\{ a = (a_k) \in \omega : \sum_k \frac{1}{r} \sum_{j=0}^{k} \left(\frac{-s}{r} \right)^{j} a_k \text{ converges} \right\}, \quad \text{where} \\
\widehat{a}_k(n) &= \lambda_k \left[\frac{a_k}{r(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r(\lambda_k - \lambda_{k-1})} + \frac{1}{s(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{n} \left(\frac{-s}{r} \right)^{n-j} a_j \right] \quad \text{for } \mathrm{k} < \mathrm{n}.
\end{aligned} \tag{5.11}$$

Then $\{c_0^{\lambda}(B)\}^{\beta} = h_2^{\lambda} \cap h_3^{\lambda} \cap h_4^{\lambda}$ and $\{c^{\lambda}(B)\}^{\beta} = h_2^{\lambda} \cap h_3^{\lambda} \cap h_4^{\lambda} \cap h_5^{\lambda}$.

Proof. Consider the equality

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \frac{1}{r} \left(\frac{-s}{r} \right)^{k-j} \left[\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] \right\} a_k \\
&= \sum_{k=0}^{n-1} \lambda_k \left[\frac{a_k}{r(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r(\lambda_k - \lambda_{k-1})} + \frac{1}{s(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{n} \left(\frac{-s}{r} \right)^{n-j} a_j \right] y_k \\
&\quad + \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n y_n \\
&= \sum_{k=0}^{n-1} \widehat{a}_k(n) y_k + \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n y_n \\
&= T_n^{\lambda}(y) \quad \forall n \in \mathbb{N},
\end{aligned} \tag{5.12}$$

where the matrix $T^{\lambda} = (t_{nk}^{\lambda})$ is defined by

$$t_{nk}^{\lambda} := \begin{cases} \widehat{a}_k(n), & k < n, \\ \dfrac{1}{r} \dfrac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n, & k = n, \\ 0, & k > n \end{cases} \tag{5.13}$$

for all $k, n \in \mathbb{N}$. Then, we deduce by (5.12) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0^{\lambda}(B)$ if and only if $T^{\lambda} y \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \{c_0^{\lambda}(B)\}^{\beta}$ if and only if $T^{\lambda} \in (c_0 : c)$. Therefore, by using Lemma 5.2, we derive from (5.4) and (5.5) that

$$\sum_{j=k+1}^{\infty} \left(\frac{-s}{r} \right)^{n-j} a_j \text{ exists for each } k \in \mathbb{N}, \tag{5.14}$$

$$\begin{gathered}
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\widehat{a}_k(n)| < \infty, \\
\sup_{n \in \mathbb{N}} \left| \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n \right| < \infty.
\end{gathered} \tag{5.15}$$

Therefore, we conclude that $\{c_0^{\lambda}(B)\}^{\beta} = h_2^{\lambda} \cap h_3^{\lambda} \cap h_4^{\lambda}$.

Similarly, we deduce from Lemma 5.3 with (5.12) that $a = (a_k) \in \{c^{\lambda}(B)\}^{\beta}$ if and only if $T^{\lambda} \in (c : c)$. Therefore, we derive from (5.4) and (5.5) that (5.14), (5.15) hold.

Further, with a simple calculation one can easily see that the equality

$$\sum_{k=0}^{n} \frac{1}{r} \sum_{j=0}^{k} \left(\frac{-s}{r} \right)^{j} a_k = \sum_{k=0}^{n-1} \widehat{a}_k(n) + \frac{1}{r} \frac{\lambda_n}{(\lambda_n - \lambda_{n-1})} a_n = \sum_k t_{nk}^{\lambda} \tag{5.16}$$

holds for all $n \in \mathbb{N}$. Consequently, from (5.6) we obtain that

$$\left\{ \frac{1}{r} \sum_{j=0}^{k} \left(\frac{-s}{r} \right)^{j} a_k \right\} \in cs. \tag{5.17}$$

Hence, we deduce that $\{c^{\lambda}(B)\}^{\beta} = h_2^{\lambda} \cap h_3^{\lambda} \cap h_4^{\lambda} \cap h_5^{\lambda}$. This completes the proof. □

Remark 5.7. We may note by combining (5.17) with the conditions (5.15) that $\{\sum_{j=0}^{k} (-s/r)^j a_k / r\} \in bs$ for every sequence $a = (a_k) \in \{c_0^{\lambda}(B)\}^{\beta}$.

Finally, we close this section with the following theorem which determines the γ-dual of the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$:

Theorem 5.8. *The γ-duals of the spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$ are the set $h_3^{\lambda} \cap h_4^{\lambda}$.*

Proof. The proof of this result follows the same lines that in the proof of Theorem 5.6 using Lemma 5.4 instead of Lemma 5.2. □

6. Certain Matrix Mappings Related to the Spaces $c_0^{\lambda}(B)$ and $c^{\lambda}(B)$

In this final section, we characterize the matrix classes $(c^{\lambda}(B) : \ell_p)$, $(c_0^{\lambda}(B) : \ell_p)$, $(c^{\lambda}(B) : c)$, $(c^{\lambda}(B) : c_0)$, $(c_0^{\lambda}(B) : c)$, and $(c_0^{\lambda}(B) : c_0)$, where $1 \leq p \leq \infty$. Also, by means of a given basic lemma, we derive the characterizations of some other classes involving the Euler, difference, Riesz, and Cesàro sequence spaces.

For an infinite matrix $A = (a_{nk})$, we write for brevity that

$$\begin{gathered}
\widehat{a}_{nk}(m) = \lambda_k \left[\frac{a_{nk}}{r(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r(\lambda_k - \lambda_{k-1})} + \frac{1}{s(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{m} \left(\frac{-s}{r} \right)^{n-j} a_{nj} \right] \quad \text{if } k < m, \\
\widehat{a}_{nk} = \lambda_k \left[\frac{a_{nk}}{r(\lambda_k - \lambda_{k-1})} + \left(\frac{1}{r(\lambda_k - \lambda_{k-1})} + \frac{1}{s(\lambda_{k+1} - \lambda_k)} \right) \sum_{j=k+1}^{\infty} \left(\frac{-s}{r} \right)^{n-j} a_{nj} \right]
\end{gathered} \tag{6.1}$$

for all $k, m, n \in \mathbb{N}$ provided the convergence of the series.

The following lemmas will be needed in proving our main results.

Lemma 6.1 (see [24, page 57]). *The matrix mappings between the BK-spaces are continuous.*

Lemma 6.2 (see [25, pages 7-8]). *$A = (a_{nk}) \in (c : \ell_p)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K} a_{nk} \right|^{p} < \infty, \quad (1 \leq p < \infty). \tag{6.2}$$

Lemma 6.3 (see [25, page 5]). *$A = (a_{nk}) \in (c : c_0)$ if and only if (5.5) holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \quad \forall k \in \mathbb{N}, \qquad \lim_{n\to\infty} \sum_k a_{nk} = 0. \tag{6.3}$$

Lemma 6.4 (see [25, page 5]). *$A = (a_{nk}) \in (c_0 : c_0)$ if and only if (5.5) and (6.3) hold.*

Now, we give the following results on the matrix transformations.

Theorem 6.5. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold.*

(i) Let $(1 \le p < \infty)$. Then, $A \in (c^\lambda(B) : \ell_p)$ if and only if

$$\sum_{j=k+1}^{\infty} \left(\frac{-s}{r}\right)^{n-j} a_{nj} \text{ exists for each fixed k} \in \mathbb{N}, \tag{6.4}$$

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \widehat{a}_{nk} \right|^p < \infty, \tag{6.5}$$

$$\sup_{m\in\mathbb{N}} \sum_{k=0}^{m-1} |\widehat{a}_{nk}(m)| < \infty, \quad \forall n \in \mathbb{N}, \tag{6.6}$$

$$\left\{ \frac{1}{r} \sum_{j=0}^{k} \left(\frac{-s}{r}\right)^{j} a_{nk} \right\}_{k=0}^{\infty} \in cs \quad \text{for each fixed n} \in \mathbb{N}, \tag{6.7}$$

$$\lim_{k\to\infty} \frac{\lambda_k}{r(\lambda_k - \lambda_{k-1})} a_{nk} = a_n \quad \text{for each fixed n} \in \mathbb{N}, \tag{6.8}$$

$$(a_n) \in \ell_p. \tag{6.9}$$

(ii) $A \in (c^\lambda(B) : \ell_\infty)$ if and only if (6.7) and (6.8) hold, and

$$\sup_{n\in\mathbb{N}} \sum_k |\widehat{a}_{nk}| < \infty, \tag{6.10}$$

$$(a_n) \in \ell_\infty. \tag{6.11}$$

Proof. Suppose that the conditions (6.4)–(6.9) hold and take any $x = (x_k) \in c^\lambda(B)$. Then, we have by Theorem 5.6 that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c^\lambda(B)\}^\beta$ for all $n \in \mathbb{N}$ and this implies that the A-transform of x exists. Also, it is clear that the associated sequence $y = (y_k)$ is in the space c and hence $y_k \to l$ as $k \to \infty$ for some suitable l. Further, it follows by combining Lemma 6.2 with (6.5) that the matrix $\widehat{A} = (\widehat{a}_{nk})$ is in the class $(c : \ell_p)$, where $1 \le p < \infty$.

Let us now consider the following equality derived by using the relation (2.11) from the mth partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \widehat{a}_{nk}(m)y_k + \frac{\lambda_m}{r(\lambda_m - \lambda_{m-1})}a_{nm}y_m, \quad \forall n, m \in \mathbb{N}. \tag{6.12}$$

Then, since $y \in c$ and $\widehat{A} \in (c : \ell_p)$, the series $\sum_k \widehat{a}_{nk}y_k$ converges for every $n \in \mathbb{N}$. Furthermore, it follows by (6.4) that the series $\sum_{j=k}^{\infty}(-s/r)^{n-j}a_{nj}$ converges for all $n, k \in \mathbb{N}$ and hence $\widehat{a}_{nk}(m) \to \widehat{a}_{nk}$ as $m \to \infty$. Therefore, if we pass to limit in (6.12) as $m \to \infty$ then we obtain by (6.8) that

$$\sum_k a_{nk}x_k = \sum_k \widehat{a}_{nk}y_k + la_n, \quad \forall n \in \mathbb{N} \tag{6.13}$$

which can be written as follows:

$$A_n(x) = \widehat{A}_n(y) + la_n, \quad \forall n \in \mathbb{N}. \tag{6.14}$$

This yields by taking p-norm that

$$\|Ax\|_p \leq \left\|\widehat{A}y\right\|_p + |l|\|(a_n)\|_p < \infty \tag{6.15}$$

which leads us to the consequence that $Ax \in \ell_p$. Hence, $A \in (c^{\lambda}(B) : \ell_p)$.

Conversely, suppose that $A \in (c^{\lambda}(B) : \ell_p)$, where $1 \leq p < \infty$. Then $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c^{\lambda}(B)\}^{\beta}$ for all $n \in \mathbb{N}$ which implies with Theorem 5.6 that the conditions (6.6) and (6.7) are necessary.

On the other hand, since $c^{\lambda}(B)$ and ℓ_p are BK-spaces, we have by Lemma 6.1 that there is a constant $M > 0$ such that

$$\|Ax\|_p \leq M\|x\|_{c^{\lambda}(B)} \tag{6.16}$$

holds for all $x \in c^{\lambda}(B)$. Now, $K \in \mathcal{F}$. Then, the sequence $z = \sum_{k\in K} b^{(k)}(\lambda)$ is in $c^{\lambda}(B)$, where the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}}$ is defined by (4.2) for every fixed $k \in \mathbb{N}$.

Since $\widehat{\Lambda}(b^{(k)}(\lambda)) = e^{(k)}$ for each fixed $k \in \mathbb{N}$, we have

$$\|z\|_{c^{\lambda}(B)} = \left\|\widehat{\Lambda}(z)\right\|_{\infty} = \left\|\sum_{k\in K}\widehat{\Lambda}\left(b^{(k)}(\lambda)\right)\right\|_{\infty} = \left\|\sum_{k\in K} e^{(k)}\right\|_{\infty} = 1. \tag{6.17}$$

Furthermore, for every $n \in \mathbb{N}$, we obtain by (4.2) that

$$A_n(z) = \sum_{k\in K} A_n\left(b^{(k)}(\lambda)\right) = \sum_{k\in K}\sum_j a_{nj}b_j^{(k)}(\lambda) = \sum_{k\in K}\widehat{a}_{nk} \tag{6.18}$$

Hence, since the inequality (6.16) is satisfied for the sequence $z \in c^{\lambda}(B)$, we have for any $K \in \mathcal{F}$ that

$$\left(\sum_{n}\left|\sum_{k\in K}\widehat{a}_{nk}\right|^{p}\right)^{1/p} \leq M \tag{6.19}$$

which shows the necessity of (6.5). Thus, it follows by Lemma 6.2 that $\widehat{A} = (\widehat{a}_{nk}) \in (c : \ell_p)$.

Now, let $y = (y_k) \in c \setminus c_0$ and consider the sequence $x = (x_k)$ defined by (2.14) for every $k \in \mathbb{N}$. Then, $x \in c^{\lambda}(B)$ such that $y = \widehat{\Lambda}(x)$, that is, the sequences x and y are connected by the relation (2.11). Therefore, Ax and $\widehat{A}y$ exist. This leads us to the convergence of the series $\sum_k a_{nk}x_k$ and $\sum_k \widehat{a}_{nk}y_k$ for every $n \in \mathbb{N}$. We thus deduce that

$$\lim_{m\to\infty}\sum_{k=0}^{m-1}\widehat{a}_{nk}(m)y_k = \sum_{k}\widehat{a}_{nk}y_k, \quad \forall n \in \mathbb{N}. \tag{6.20}$$

Consequently, we obtain from (6.12) as $m \to \infty$ that

$$\lim_{m\to\infty}\frac{\lambda_m}{r(\lambda_m-\lambda_{m-1})}a_{nm}y_m \quad \text{exists for each fixed } n \in \mathbb{N} \tag{6.21}$$

and since $y = (y_k) \in c \setminus c_0$, we conclude that

$$\lim_{m\to\infty}\frac{\lambda_m}{r(\lambda_m-\lambda_{m-1})}a_{nm} \quad \text{exists for each fixed } n \in \mathbb{N} \tag{6.22}$$

which shows the necessity of (6.8). Then relation (6.14) holds.

Finally, since $Ax \in \ell_p$ and $\widehat{A}y \in \ell_p$, the necessity of (6.9) is immediate by (6.14). This completes the proof of Part (i) of the theorem.

Since Part (ii) can be proved by using the similar way that used in the proof of Part (i) with Lemma 5.4 instead of Lemma 6.2, we leave the details to the reader. □

Remark 6.6. It is clear by (6.10) that the limit

$$\lim_{m\to\infty}\sum_{k=0}^{m-1}|\widehat{a}_{nk}(m)| = \sum_{k}|\widehat{a}_{nk}| \tag{6.23}$$

exists for each $n \in \mathbb{N}$. This just tells us that condition (6.10) implies condition (6.6).

Now, we may note that $(c_0 : \ell_p) = (c : \ell_p)$ for $1 \leq p \leq \infty$, (see [25, pages 7-8]). Thus, by means of Theorem 5.6 and Lemmas 6.2 and 5.4, we immediately conclude the following theorem.

Theorem 6.7. *Let $A = (a_{nk})$ be an infinite matrix over the complex field. Then, the following statements hold.*

(i) Let $1 \leq p < \infty$. *Then,* $A \in (c_0^\lambda(B) : \ell_p)$ *if and only if* (6.5) *and* (6.6) *hold, and*

$$\sum_{j=k}^{\infty}\left(\frac{-s}{r}\right)^{n-j} a_{nj} \quad \text{exists for all } n, k \in \mathbb{N},$$
$$\left\{\frac{\lambda_k}{r(\lambda_k - \lambda_{k-1})} a_{nk}\right\}_{k=0}^{\infty} \in \ell_\infty, \quad \forall n \in \mathbb{N}. \tag{6.24}$$

(ii) $A \in (c_0^\lambda(B) : \ell_\infty)$ *if and only if* (6.10) *and* (6.24) *hold.*

Proof. It is natural that the present theorem can be proved by the same technique used in the proof of Theorem 6.5, above, and so we omit the proof. □

Theorem 6.8. $A = (a_{nk}) \in (c^\lambda(B) : c)$ *if and only if* (6.7), (6.8), *and* (6.10) *hold, and*

$$\lim_{n\to\infty} a_n = a, \tag{6.25}$$

$$\lim_{n\to\infty} \widehat{a}_{nk} = \alpha_k, \quad \text{for each k} \in \mathbb{N}, \tag{6.26}$$

$$\lim_{n\to\infty} \sum_k \widehat{a}_{nk} = \alpha. \tag{6.27}$$

Proof. Suppose that A satisfies the conditions (6.7), (6.8), (6.10), (6.25), (6.26), and (6.27) and take any $x \in c^\lambda(B)$. Then, since (6.10) implies (6.6), we have by Theorem 5.6 that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{c^\lambda(B)\}^\beta$ for all $n \in \mathbb{N}$ and hence Ax exists. We also observe from (6.10) and (6.26) that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_{m\in\mathbb{N}} \sum_j |\widehat{a}_{nk}| < \infty \tag{6.28}$$

holds for every $k \in \mathbb{N}$. This implies that $(\alpha_k) \in \ell_1$ and hence the series $\sum_k \alpha_k(y_k - l)$ converges, where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ by the relation (2.11) such that $y_k \to l$ as $k \to \infty$. Further it is obvious by combining Lemma 5.3 with the condition (6.10), (6.26), and (6.27) that the matrix $\widehat{A} = (\widehat{a}_{nk})$ is in the class $(c : c)$.

Now reasoning as in the proof of Theorem 6.5, we obtain that the relation (6.13) holds which can be written as follows:

$$\sum_k a_{nk} x_k = \sum_k \widehat{a}_{nk}(y_k - l) + l \sum_k \widehat{a}_{nk} + l a_n, \quad \forall n \in \mathbb{N}. \tag{6.29}$$

In this situation, we see by passing to the limit in (6.29) as $n \to \infty$ that the first term on the right tends to $\sum_k \alpha_k(y_k - l)$ by (6.10) and (6.26). The second term tends to $l\alpha$ by (6.27) and the last term to la by (6.25). Consequently, we have

$$\lim_{n\to\infty} A_n(x) = \sum_k \alpha_k(y_k - l) + l(\alpha + a), \tag{6.30}$$

which shows that $Ax \in c$, that is to say that $A \in (c^\lambda(B) : c)$.

Conversely, Suppose that $A \in (c^{\lambda}(B) : c)$. Then, since the inclusion $c \subset \ell_{\infty}$ holds, it is trivial that $A \in (c^{\lambda}(B) : \ell_{\infty})$. Therefore, the necessity of the conditions (6.7), (6.8), and (6.10) are obvious from Theorem 6.5. Further, consider the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}} \in c^{\lambda}(B)$ defined by (4.2) for every fixed $k \in \mathbb{N}$. Then, it is easily seen that $Ab^{(k)}(\lambda) = \{\widehat{a}_{nk}\}_{n\in\mathbb{N}}$ and hence $\{\widehat{a}_{nk}\}_{n\in\mathbb{N}} \in c$ for every $k \in \mathbb{N}$ which shows the necessity of (6.26). Let $z = \sum_k b^{(k)}(\lambda)$. Then, since the linear transformation $T : c^{\lambda}(B) \to c$, defined as in the proof of Theorem 2.3 by analogy, is continuous and $\widehat{\Lambda}(b^{(k)}(\lambda)) = e^{(k)}$ for each fixed $k \in \mathbb{N}$, we obtain that

$$\widehat{\Lambda}_n(z) = \sum_k \widehat{\Lambda}_n\left(b^{(k)}(\lambda)\right) = \sum_k \delta_{nk} = 1 \quad \text{for each } n \in \mathbb{N} \tag{6.31}$$

which shows that $\widehat{\Lambda}(z) = e \in c$ and hence $z \in c^{\lambda}(B)$. On the other hand, since $c^{\lambda}(B)$ and c are the BK-spaces, Lemma 6.1 implies the continuity of the matrix mapping $A : c^{\lambda}(B) \to c$. Thus, we have for every $n \in \mathbb{N}$ that

$$A_n(z) = \sum_k A_n\left(b^{(k)}(\lambda)\right) = \sum_k \widehat{a}_{nk}. \tag{6.32}$$

This shows the necessity of (6.27).

Now, it follows by (6.10), (6.26), and (6.27) with Lemma 5.3 that $\widehat{A} = (\widehat{a}_{nk}) \in (c : c)$. So by (6.7), and (6.8), relation (6.14) holds for all $x \in c^{\lambda}(B)$ and $y \in c$, and x and y are connected by relation (2.11), where $y_k \to l$ $(k \to \infty)$.

Lastly, since $Ax \in c$ and $\widehat{A}x \in c$; the necessity of (6.25) is immediate by (6.14). This step concludes the proof. □

Theorem 6.9. *$A = (a_{nk}) \in (c^{\lambda}(B) : c_0)$ if and only if (6.7), (6.8), and (6.10) hold, and*

$$\begin{gathered}\lim_{n\to\infty} a_n = 0, \\ \lim_{n\to\infty} \widehat{a}_{nk} = 0 \quad \text{for each k} \in \mathbb{N}, \\ \lim_{n\to\infty} \sum_k \widehat{a}_{nk} = 0.\end{gathered} \tag{6.33}$$

Proof. Since the present theorem can be proved by the similar way used in the proof of Theorem 6.8 with Lemma 6.3 instead of Lemma 5.3, we omit the detailed proof. □

Theorem 6.10. *$A = (a_{nk}) \in (c_0^{\lambda}(B) : c)$ if and only if (6.10), (6.24), and (6.26) hold.*

Proof. This is similarly obtained by using Lemma 5.2, Theorem 5.6, and Part (ii) of Theorem 6.7. □

Theorem 6.11. *$A = (a_{nk}) \in (c_0^{\lambda}(B) : c_0)$ if and only if (6.10) and (6.24) hold, and (6.26) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Proof. This is immediate by Lemma 6.4, Theorems 5.6 and 6.10. □

7. Conclusion

Let ν denotes any of the classical sequence spaces ℓ_∞, c, or c_0. Then, $\nu(\Delta)$ consisting of the sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k+1}) \in \nu$ is called as the *difference sequence space* which was introduced by Kızmaz [19]. Kızmaz [19] proved that $\nu(\Delta)$ is a Banach space with the norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$, where $x = (x_k) \in \nu(\Delta)$ and the inclusion relation $\nu \subset \nu(\Delta)$ strictly holds. He also determined the α-, β-, and γ-duals of the difference spaces and characterized the classes $(\nu(\Delta) : \mu)$ and $(\mu : \nu(\Delta))$ of infinite matrices, where $\nu, \mu \in \{\ell_\infty, c\}$. Following Kızmaz [19], Sarıgöl [26] extended the difference space $\nu(\Delta)$ to the space $\nu(\Delta_r)$ defined by

$$\nu(\Delta_r) := \{x = (x_k) \in \omega : \Delta_r x = \{k^r (x_k - x_{k+1})\} \in \nu \text{ for } r < 1\} \tag{7.1}$$

and computed the α-, β-, and γ-duals of the space $\nu(\Delta_r)$, where $\nu \in \{\ell_\infty, c, c_0\}$. It is easily seen that $\nu(\Delta_r) \subset \nu(\Delta)$, if $0 < r < 1$ and $\nu(\Delta) \subset \nu(\Delta_r)$, if $r < 0$. Recently, the difference spaces bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [27] and in the case $1 \le p \le \infty$ by Başar and Altay [18], Çolak et al. [28], and Malkowsky et al. [29]. Quite recently, Mursaleen and Noman have introduced the spaces c^λ and c_0^λ of λ-convergent and λ-null sequences and nextly studied the difference spaces $c^\lambda(\Delta)$ and $c_0^\lambda(\Delta)$ in [21, 22], respectively. Of course, there is a wide literature concerning the difference sequence spaces. By the domain of the generalized difference matrix $B(r,s)$ in the spaces of λ-convergent and λ-null sequences we have generalized the difference spaces $c^\lambda(\Delta)$ and $c_0^\lambda(\Delta)$ defined by Mursaleen and Noman [21]. Since the generalized difference matrix $B(r,s)$ reduces, in the special case $r = 1$, $s = -1$, to the usual difference matrix Δ; our results are more general and more comprehensive than the corresponding results of Mursaleen and Noman [21].

Since the difference spaces of λ-bounded and absolutely λp-summable sequences are not studied, the domain of both the difference matrix Δ, and the generalized difference matrix $B(r,s)$ in those spaces are still open. So, it is meaningful to fill this gap.

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