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Research Article

On an Integral Transform of a Class of Analytic Functions

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For $\alpha, \gamma \geq 0$ and $\beta < 1$, let $\mathcal{W}_\beta(\alpha, \gamma)$ denote the class of all normalized analytic functions f in the open unit disc $E = \{z : |z| < 1\}$ such that $\Re e^{i\phi}((1-\alpha+2\gamma)(f(z)/z) + (\alpha - 2\gamma)f'(z) + \gamma z f''(z) - \beta) > 0$, $z \in E$ for some $\phi \in \mathbb{R}$. It is known (Noshiro (1934) and Warschawski (1935)) that functions in $\mathcal{W}_\beta(1,0)$ are close-to-convex and hence univalent for $0 \leq \beta < 1$. For $f \in \mathcal{W}_\beta(\alpha, \gamma)$, we consider the integral transform $F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)(f(tz)/t)dt$, where λ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. The aim of present paper is, for given $\delta < 1$, to find sharp values of β such that (i) $V_\lambda(f) \in \mathcal{W}_\delta(1,0)$ whenever $f \in \mathcal{W}_\beta(\alpha, \gamma)$ and (ii) $V_\lambda(f) \in \mathcal{W}_\delta(\alpha, \gamma)$ whenever $f \in \mathcal{W}_\beta(\alpha, \gamma)$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$ with the normalizations $f(0) = f'(0) - 1 = 0$, and let S be the subclass of $\mathcal{A}$ consisting of functions univalent in E. For any two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.1}$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [1] introduced the integral operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \tag{1.2}$$

where λ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. This operator contains some well-known operators such as Libera, Bernardi, and Komatu as its special cases. Fournier and Ruscheweyh [1] applied the famous duality theory to show that for a function f in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left(f'(z) - \beta\right) > 0,\ z \in E \right\}, \tag{1.3}$$

the linear integral operator $V_\lambda(f)$ is univalent in E. Since then, this operator has been studied by a number of authors for various choices of $\lambda(t)$. In another remarkable paper, Barnard et al. in [2] obtained conditions such that $V_\lambda(f) \in \mathcal{P}_1(\beta)$ whenever f is in the class

$$\mathcal{P}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right) > 0,\ z \in E \right\}, \tag{1.4}$$

with $\beta < 1$, $\gamma \geq 0$. Note that for $0 \leq \beta < 1$, functions in $\mathcal{P}_1(\beta) \equiv \mathcal{P}(\beta)$ satisfy the condition $\Re f'(z) > \beta$ in E and thus are close-to-convex in E. A domain D in $\mathbb{C}$ is close-to-convex if its compliment in $\mathbb{C}$ can be written as union of nonintersecting half lines.

In 2008, Ponnusamy and Rønning [3] discussed the univalence of $V_\lambda(f)$ for the functions in the class

$$\mathcal{R}_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left(f'(z) + \gamma z f''(z) - \beta\right) > 0,\ z \in E \right\}. \tag{1.5}$$

In a very recent paper, Ali et al. [4] studied the class

$$\begin{aligned}\mathcal{W}_\beta(\alpha,\gamma) \\ := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\right) > 0,\ z \in E \right\},\end{aligned} \tag{1.6}$$

where $\alpha, \gamma \geq 0$ and $\beta < 1$. In this paper, they obtained sufficient conditions so that the integral transform $V_\lambda(f)$ maps normalized analytic functions $f \in \mathcal{W}_\beta(\alpha,\gamma)$ into the class of starlike functions. It is evident that $\mathcal{W}_\beta(1,0) \equiv \mathcal{P}(\beta)$, $\mathcal{W}_\beta(\alpha,0) \equiv \mathcal{P}_\alpha(\beta)$ and $\mathcal{W}_\beta(1+2\gamma,\gamma) \equiv \mathcal{R}_\gamma(\beta)$.

In the present paper, we shall mainly tackle the following problems.

(1) For given $\delta < 1$, find sharp values of $\beta = \beta(\delta,\alpha)$ such that $V_\lambda(f) \in \mathcal{W}_\delta(1,0)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

(2) For given $\delta < 1$, find sharp values of $\beta = \beta(\delta)$ such that $V_\lambda(f) \in \mathcal{W}_\delta(\alpha,\gamma)$ whenever $f \in \mathcal{W}_\beta(\alpha,\gamma)$.

To prove one of our results, we shall need the generalized hypergeometric function ${}_pF_q$, so we define it here.

Let α_j $(j = 1,2,\ldots,p)$ and β_j $(j = 1,2,\ldots,q)$ be complex numbers with $\beta_j \neq 0,-1,-2,\ldots$ $(j = 1,2,\ldots,q)$. Then the generalized hypergeometric function ${}_pF_q$ is defined by

$$ {}_pF_q(z) = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q+1), \tag{1.7} $$

where $(a)_n$ is the Pochhammer symbol, defined in terms of the Gamma function, by

$$ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a+1)\cdots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.8} $$

In particular, ${}_2F_1$ is called the Gaussian hypergeometric function. We note that the ${}_pF_q$ series in (1.7) converges absolutely for $|z| < \infty$ if $p < q+1$ and for $z \in E$ if $p = q+1$.

We shall also need the following lemma.

Lemma 1.1 (see [5]). *Let $\beta_1 < 1$, $\beta_2 < 1$, and $\eta \in \mathbb{R}$. Then, for p, q analytic in E with $p(0) = q(0) = 1$, the conditions $\Re p(z) > \beta_1$ and $\Re e^{i\eta}(q(z) - \beta_2) > 0$ imply $\Re e^{i\eta}((p * q)(z) - \delta) > 0$, where $1 - \delta = 2(1-\beta_1)(1-\beta_2)$.*

2. Main Results

We use the notations introduced in [4]. Let $\mu \geq 0$ and $\nu \geq 0$ satisfy

$$ \mu + \nu = \alpha - \gamma, \qquad \mu\nu = \gamma. \tag{2.1} $$

When $\gamma = 0$, then μ is chosen to be 0, in which case, $\nu = \alpha \geq 0$. When $\alpha = 1 + 2\gamma$, (2.1) yields $\mu + \nu = 1 + \gamma = 1 + \mu\nu$ or $(\mu - 1)(1 - \nu) = 0$.

(i) For $\gamma > 0$, then choosing $\mu = 1$ gives $\nu = \gamma$.

(ii) For $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha = 1$.

Theorem 2.1. *Let $\mu \geq 0$, $\nu \geq 0$ satisfy* (2.1)*. Further, let $\delta < 1$ be given, and define $\beta = \beta(\delta, \mu, \nu)$ by*

$$ \begin{aligned} &1 - \frac{1-\delta}{2}\left\{1 - \frac{1}{\nu}\int_0^1 \lambda(t)\left(\int_0^1 \frac{ds}{1+ts^{\mu}}\right)dt + \left(\frac{1}{\nu} - 1\right)\int_0^1 \lambda(t)\left(\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\}^{-1}, \quad \gamma \neq 0, \\ &1 - \frac{1-\delta}{2}\left\{1 - \frac{1}{\alpha}\int_0^1 \frac{\lambda(t)}{1+t}dt + \left(\frac{1}{\alpha} - 1\right)\int_0^1 \lambda(t)\left(\int_0^1 \frac{d\eta}{1+t\eta^{\alpha}}\right)dt\right\}^{-1}, \quad \gamma = 0 \ (\mu = 0,\ \nu = \alpha > 0). \end{aligned} \tag{2.2} $$

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then $F = V_\lambda(f) \in \mathcal{W}_\delta(1, 0) \subset S$. The value of β is sharp.

Proof. The case $\gamma = 0$ $(\mu = 0, \nu = \alpha > 0)$ corresponds to Theorem 1.5 in [2]. So we assume that $\gamma > 0$.

Define

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)=H(z). \tag{2.3}$$

Writing $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, it follows that

$$H(z)=1+\sum_{n=1}^{\infty}a_{n+1}(n\nu+1)(n\mu+1)z^n. \tag{2.4}$$

It is a simple exercise to see that

$$f'(z)=H(z)*{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right). \tag{2.5}$$

Let $F(z)=V_\lambda(f)(z)$, where $V_\lambda(f)$ is defined by (1.2). Then for $\gamma\neq 0$, we can write

$$\begin{aligned}F'(z)&=f'(z)*\int_0^1\frac{\lambda(t)}{1-tz}dt\\&=H(z)*{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right)*\int_0^1\frac{\lambda(t)}{1-tz}dt\\&=H(z)*\int_0^1\lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt.\end{aligned} \tag{2.6}$$

Since $f\in\mathcal{W}_\beta(\alpha,\gamma)$, it follows that $\Re\{e^{i\phi}(H(z)-\beta)\}>0$ for some $\phi\in\mathbb{R}$. Now, for each $\gamma>0$, we first claim that

$$\Re\left[\int_0^1\lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt\right]>1-\frac{1-\delta}{2(1-\beta)},\quad z\in E, \tag{2.7}$$

which, by Lemma 1.1, implies that $F\in\mathcal{W}_\delta(1,0)$. Therefore, it suffices to verify the inequality (2.7). Using the identity (which can be checked by comparing the coefficients of z^n on both sides)

$${}_3F_2(2,b,c;d,e;z)=(d-1)\,{}_3F_2(1,b,c;d-1,e;z)-(d-2)\,{}_3F_2(1,b,c;d,e;z), \tag{2.8}$$

it follows that

$${}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};z\right)=\frac{1}{\nu}\int_0^1\frac{ds}{1-zs^\mu}+\left(1-\frac{1}{\nu}\right)\iint_0^1\frac{d\eta\,d\zeta}{1-z\eta^\nu\zeta^\mu}. \tag{2.9}$$

Thus,

$$\int_0^1 \lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt$$
$$=\int_0^1 \lambda(t)\left\{\frac{1}{\nu}\int_0^1\frac{ds}{1-tzs^{\mu}}+\left(1-\frac{1}{\nu}\right)\iint_0^1\frac{d\eta\,d\zeta}{1-tz\eta^{\nu}\zeta^{\mu}}\right\}dt. \tag{2.10}$$

Therefore, for $\gamma>0$, we have

$$\Re\left[\int_0^1 \lambda(t)\,{}_3F_2\left(2,\frac{1}{\nu},\frac{1}{\mu};\frac{1}{\nu+1},\frac{1}{\mu+1};tz\right)dt\right]$$
$$>\frac{1}{\nu}\int_0^1\lambda(t)\left(\int_0^1\frac{ds}{1+ts^{\mu}}\right)dt+\left(\frac{1}{\nu}-1\right)\int_0^1\lambda(t)\left(\iint_0^1\frac{d\eta\,d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt \tag{2.11}$$
$$=1-\frac{1-\delta}{2(1-\beta)},$$

in the view of (2.2).

To prove the sharpness, let $f\in\mathcal{W}_{\beta}(\alpha,\gamma)$ be the function determined by

$$(1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)=\beta+(1-\beta)\frac{1+z}{1-z}. \tag{2.12}$$

Using a series expansion, we see that we can write

$$f(z)=z+\sum_{n=2}^{\infty}\frac{2(1-\beta)}{(n\nu+1-\nu)(n\mu+1-\mu)}z^n. \tag{2.13}$$

Then,

$$F(z)=V_{\lambda}(f)(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{\psi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}z^n, \tag{2.14}$$

where $\psi_n = \int_0^1 \lambda(t)t^{n-1}dt$. Equation (2.2) can be restated as

$$\begin{aligned}
\frac{1}{1-\beta} &= \frac{2}{1-\delta}\left\{1 - \frac{1}{\nu}\int_0^1 \lambda(t)\left(\int_0^1 \frac{ds}{1+ts^{\mu}}\right)dt + \left(\frac{1}{\nu}-1\right)\int_0^1 \lambda(t)\left(\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\} \\
&= \frac{2}{1-\delta}\left\{1 + \int_0^1 \lambda(t)\left(-\frac{1}{\nu}\int_0^1 \frac{ds}{1+ts^{\mu}} + \left(\frac{1}{\nu}-1\right)\iint_0^1 \frac{d\eta\, d\zeta}{1+t\eta^{\nu}\zeta^{\mu}}\right)dt\right\} \\
&= \frac{2}{1-\delta}\int_0^1 \lambda(t)\left\{\sum_{n=2}^{\infty}\frac{(-1)^{n-1}t^{n-1}}{(n\mu+1-\mu)}\left(-\frac{1}{\nu} + \left(\frac{1}{\nu}-1\right)\frac{1}{(n\nu+1-\nu)}\right)\right\}dt \\
&= -\frac{2}{1-\delta}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\psi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}.
\end{aligned} \tag{2.15}$$

Finally,

$$F'(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{n\psi_n}{(n\nu+1-\nu)(n\mu+1-\mu)}z^{n-1}, \tag{2.16}$$

which for $z = -1$ takes the value

$$F'(-1) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}n\psi_n}{(n\nu+1-\nu)(n\mu+1-\mu)} = 1 + 2(1-\beta)\left\{\frac{-(1-\delta)}{2(1-\beta)}\right\} = \delta. \tag{2.17}$$

This shows that the result is sharp. □

Letting $\gamma = 0$ and $\alpha = 1$ in Theorem 1.1, we obtain the following result of Ruscheweyh [6].

Corollary 2.2. *Let $\delta < 1$, and define $\beta = \beta(\delta, 1) < 1$ by*

$$\beta(\delta) = 1 - \frac{1-\delta}{2}\left\{1 - \int_0^1 \frac{\lambda(t)}{1+t}dt\right\}^{-1}. \tag{2.18}$$

If $f \in \mathcal{W}_{\beta}(1,0) \equiv \mathcal{P}_1(\beta)$, then $F = V_{\lambda}(f) \in \mathcal{W}_{\delta}(1,0) \subset S$. The value of β is sharp.

Theorem 2.3. *Let $\delta < 1$ and $\alpha, \gamma \geq 0$, and define $\beta = \beta(\delta) < 1$ by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\frac{(1-((1+\delta)/(1-\delta))t)}{(1+t)}dt. \tag{2.19}$$

If $f \in \mathcal{W}_{\beta}(\alpha, \gamma)$, then $V_{\lambda}(f) \in \mathcal{W}_{\delta}(\alpha, \gamma)$. The value of β is sharp.

Proof. The idea of the proof is similar to the one used to prove Theorem 2 in [1].

Let $F(z) = V_\lambda(f)(z) = \int_0^1 \lambda(t)(f(tz)/t)dt$. Clearly,

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}dt * f'(z). \tag{2.20}$$

Since, $f \in \mathcal{W}_\beta(\alpha,\gamma)$, so with

$$g(z) = \frac{(1-\alpha+2\gamma)(f(z)/z) + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta}{1-\beta}, \tag{2.21}$$

we have $\Re[e^{i\phi}g(z)] > 0$, where $\phi \in \mathbb{R}$.

For $\gamma \neq \alpha/2$,

$$f'(z) = \frac{1}{\alpha-2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z). \tag{2.22}$$

Putting this value in (2.20),

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left(\frac{1}{\alpha-2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{f(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zf''(z)\right). \tag{2.23}$$

Equivalently,

$$F'(z) = \frac{1}{\alpha-2\gamma}g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] - \frac{1-\alpha+2\gamma}{\alpha-2\gamma}\frac{F(z)}{z} - \frac{\gamma}{\alpha-2\gamma}zF''(z). \tag{2.24}$$

Thus

$$(1-\alpha+2\gamma)(F(z)/z) + (\alpha-2\gamma)F'(z) + \gamma zF''(z) = g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right]. \tag{2.25}$$

In the case when $\gamma = \alpha/2$,

$$g(z) = \frac{f(z)/z + \gamma z f''(z) - \beta}{1-\beta}. \tag{2.26}$$

Since

$$\frac{f(z)}{z} = \beta + (1-\beta)g(z) - \gamma z f''(z), \tag{2.27}$$

This leads to,

$$\frac{F(z)}{z} + \gamma z F''(z) = g(z) * \left[\beta + (1-\beta)\int_0^1 \frac{\lambda(t)}{1-tz}dt\right], \tag{2.28}$$

which is clearly (2.25) with $\gamma = \alpha/2$.

Further $F \in \mathcal{W}_\delta(\alpha,\gamma)$ if and only if $G(z) := (F(z)-\delta z)/(1-\delta) \in \mathcal{W}_0(\alpha,\gamma)$. Now using (2.25), we obtain

$$(1-\alpha+2\gamma)\frac{G(z)}{z} + (\alpha-\gamma)G'(z) + \gamma z G''(z) = g(z) * \left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right]. \tag{2.29}$$

Since $\Re e^{i\phi} g(z) > 0$ for some $\phi \in \mathbb{R}$, it follows by duality principle [8, page 23] that

$$(1-\alpha+2\gamma)\frac{G(z)}{z} + (\alpha-2\gamma)G'(z) + \gamma z G''(z) \neq 0 \tag{2.30}$$

if, and only if,

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1}{2}. \tag{2.31}$$

Using $\Re(1/(1-tz)) > 1/(1+t)$, we get

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1-\beta}{1-\delta}\left[\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt\right]. \tag{2.32}$$

By using (2.19), we have

$$\frac{\beta-(1+\delta)/2}{1-\beta} = -\int_0^1 \frac{\lambda(t)}{(1+t)}dt. \tag{2.33}$$

Thus,

$$\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt = \frac{1}{2}\frac{1-\delta}{1-\beta}, \tag{2.34}$$

which implies that

$$\Re\left[\frac{\beta-\delta}{1-\delta} + \frac{1-\beta}{1-\delta}\int_0^1 \frac{\lambda(t)}{1-tz}dt\right] > \frac{1-\beta}{1-\delta}\left[\frac{\beta-\delta}{1-\beta} + \int_0^1 \frac{\lambda(t)}{1+t}dt\right] = \frac{1}{2}. \tag{2.35}$$

Thus, we deduce, using duality principle, that $(1-\alpha+2\gamma)(G(z)/z)+(\alpha-\gamma)G'(z)+\gamma zG''(z)$ is contained in a half plane not containing the origin. So, $G \in \mathcal{W}_0(\alpha,\gamma)$ and hence $F \in \mathcal{W}_\delta(\alpha,\gamma)$.

To prove the sharpness, let $f(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}(z^n/(n\mu+1-\mu)(n\nu+1-\nu))$.

$$F(z) = V_\lambda(f)(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}\frac{z^n\omega_n}{(n\mu+1-\mu)(n\nu+1-\nu)}, \quad \text{where } \omega_n = \int_0^1 \lambda(t)t^{n-1}dt. \tag{2.36}$$

Further,

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\frac{(1-((1+\delta)/(1-\delta))t)}{(1+t)}dt \tag{2.37}$$

gives

$$\frac{\beta}{1-\beta} = -1 + \int_0^1 \lambda(t)\frac{(1+(1+\delta)/(1-\delta))}{(1+t)}t\,dt, \tag{2.38}$$

or

$$\frac{1}{1-\beta} = \frac{2}{1-\delta}\int_0^1 \frac{t\lambda(t)}{1+t}dt = \frac{2}{1-\delta}\sum_{n=2}^{\infty}(-1)^n\omega_n. \tag{2.39}$$

Further, assume that

$$H(z) = (1-\alpha+2\gamma)\frac{F(z)}{z} + (\alpha-\gamma)F'(z) + \gamma zF''(z). \tag{2.40}$$

Since $F(z) = z + 2(1-\beta)\sum_{n=2}^{\infty}(\omega_n z^n/(n\mu+1-\mu)(n\nu+1-\nu))$,
so,

$$H(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\omega_n z^{n-1}. \tag{2.41}$$

Therefore, for $z = -1$,

$$H(-1) = 1 - 2(1-\beta)\sum_{n=2}^{\infty}\omega_n(-1)^n = 1 - 2(1-\beta)\frac{1-\delta}{2(1-\beta)} = \delta. \tag{2.42}$$

This shows that the result is sharp. □

Letting $\gamma = 0$ in Theorem 2.3 above, we obtain the following result of Kim and Rønning [9].

Corollary 2.4. *Let $\delta < 1$ and $\alpha \geq 0$, and define $\beta = \beta(\delta)$ by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{(1 - ((1+\delta)/(1-\delta))t)}{(1+t)} dt. \tag{2.43}$$

If $f \in \mathcal{W}_\beta(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$, then $V_\lambda(f) \in \mathcal{W}_\delta(\alpha, 0) \equiv \mathcal{P}_\alpha(\delta)$. The value of β is sharp.

Upon setting $\lambda(t) = (1+c)t^c$ with $-1 < c$, we have the following corollary.

Corollary 2.5. *Let $\delta < 1$, $\alpha, \gamma \geq 0$, and $-1 < c \leq 0$ be given, and let $G(z)$ be defined by*

$$G(z) = \frac{(1+c)}{z^c} \int_0^z u^{c-1} f(u) du. \tag{2.44}$$

Suppose that $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then $G \in \mathcal{W}_\delta(\alpha, 0)$, where

$$\beta = \frac{2(1+c)\,{}_2F_1(1, 2+c; 3+c, -1) - (2+c)}{2(1+c)\,{}_2F_1(1, 2+c; 3+c, -1)}. \tag{2.45}$$

The constant β is sharp.

The special case of Corollary 2.5 (with $\gamma = 0$) has been obtained by Aghalary et al. [11].

References

[1] R. Fournier and S. Ruscheweyh, “On two extremal problems related to univalent functions,” *The Rocky Mountain Journal of Mathematics*, vol. 24, no. 2, pp. 529–538, 1994.

[2] R. W. Barnard, S. Naik, and S. Ponnusamy, “Univalency of weighted integral transforms of certain functions,” *Journal of Computational and Applied Mathematics*, vol. 193, no. 2, pp. 638–651, 2006.

[3] S. Ponnusamy and F. Rønning, “Integral transforms of a class of analytic functions,” *Complex Variables and Elliptic Equations*, vol. 53, no. 5, pp. 423–434, 2008.

[4] R. M. Ali, A. O. Badghaish, V. Ravichandran, and A. Swaminathan, “Starlikeness of integral transforms and duality,” *Journal of Mathematical Analysis and Applications*, vol. 385, no. 2, pp. 808–822, 2012.

[5] S. Ponnusamy, “Inclusion theorems for convolution product of second order polylogarithms and functions with the derivative in a halfplane,” *The Rocky Mountain Journal of Mathematics*, vol. 28, no. 2, pp. 695–733, 1998, Reports of the Department of Mathematics, Preprint 92 , University of Helsinki, Finland, 1995.

[6] S. Ruscheweyh, “Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc,” *Transactions of the American Mathematical Society*, vol. 210, pp. 63–74, 1975.

[7] S. E. Warschawski, “On the higher derivatives at the boundary in conformal mapping,” *Transactions of the American Mathematical Society*, vol. 38, no. 2, pp. 310–340, 1935.

[8] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, vol. 83 of *Séminaire de Mathématiques Supérieures*, Presses de l’Université de Montréal, Montreal, Canada, 1982.

[9] Y. C. Kim and F. Rønning, “Integral transforms of certain subclasses of analytic functions,” *Journal of Mathematical Analysis and Applications*, vol. 258, no. 2, pp. 466–489, 2001.

[10] K. Noshiro, “On the theory of schlicht functions,” *Journal of the faculty of science, Hokkaido University*, vol. 2, pp. 129–155, 1934.

[11] R. Aghalary, A. Ebadian, and S. Shams, “Geometric properties of some linear operators defined by convolution,” *Tamkang Journal of Mathematics*, vol. 39, no. 4, pp. 325–334, 2008.