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Research Article

Existence of Solutions for the $p(x)$-Laplacian Problem with the Critical Sobolev-Hardy Exponent

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This paper deals with the $p(x)$-Laplacian equation involving the critical Sobolev-Hardy exponent. Firstly, a principle of concentration compactness in $W_0^{1,p(x)}(\Omega)$ space is established, then by applying it we obtain the existence of solutions for the following $p(x)$-Laplacian problem: $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = (h(x)|u|^{p_s^*(x)-2}u/|x|^{s(x)}) + f(x,u)$, $x \in \Omega$, $u = 0$, $x \in \partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 \in \Omega$, $1 < p^- \le p(x) \le p^+ < N$, and $f(x,u)$ satisfies $p(x)$-growth conditions.

1. Introduction

In this paper we are concerned with the following $p(x)$-Laplacian problem:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) &+ |u|^{p(x)-2}u \\ &= \frac{h(x)|u|^{p_s^*(x)-2}u}{|x|^{s(x)}} + f(x,u), \quad x \in \Omega,\ u = 0,\ x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $0 \in \Omega \subset \mathbb{R}^N$ is a bounded domain, $p(x)$ is Lipschitz continuous, radially symmetric on $\overline{\Omega}$, and $1 < p^- \le p(x) \le p^+ < N$. $s(x)$ is Lipschitz continuous, radially symmetric on $\overline{\Omega}$ and $0 \le s(x) \ll p(x)$. $p_s^*(x) = ((N-s(x))/(N-p(x)))p(x)$ is the critical Sobolev-Hardy exponent, and $p_0^*(x) = Np(x)/(N-p(x)) = p^*(x)$ is the critical Sobolev exponent. Throughout this paper we assume the following:

(F-1) $f(x,t)$ satisfies the Carathéodory condition.

(F-2) $|f(x,t)| \le c_1 + c_2|t|^{q(x)-1}$, $q : \overline{\Omega} \to \mathbb{R}$ is measurable and satisfies $p(x) \ll q(x) \ll p_s^*(x)$ or $1 < q^- \le q(x) \ll p(x)$, for any $x \in \overline{\Omega}$.

(F-3) $f(x,t) = f(|x|,t)$, for any $(x,t) \in \Omega \times \mathbb{R}$.

(F-4) $f(x,t) = -f(x,-t)$, for any $(x,t) \in \Omega \times \mathbb{R}$.

(F-5) $h(x) \in C(\overline{\Omega})$, $h(x) = h(|x|) > 0$ for any $0 \neq x \in \Omega$ and $h(0) = 0$.

In this paper, we mainly consider the singularity, that is, $\lim_{x\to 0} h(x) \cdot (1/|x|^{s(x)}) = \infty$. For example, let $h(x) = 1/|\ln|x||$ for $x \neq 0$; $h(x) = 0$ for $x = 0$; $s_0 = \inf_{x\in\overline{\Omega}} s(x) > 0$. It is easy to get $\lim_{x\to 0}(1/|\ln|x||) \cdot (1/|x|^{s(x)}) = \infty$.

Here we explain some notations employed in this paper: Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1,\infty)$. For all $p(x) \in \mathbf{P}(\Omega)$, we denote $p^+ = \sup_{x\in\overline{\Omega}} p(x)$, $p^- = \inf_{x\in\overline{\Omega}} p(x)$, $p_s^{*+} = \sup_{x\in\overline{\Omega}} p_s^*(x)$, $p_s^{*-} = \inf_{x\in\overline{\Omega}} p_s^*(x)$ and denote by $p_1(x) \ll p_2(x)$ the fact that $\inf\{p_2(x) - p_1(x)\} > 0$. Denote by c_i, C, and k_i the generic positive constants. Denote by $|\Omega|$ the Lebesgue measure of Ω.

When $p(x) \equiv p$ is a constant function, the p-Laplacian problem related to Sobolev-Hardy inequality had been studied by many authors, either is the bounded domain or in the whole space $\mathbb{R}^N$, see, for example, [1–4]. In recent years, along with variable Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ being used, there are a lot of studies on $p(x)$-Laplacian problems, see [5–8], and the theory on problems with $p(x)$-growth conditions has important applications in nonlinear elastic mechanics and electrorheological fluids, see [9–12]. In [13], Fu discussed the existence of solutions for a class of $p(x)$-Laplacian equation with critical growth by establishing a principle of concentration compactness. The method employed in this paper is a extension of the argument in [13, 14].

This paper is organized as follows: in Section 2 we deal with some preliminary materials and technical results; in Section 3 we give the proof of a principle of concentration compactness; in Section 4 we study the problem of $p(x)$-Laplacian equation with the critical Sobolev-Hardy exponent.

2. Preliminaries

In this section we first recall some facts on variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open set, see [15–19] for the details.

Let $p(x) \in \mathbf{P}(\Omega)$ and

$$\|u\|_p = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of functions u such that $\int_\Omega |u(x)|^{p(x)}\ dx < \infty$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.1).

For a given $p(x) \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as:

$$p' = \frac{p(x)}{p(x) - 1}. \tag{2.2}$$

Theorem 2.1. *Let $p(x) \in \mathbf{P}(\Omega)$. Then the inequality*

$$\int_\Omega \left|f(x) \cdot g(x)\right| dx \le 2\|f\|_p \|g\|_{p'} \tag{2.3}$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$.

Theorem 2.2. *Suppose that $p(x)$ satisfies*

$$1 < p^- \leq p^+ < \infty. \tag{2.4}$$

Let meas $\Omega < \infty$, $p_1(x), p_2(x) \in \mathbf{P}(\Omega)$, *then the necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ we have $p_1(x) \leq p_2(x)$, and in this case, the imbedding is continuous.*

Theorem 2.3. *Suppose that $p(x)$ satisfies* (2.4). *Let $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$. If $u, u_k \in L^{p(x)}(\Omega)$, then*

(1) $\|u\|_p < 1(= 1; > 1)$ *if and only if* $\rho(u) < 1(= 1; > 1)$.

(2) *If* $\|u\|_p > 1$, *then* $\|u\|_p^{p^-} \leq \rho(u) \leq \|u\|_p^{p^+}$.

(3) *If* $\|u\|_p < 1$, *then* $\|u\|_p^{p^+} \leq \rho(u) \leq \|u\|_p^{p^-}$.

(4) $\lim_{k\to\infty} \|u_k\|_p = 0$ *if and only if* $\lim_{k\to\infty} \rho(u_k) = 0$.

(5) $\|u_k\|_p \to \infty$ *if and only if* $\rho(u_k) \to \infty$.

We assume that k is a given positive integer.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of functions f on Ω such that $D^\alpha f \in L^{p(x)}$ for every multi-index α with $|\alpha| \leq k$. $W^{k,p(x)}(\Omega)$ is endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p. \tag{2.5}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.5).

In this paper we use the following equivalent norm of $W^{1,p(x)}(\Omega)$:

$$\|u\|_{1,p} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\right\}. \tag{2.6}$$

Then we have the inequality $(1/2)(\|\nabla u\|_p + \|u\|_p) \leq \|u\|_{1,p} \leq 2(\|\nabla u\|_p + \|u\|_p)$.

Theorem 2.4. *The spaces $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable reflexive Banach spaces if $p(x)$ satisfies* (2.4).

Theorem 2.5. *Suppose that $p(x)$ satisfies* (2.4). *Let $\varphi(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u(x)|^{p(x)} dx$. If $u, u_k \in W^{1,p(x)}(\Omega)$, then*

(1) $\|u\|_{1,p} < 1(= 1; > 1)$ *if and only if* $\varphi(u) < 1(= 1; > 1)$.

(2) *If* $\|u\|_{1,p} > 1$, *then* $\|u\|_{1,p}^{p^-} \leq \varphi(u) \leq \|u\|_{1,p}^{p^+}$.

(3) *If* $\|u\|_{1,p} < 1$, *then* $\|u\|_{1,p}^{p^+} \leq \varphi(u) \leq \|u\|_{1,p}^{p^-}$.

(4) $\lim_{k\to\infty}\|u_k\|_{1,p} = 0$ *if and only if* $\lim_{k\to\infty}\varphi(u_k) = 0$.

(5) $\|u_k\|_{1,p} \to \infty$ *if and only if* $\varphi(u_k) \to \infty$.

Theorem 2.6. *Let Ω be a bounded in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ and satisfies* (2.4)*. Then for any measurable function $q(x)$ with $1 \le q(x) \ll p^*(x)$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Theorem 2.7. *If $p : \overline{\Omega} \to R$ is Lipschitz continuous and satisfies* (2.4)*, then for any measurable function $q(x)$ with $p(x) \le q(x) \le p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Next let us consider the weighted variable exponent Lebesgue space. Let $a(x) \in \mathbf{P}(\Omega)$ and $a(x) > 0$ for $x \in \Omega$. Define

$$L^{p(x)}_{a(x)}(\Omega) = \left\{ u \in P(\Omega) : \int_\Omega a(x)|u(x)|^{p(x)}dx < \infty \right\} \tag{2.7}$$

with the norm

$$|u|_{L^{p(x)}_{a(x)}(\Omega)} = \|u\|_{p,a} = \inf\left\{ \lambda > 0 : \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}, \tag{2.8}$$

then $L^{p(x)}_{a(x)}(\Omega)$ is a Banach space.

Theorem 2.8. *Suppose that $p(x)$ satisfies* (2.4)*. Let $\rho(u) = \int_\Omega a(x)|u(x)|^{p(x)}dx$. If $u, u_k \in L^{p(x)}_{a(x)}(\Omega)$, then*

(1) *For $u \neq 0$, $\|u\|_{p,a} = \lambda$ if and only if $\rho(u/\lambda) = 1$.*

(2) $\|u\|_{p,a} < 1(= 1; > 1)$ *if and only if* $\rho(u) < 1(= 1; > 1)$.

(3) *If $\|u\|_{p,a} > 1$, then $\|u\|^{p^-}_{p,a} \le \rho(u) \le \|u\|^{p^+}_{p,a}$.*

(4) *If $\|u\|_{p,a} < 1$, then $\|u\|^{p^+}_{p,a} \le \rho(u) \le \|u\|^{p^-}_{p,a}$.*

(5) $\lim_{k\to\infty}\|u_k\|_{p,a} = 0$ *if and only if* $\lim_{k\to\infty}\rho(u_k) = 0$.

(6) $\|u_k\|_{p,a} \to \infty$ *if and only if* $\rho(u_k) \to \infty$.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a measurable subset. Suppose that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caracheodory function and satisfies*

$$\left|g(x,u)\right| \le \alpha(x) + \beta|u|^{(p_1(x))/(p_2(x))} \quad \textit{for any } x \in \Omega,\ t \in \mathbb{R}, \tag{2.9}$$

where $p_i(x) \ge 1$, $i = 1,2$, $\alpha(x) \in L^{p_2(x)}(\Omega)$, $\alpha(x) \ge 0$, $\beta \ge 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_g u)(x) = g(x,u(x))$ is a continuous and bounded operator.

Theorem 2.10. *Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \le s(x) < N$ for $x \in \overline{\Omega}$. If $q(x)$ satisfies $1 \le q(x) < p^*_s(x)$ for $x \in \overline{\Omega}$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{|x|^{-s(x)}}(\Omega)$.*

Theorem 2.11. *Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \le s(x) \ll p(x)$ for $x \in \overline{\Omega}$. There is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_s^*(x)}_{|x|^{-s(x)}}(\Omega)$.*

Proof. Let $u \in W^{1,p(x)}(\Omega)$. Note that

$$\begin{aligned}\int_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}}dx &= \int_\Omega \frac{|u|^{s(x)}|u|^{p_s^*(x)-s(x)}}{|x|^{s(x)}}dx \\ &\le C_1 \left\| \left|\frac{u}{x}\right|^{s(x)} \right\|_{p/s} \left\| |u|^{N(p(x)-s(x))/(N-p(x))} \right\|_{p/(p-s)}.\end{aligned} \tag{2.10}$$

By Theorems 2.7 and 2.10, we have $\|u\|_{p,|x|^{-p}} \le C_2\|u\|_{1,p} < \infty$ and $\|u\|_{p^*} \le C_3\|u\|_{1,p} < \infty$. So we get

$$\begin{aligned}\int_\Omega \left(\left|\frac{u}{x}\right|^{s(x)}\right)^{p(x)/s(x)} dx &= \int_\Omega \left|\frac{u}{x}\right|^{p(x)} dx < \infty, \\ \int_\Omega |u|^{(N(p(x)-s(x))/(N-p(x)))\cdot(p(x)/(p(x)-s(x)))}dx &= \int_\Omega |u|^{p^*(x)}dx < \infty.\end{aligned} \tag{2.11}$$

Furthermore, we obtain $\int_\Omega |u|^{p_s^*(x)}/|x|^{s(x)}dx < \infty$. This shows $W^{1,p(x)}(\Omega) \subset L^{p_s^*(x)}_{|x|^{-s(x)}}(\Omega)$, then by the closed graph theorem in Banach space, we get the continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p_s^*(x)}_{|x|^{-s(x)}}(\Omega)$. □

3. The Principle of Concentration Compactness

In this section, we will establish the principle of concentration compactness in $W_0^{1,p(x)}(\Omega)$.

We denote by $\mathcal{M}(\overline{\Omega})$ the space of finite nonnegative Borel measures on $\overline{\Omega}$. A sequence $\mu_n \to \mu$ weakly-$*$ in $\mathcal{M}(\overline{\Omega})$ is defined by $(\mu_n, u) \to (\mu, u)$, for any $u \in C(\overline{\Omega}) \bigcap C^\infty(\Omega)$.

We first give two lemmas. From [13] we can obtain the proof of the following lemmas. Assume that $p(x)$ is Lipschitz continuous satisfying (2.4) and $s(x)$ is continuous on $\overline{\Omega}$.

Lemma 3.1. *Let $\{u_n\} \subset L^{p(x)}_{|x|^{-s(x)}}(\Omega)$ be bounded, and $u_n \to u \in L^{p(x)}_{|x|^{-s(x)}}(\Omega)$ a.e. on Ω, then*

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p(x)}}{|x|^{s(x)}} - \frac{|u_n-u|^{p(x)}}{|x|^{s(x)}}dx = \int_\Omega \frac{|u|^{p(x)}}{|x|^{s(x)}}dx. \tag{3.1}$$

Lemma 3.2. *Let $\delta > 0$, $0 < r < R < 1$, and $r/R \le k(\delta) = \min\{\exp(-(\delta/(2\widetilde{C}))^{n/p^-(1-n)}), e^{-|s^{n-1}|^{1/(n-1)}}\}$, where $\widetilde{C} = ((1/((1+(\delta/2))^{1/(p^+-1)}-1))+1)^{p^+-1}\max\{2C^{p^+}, 2C^{p^-}\}|s^{n-1}|^{p^-/n}$, $|s^{n-1}|$*

denotes the surface area of the unit sphere in $\mathbb{R}^n$ *and* C *satisfies the inequality* $\|u\|_{p^*(x)} \leq C\|\nabla u\|_{p(x)}$. *Then for every* $u \in W_0^{1,p(x)}(\Omega)$,

$$\int_{B_r(x_0)} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx \leq C^* \max\left\{ \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} + |u|^{p(x)} dx + \delta \max\left\{ \|u\|_{1,p}^{p^+}, \|u\|_{1,p}^{p^-} \right\} \right)^{p_s^{*-}/p^+}, \left(\int_{B_R(x_0)} |\nabla u|^{p(x)} + |u|^{p(x)} dx + \delta \max\left\{ \|u\|_{1,p}^{p^+}, \|u\|_{1,p}^{p^-} \right\} \right)^{p_s^{*+}/p^-} \right\}, \tag{3.2}$$

where $C^* = \sup\{\int_\Omega |u|^{p_s^*(x)}/|x|^{s(x)} dx : \|u\|_{1,p} \leq 1,\ u \in W_0^{1,p(x)}(\Omega)\}$.

Theorem 3.3. *Let* $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ *with* $\|u_n\|_{1,p} \leq 1$ *such that*

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega), \\ |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \longrightarrow \mu \quad \text{weakly-}* \text{ in } \mathcal{M}\left(\overline{\Omega}\right), \\ \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \longrightarrow \nu \quad \text{weakly-}* \text{ in } \mathcal{M}\left(\overline{\Omega}\right), \end{gathered} \tag{3.3}$$

as $n \to \infty$. *Then the limit measures are of the form*

$$\begin{gathered} \mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 + \tilde{\mu}, \quad \mu\left(\overline{\Omega}\right) \leq 1, \\ \nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0, \quad \nu\left(\overline{\Omega}\right) \leq C^*, \end{gathered} \tag{3.4}$$

where J *is a countable set,* $\{\mu_j\} \subset [0, \infty)$, $\{\nu_j\} \subset [0, \infty)$, $\mu_0 \geq 0$, $\nu_0 \geq 0$, $\{x_j\} \in \overline{\Omega}$, $\tilde{\mu} \in \mathcal{M}(\overline{\Omega})$ *is a nonatomic positive measure.* δ_{x_j} *and* δ_0 *are atomic measures which concentrate on* x_j *and* 0, *respectively.* C^* *is as defined in Lemma 3.2. The atoms and the regular part satisfy the generalized Sobolev inequalities*

$$\begin{gathered} \nu\left(\overline{\Omega}\right) \leq C^* \max\left\{ \mu\left(\overline{\Omega}\right)^{p_s^{*+}/p^-}, \mu\left(\overline{\Omega}\right)^{p_s^{*-}/p^+} \right\}, \\ \nu_j \leq C^* \max\left\{ \mu_j^{p_s^{*+}/p^-}, \mu_j^{p_s^{*-}/p^+} \right\}, \\ \nu_0 \leq C^* \max\left\{ \mu_0^{p_s^{*+}/p^-}, \mu_0^{p_s^{*-}/p^+} \right\}. \end{gathered} \tag{3.5}$$

Proof. By Lemma 3.2, for every $\delta > 0$, there exists $k(\delta) > 0$ such that for $0 < r < R$ with $r/R \le k(\delta)$,

$$\begin{aligned}
&\int_{B_r(0)} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} dx \\
&\quad \le C^* \max\left\{ \left(\int_{B_R(0)} |\nabla u_n|^{p(x)} + |u_n|^{p(x)} dx + \delta \max\left\{ \|u_n\|_{1,p}^{p^+}, \|u_n\|_{1,p}^{p^-} \right\} \right)^{p_s^{*-}/p^+} \right. \\
&\qquad \left. \left(\int_{B_R(0)} |\nabla u_n|^{p(x)} + |u_n|^{p(x)} dx + \delta \max\left\{ \|u_n\|_{1,p}^{p^+}, \|u_n\|_{1,p}^{p^-} \right\} \right)^{p_s^{*+}/p^-} \right\}.
\end{aligned} \tag{3.6}$$

Let $\eta_1 \in C_0^\infty(B_r(0))$ and $\eta_2 \in C_0^\infty(B_{2R}(0))$ such that $0 \le \eta_1, \eta_2 \le 1$, $\eta_1 \equiv 1$ in $B_{r/2}(0)$ and $\eta_2 \equiv 1$ in $B_R(0)$. Then we have

$$\begin{gathered}
\int_{B_r(0)} \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \eta_1 dx \longrightarrow \int_{B_r(0)} \eta_1 d\nu, \\
\int_{B_{2R}(0)} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \eta_2 dx \longrightarrow \int_{B_{2R}(0)} \eta_2 d\mu.
\end{gathered} \tag{3.7}$$

Thus,

$$\int_{B_r(0)} \eta_1 d\nu \le C^* \max\left\{ \left(\int_{B_{2R}(0)} \eta_2 d\mu + \delta \right)^{p_s^{*-}/p^+}, \left(\int_{B_{2R}(0)} \eta_2 d\mu + \delta \right)^{p_s^{*+}/p^-} \right\}. \tag{3.8}$$

Furthermore,

$$\nu(\{0\}) \le \nu(B_{r/2}(0)) \le C^* \max\left\{ (\mu(B_{2R}(0)) + \delta)^{p_s^{*-}/p^+}, (\mu(B_{2R}(0)) + \delta)^{p_s^{*+}/p^-} \right\}. \tag{3.9}$$

Let $\delta \to 0$ and $R \to 0$, then we get

$$\nu(\{0\}) \le C^* \max\left\{ \mu(\{0\})^{p_s^{*-}/p^+}, \mu(\{0\})^{p_s^{*+}/p^-} \right\}, \tag{3.10}$$

that is,

$$\nu_0 \le C^* \max\left\{ \mu_0^{p_s^{*-}/p^+}, \mu_0^{p_s^{*+}/p^-} \right\}. \tag{3.11}$$

By Theorem 2.11 and the definition of C^*, we have

$$\int_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx \le C^* \max\left\{ \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx \right)^{p_s^{*-}/p^+}, \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx \right)^{p_s^{*+}/p^-} \right\}. \tag{3.12}$$

Similar to the proof of Theorem 3.1 in [13], we get

$$\nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j\in J} \nu_j \delta_{x_j} + \nu_0 \delta_0 \tag{3.13}$$

and the other results. □

4. Existence of Solutions

Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$, and G is a subgroup of $O(N)$. For $x \neq 0$, we denote the cardinality of $G_x = \{gx : g \in G\}$ by $|G_x|$ and set $|G| = \inf_{x\in\mathbb{R}^N, x\neq 0}|G_x|$. An open subset $\Omega \subset \mathbb{R}^N$ is G-invariant if $g\Omega = \Omega$ for any $g \in G$.

Definition 4.1. Let Ω be a G-invariant open subset of $\mathbb{R}^N$. The action of G on $W_0^{1,p(x)}(\Omega)$ is defined by $gu(x) = u(g^{-1}x)$ for any $u \in W_0^{1,p(x)}(\Omega)$. The subspace of invariant functions is defined by

$$W_{0,G}^{1,p(x)}(\Omega) = \left\{ u \in W_0^{1,p(x)}(\Omega) : gu = u,\ \forall g \in G \right\}. \tag{4.1}$$

A functional $I : W_0^{1,p(x)}(\Omega) \to \mathbb{R}^N$ is G-invariant if $I \circ g = I$ for any $g \in G$.

Set

$$\begin{gathered} I(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) - \frac{h(x)}{p_s^*(x)} \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} - F(x,u)dx, \\ F(x,t) = \int_0^t f(x,s)ds. \end{gathered} \tag{4.2}$$

The critical points of $I(u)$, that is,

$$0 = I'(u)\varphi = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi + |u|^{p(x)-2}u\varphi - h(x)\frac{|u|^{p_s^*(x)-2}u}{|x|^{s(x)}}\varphi - f(x,u)\varphi\, dx \tag{4.3}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, are weak solutions of the problem (1.1). So next we need only to consider the existence of nontrivial critical points of $I(u)$.

In this paper, assume that $G = O(N)$ and Ω is $O(N)$-invariant. By (F-3) and (F-5), we get that I is $O(N)$-invariant. By the principle of symmetric criticality of Krawcewicz and Marzantowicz [20], u is a critical point of I if and only if u is a critical point of $\widetilde{I} = I|_{W_{0,O(N)}^{1,p(x)}(\Omega)}$. So we only need to prove the existence of critical points of $\widetilde{I}$ on $W_{0,O(N)}^{1,p(x)}(\Omega)$.

Lemma 4.2. *Any* $(PS)_c$ *sequence* $\{u_n\} \subset W_{0,O(N)}^{1,p(x)}(\Omega)$ *possesses a convergent subsequence.*

Proof. Suppose that $\widetilde{I}(u_n) \to c$, $c \in \mathbb{R}$, and $\widetilde{I}'(u_n) \to 0$ in $(W_{0,O(N)}^{1,p(x)}(\Omega))^*$. Let $l(x) = (p(x) + p_s^*(x))/2$ and $|\nabla(1/l(x))| \leq C$. Denote $a = \inf_{x\in\overline{\Omega}}((1/p(x)) - (1/l(x))) > 0$ and $b = \inf_{x\in\overline{\Omega}}((1/(l(x))) - (1/(p_s^*(x)))) > 0$. Then we have

$$\begin{aligned}
&\widetilde{I}(u_n) - \left\langle \widetilde{I}'(u_n), \frac{u_n}{l(x)} \right\rangle \\
&\quad = \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{l(x)}\right)\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) + h(x)\left(\frac{1}{l(x)} - \frac{1}{p_s^*(x)}\right)\frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \\
&\qquad + \frac{1}{l(x)} f(x,u_n)u_n - F(x,u_n)dx - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\left(\frac{1}{l(x)}\right)u_n dx \\
&\quad \geq \int_\Omega a\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) + bh(x)\frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} + \frac{1}{l(x)} f(x,u_n)u_n - F(x,u_n)dx \\
&\qquad - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\left(\frac{1}{l(x)}\right)u_n dx.
\end{aligned} \tag{4.4}$$

By Young's inequality, for $\varepsilon_1 \in (0,1)$, we get

$$\left||\nabla u_n|^{p(x)-2}\nabla u_n \cdot u_n\right| \leq \varepsilon_1 |\nabla u_n|^{p(x)} + \varepsilon_1 |u_n|^{p_s^*(x)} + C(\varepsilon_1). \tag{4.5}$$

By (F-2), $|(1/l(x))f(x,u_n)u_n - F(x,u_n)| \leq C(|u_n| + |u_n|^{q(x)})$, then we have for $\varepsilon_2 \in (0,1)$

$$|u_n| + |u_n|^{q(x)} \leq \varepsilon_2 |u_n|^{p_s^*(x)} + C(\varepsilon_2). \tag{4.6}$$

From $h(x)/|x|^{s(x)} \to \infty$ as $x \to 0$, we get that there exists $\overline{H} > 0$ such that $h(x)/|x|^{s(x)} > \overline{H}$ for any $x \in \Omega$, so we have

$$\begin{aligned}
&\widetilde{I}(u_n) - \left\langle \widetilde{I}'(u_n), \frac{u_n}{l(x)} \right\rangle \\
&\quad \geq \int_\Omega a\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \int_\Omega b\overline{H}|u_n|^{p_s^*(x)}dx - C\varepsilon_1\int_\Omega |\nabla u_n|^{p(x)}dx \\
&\qquad - C(\varepsilon_1 + \varepsilon_2)\int_\Omega |u_n|^{p_s^*(x)}dx - C(\varepsilon_1) - C(\varepsilon_2).
\end{aligned} \tag{4.7}$$

Take ε_1 and ε_2 sufficiently small such that $C\varepsilon_1 < a/2$ and $C(\varepsilon_1 + \varepsilon_2) \leq b\overline{H}$, thus,

$$c + 1 > I(u_n) \geq \int_\Omega \frac{a}{2}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - C, \tag{4.8}$$

if n is sufficiently large. Furthermore, we obtain $\|u_n\|_{1,p} < \infty$.

Note that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)dx \\
&\quad \le \left|\left\langle \widetilde{I}'(u_n), u_n - u\right\rangle\right| + \int_\Omega \left|\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\right|dx \\
&\qquad + \left|\left\langle \widetilde{I}'(u), u_n - u\right\rangle\right| + \int_\Omega \left|h(x)\left(\frac{|u_n|^{p_s^*(x)-2}u_n}{|x|^{s(x)}} - \frac{|u|^{p_s^*(x)-2}u}{|x|^{s(x)}}\right)(u_n - u)\right|dx \\
&\qquad + \int_\Omega \left|\left(f(x,u_n) - f(x,u)\right)(u_n - u)\right|dx \\
&\quad \triangleq \sum_{i=1}^{5} I_i.
\end{aligned} \tag{4.9}$$

Because $\{u_n\}$ is bounded in $W_{0,O(N)}^{1,p(x)}(\Omega)$, there exists a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ weakly in $W_{0,O(N)}^{1,p(x)}(\Omega)$. Then we have $u_n \to u$ in $L^{q(x)}(\Omega)$. It is easy to get $I_1 \to 0$, $I_2 \to 0$, and $I_3 \to 0$. By (F-2)

$$\begin{aligned}
&\int_\Omega \left|f(x,u_n)\right|^{q'(x)}dx \\
&\quad \le \int_\Omega \left(c_1 + c_2|u_n|^{q(x)-1}\right)^{q'(x)}dx \\
&\quad \le C\int_\Omega (1 + |u_n|)^{(q(x)-1)q'(x)}dx \\
&\quad \le C\left(|\Omega| + \int_\Omega |u_n|^{q(x)}dx\right).
\end{aligned} \tag{4.10}$$

Then we have that $\|f(x,u_n)\|_{q'}$ is bounded. By

$$I_5 \le 2\left\|f(x,u_n)\right\|_{q'}\|u_n - u\|_q + 2\left\|f(x,u)\right\|_{q'}\|u_n - u\|_q, \tag{4.11}$$

we get $I_5 \to 0$.

Next we show that $I_4 \to 0$. Note that

$$\begin{aligned}
I_4 &\le h^0\left(\int_\Omega \frac{|u_n|^{p_s^*(x)-1}}{|x|^{s(x)}}|u_n - u|dx + \int_\Omega \frac{|u|^{p_s^*(x)-1}}{|x|^{s(x)}}|u_n - u|dx\right) \\
&\le 2h^0\left(\left\|\frac{|u_n|^{p_s^*(x)-1}}{|x|^{s(x)/p_s^{*\prime}(x)}}\right\|_{p_s^{*\prime}}\left\|\frac{u_n - u}{|x|^{s(x)/p_s^*(x)}}\right\|_{p_s^*} + \left\|\frac{|u|^{p_s^*(x)-1}}{|x|^{s(x)/p_s^{*\prime}(x)}}\right\|_{p_s^{*\prime}}\left\|\frac{u_n - u}{|x|^{s(x)/p_s^*(x)}}\right\|_{p_s^*}\right),
\end{aligned} \tag{4.12}$$

where $h^0 = \max_{x\in\overline{\Omega}} h(x)$. By Theorem 2.11, $\| |u_n|^{p_s^*(x)-1}/|x|^{s(x)/p_s^{*\prime}(x)} \|_{p_s^{*\prime}}$ is bounded. If we show that there exists a subsequence (still denoted by $\{u_n\}$) such that $\int_\Omega |u_n - u|^{p_s^*(x)}/|x|^{s(x)} dx \to 0$ as $n \to \infty$, then $I_4 \to 0$.

As $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}_{0,O(N)}(\Omega)$, passing to a subsequence, still denoted by $\{u_n\}$, by Theorem 3.3 we assume that there exist $\mu, \nu \in \mathcal{M}(\overline{\Omega})$ and $\{x_j\}_{j\in J}$ in $\overline{\Omega}$ such that $|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \to \mu$ weakly-$*$ in $\mathcal{M}(\overline{\Omega})$ and $|u_n|^{p_s^*(x)}/|x|^{s(x)} \to \nu$ weakly-$*$ in $\mathcal{M}(\overline{\Omega})$, where

$$\begin{gathered}\mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j\in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 + \widetilde{\mu}, \\ \nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j\in J} \nu_j \delta_{x_j} + \nu_0 \delta_0.\end{gathered} \tag{4.13}$$

J is a countable set, $\{\mu_j\} \subset [0,\infty)$, $\{\nu_j\} \subset [0,\infty)$, $\mu_0 \ge 0$, $\nu_0 \ge 0$, $\widetilde{\mu} \in \mathcal{M}(\overline{\Omega})$ is a nonatomic positive measure. Take $\eta \equiv 1$, then

$$\lim_{n\to\infty} \int_\Omega \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \eta \, dx = \int_\Omega \eta \, d\nu = \int_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx + \sum_{j\in J} \nu_j + \nu_0. \tag{4.14}$$

We claim $\nu_0 = 0$ and $\nu_j = 0$ for any $j \in J$. First we consider ν_0.

For any $\varepsilon > 0$, choose $\varphi_0 \in C_0^\infty(B_{2\varepsilon}(0))$ such that $0 \le \varphi_0 \le 1$, $\varphi_0 = 1$ on $B_\varepsilon(0)$ and $|\nabla \varphi_0| \le 2/\varepsilon$. Then

$$\begin{aligned}\left\langle \widetilde{I}'(u_n), u_n\varphi_0 \right\rangle = {} & \int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \varphi_0 dx - \int_\Omega h(x) \frac{|u_n|^{p_s^*(x)} \varphi_0}{|x|^{s(x)}} dx \\ & - \int_\Omega f(x,u_n) u_n \varphi_0 dx + \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n dx.\end{aligned} \tag{4.15}$$

We have

$$\begin{gathered}\lim_{n\to\infty} \int_{B_{2\varepsilon}(0)} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) \varphi_0 dx = \int_{B_{2\varepsilon}(0)} \varphi_0 d\mu, \\ \lim_{n\to\infty} \int_{B_{2\varepsilon}(0)} \frac{|u_n|^{p_s^*(x)} \varphi_0}{|x|^{s(x)}} dx = \int_{B_{2\varepsilon}(0)} \varphi_0 d\nu.\end{gathered} \tag{4.16}$$

By Theorem 2.1,

$$\begin{aligned}& \int_{B_{2\varepsilon}(0)} \left| |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi_0 u_n \right| dx \\ & \quad \le 2 \left\| u_n \nabla \varphi_0 \right\|_{p, B_{2\varepsilon}(0)} \left\| |\nabla u_n|^{p(x)-1} \right\|_{p', B_{2\varepsilon}(0)} \\ & \quad \le C \left\| u_n \nabla \varphi_0 \right\|_{p, B_{2\varepsilon}(0)}.\end{aligned} \tag{4.17}$$

By Theorem 2.6, we have $u_n \to u$ in $L^{p(x)}(\Omega)$, then

$$\lim_{n\to\infty}\int_{B_{2\varepsilon}(0)}\left|u_n\nabla\varphi_0\right|^{p(x)}dx=\int_{B_{2\varepsilon}(0)}\left|u\nabla\varphi_0\right|^{p(x)}dx. \tag{4.18}$$

Furthermore,

$$\begin{gathered}\int_{B_{2\varepsilon}(0)}\left|u\nabla\varphi_0\right|^{p(x)}dx\le 2\left\|\left|\nabla\varphi_0\right|^{p(x)}\right\|_{N/p,B_{2\varepsilon}(0)}\left\||u|^{p(x)}\right\|_{N/(N-p),B_{2\varepsilon}(0)},\\ \int_{B_{2\varepsilon}(0)}\left|\nabla\varphi_0\right|^{N}dx\le 4^N\omega_N,\end{gathered} \tag{4.19}$$

where ω_N is the volume of the unit boll. By $\lim_{\varepsilon\to 0}\int_{B_{2\varepsilon}(0)}|u|^{p^*(x)}dx=0$, then we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\Omega}\left||\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi_0 u_n\right|dx=0. \tag{4.20}$$

Since $\|f(x,u_n)\|_{q'}$ is bounded and by Theorem 2.9 we have

$$\lim_{n\to\infty}\int_{B_{2\varepsilon}(0)}\left|f(x,u_n)-f(x,u)\right|^{q'(x)}dx=0. \tag{4.21}$$

From

$$\begin{aligned}&\int_{B_{2\varepsilon}(0)}\left|f(x,u_n)u_n-f(x,u)u\right|dx\\ &\quad\le 2\left\|f(x,u_n)\right\|_{q'}\|u_n-u\|_q+2\left\|f(x,u_n)-f(x,u)\right\|_{q'}\|u\|_{q},\end{aligned} \tag{4.22}$$

we have

$$\lim_{n\to\infty}\int_{B_{2\varepsilon}(0)}f(x,u_n)u_n\varphi_0dx=\int_{B_{2\varepsilon}(0)}f(x,u)u\varphi_0dx. \tag{4.23}$$

Therefore,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B_{2\varepsilon}(0)}f(x,u_n)u_n\varphi_0dx=\lim_{\varepsilon\to 0}\int_{B_{2\varepsilon}(0)}f(x,u)u\varphi_0dx=0. \tag{4.24}$$

Thus, we have

$$\begin{aligned}0=\lim_{n\to\infty}\left\langle\widetilde{I}'(u_n),u_n\varphi_0\right\rangle=&\int_{B_{2\varepsilon}(0)}\varphi_0d\mu-\int_{B_{2\varepsilon}(0)}h(x)\varphi_0d\nu-\int_{B_{2\varepsilon}(0)}f(x,u)u\varphi_0dx\\ &+\lim_{n\to\infty}\int_{B_{2\varepsilon}(0)}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\varphi_0\cdot u_ndx.\end{aligned} \tag{4.25}$$

Furthermore, we obtain

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left\langle \widetilde{I}'(u_n), u_n \varphi_0 \right\rangle = \mu_0 - h(0)\nu_0. \tag{4.26}$$

As $h(0) = 0$, $\mu_0 = 0$, thus, $\nu_0 = 0$.

Next we consider ν_j for any $j \in J$. Suppose $\exists j_0 \in J$ such that $\nu_{j_0} > 0$. Note that $u_n \in W^{1,p(x)}_{0,O(N)}(\Omega)$, then for any $g \in O(N)$, $\nu(gx_{j_0}) = \nu(x_{j_0}) > 0$. By $|O(N)| = \infty$, we get $\nu(\{gx_{j_0} : g \in O(N)\}) = \infty$. As the measure ν is finite, that is a contradiction. So we obtain that $\nu_0 = 0$ and $\nu_j = 0$ for any $j \in J$. Thus,

$$\lim_{n \to \infty} \int_\Omega \frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} dx = \int_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} dx. \tag{4.27}$$

By Lemma 3.1, we obtain $\lim_{n \to \infty} \int_\Omega |u_n - u|^{p_s^*(x)} / |x|^{s(x)} dx = 0$, that is, $u_n \to u$ strongly in $L^{p_s^*(x)}_{|x|^{-s(x)}}(\Omega)$.

We obtain that $\{u_n\}$ possesses a subsequence (still denoted by $\{u_n\}$), such that $I_i \to 0$, $i = 1, \ldots, 5$, as $n \to \infty$. Thus, $\int_\Omega (|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u)(\nabla u_n - \nabla u) dx \to 0$, as $n \to \infty$. As in the proof of Theorem 3.1 in [5], we divide Ω into two parts:

$$\Omega_1 = \{x \in \Omega : p(x) \geq 2\}, \qquad \Omega_2 = \{x \in \Omega : p(x) < 2\}. \tag{4.28}$$

We have

$$\int_{\Omega_1} |\nabla u_n - \nabla u|^{p(x)} dx + \int_{\Omega_2} |\nabla u_n - \nabla u|^{p(x)} dx \longrightarrow 0, \tag{4.29}$$

that is, $\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \to 0$. Then $u_n \to u$ in $W^{1,p(x)}_{0,O(N)}(\Omega)$. □

Since $W^{1,p(x)}_0(\Omega)$ is a separable and reflexive Banach space, $W^{1,p(x)}_{0,O(N)}(\Omega)$ is also a separable and reflexive Banach space. So there exist $\{e_n\}_{n=1}^\infty \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ and $\{e_n^*\}_{n=1}^\infty \subset (W^{1,p(x)}_{0,O(N)}(\Omega))^*$ such that

$$\begin{gathered}
\left\langle e_j^*, e_i \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \\
W^{1,p(x)}_{0,O(N)}(\Omega) = \overline{\operatorname{span}\{e_n : n = 1, 2, \ldots\}}, \\
\left(W^{1,p(x)}_{0,O(N)}(\Omega)\right)^* = \overline{\operatorname{span}\{\mathrm{e}_n^* : n = 1, 2, \ldots\}}.
\end{gathered} \tag{4.30}$$

For $k = 1, 2, \ldots$, denote $X_k = \operatorname{span}\{e_k\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$.

Theorem 4.3. *Under assumptions (F-1)–(F-5), the problem (1.1) admits a sequence of solutions* $\{u_n\} \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ *such that* $I(u_n) \to \infty$.

Proof. Set $\varphi(u) = \int_\Omega F(x,u)\,dx$. We first show that $\varphi(u)$ is weakly strongly continuous. Let $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}_{0,O(N)}(\Omega)$. So we have $u_n \to u$ in $L^{q(x)}(\Omega)$. Note that

$$|F(x,u)| \le C\left(|u| + |u|^{q(x)}\right) \le C\left(1 + |u|^{q(x)}\right), \tag{4.31}$$

then by Theorem 2.9 we obtain $F(x,u_n) \to F(x,u)$ in $L^1(\Omega)$. By Proposition 3.5 in [18],

$$\beta_k = \beta_k(r) = \sup_{u\in Z_k, \|u\|_{1,p}\le r} \int_\Omega |F(x,u)|dx \longrightarrow 0, \tag{4.32}$$

as $k \to \infty$ for $r > 0$.

Set

$$\theta_k = \theta_k(r) = \sup_{u\in Z_k, \|u\|_{1,p}\le r} \int_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}}dx. \tag{4.33}$$

Next we show $\theta_k \to \sum_{j\in J} \nu_j + \nu_0$ as $k \to \infty$. Note that $0 \le \theta_{k+1} \le \theta_k$, then $\theta_k \to \theta \ge 0$, as $k \to \infty$. There exists $u_k \in Z_k$ with $\|u_k\|_{1,p} \le r$ such that $0 \le \theta_k - \int_\Omega (|u_k|^{p_s^*(x)}/|x|^{s(x)})dx < 1/k$, for each $k = 1, 2, \ldots$. As $W^{1,p(x)}_{0,O(N)}(\Omega)$ is reflexive, passing to a subsequence, still denoted by $\{u_k\}$, we assume $u_k \rightharpoonup u$ weakly in $W^{1,p(x)}_{0,O(N)}(\Omega)$. We claim $u = 0$. In fact, for any e_m^*, we have $e_m^*(u_k) = 0$, when $k >$ m, then $e_m^*(u_k) \to 0$ as $k \to \infty$. It is immediate to get $e_m^*(u) = 0$ for any $m \in \mathbb{N}$. Then we have $u = 0$. By Theorem 3.3, there exist a finite measure ν and a sequence $\{x_j\} \subset \overline{\Omega}$ such that

$$\frac{|u_k|^{p_s^*(x)}}{|x|^{s(x)}} \rightharpoonup \nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j\in J} \nu_j \delta_{x_j} + \nu_0\delta_0, \tag{4.34}$$

where J is countable. Set $\eta \equiv 1$, we obtain $\int_\Omega (|u_k|^{p_s^*(x)}/|x|^{s(x)})\eta dx \to \sum_{j\in J} \nu_j + \nu_0$. So we have $\lim_{k\to\infty}\theta_k = \sum_{j\in J} \nu_j + \nu_0 \le \nu(\overline{\Omega}) < \infty$.

For any $n \in \mathbb{N}$, there exists a positive integer k_n such that $\beta_k(n) \le 1$ and $\theta_k(n) \le \sum_{j\in J} \nu_j + \nu_0 + 1$ for all $k \ge k_n$. Assume that $k_n < k_{n+1}$ for each n. Define $\{r_k : k = 1, 2, \ldots\}$ in the following way:

$$r_k = \begin{cases} n, & k_n \le k < k_n + 1, \\ 1, & 1 \le k < k_1. \end{cases} \tag{4.35}$$

Then we get $r_k \to \infty$ as $k \to \infty$. Hence, for $u \in Z_k$ with $\|u\|_{1,p} = r_k$, we get

$$\begin{aligned}\widetilde{I}(u) &\geq \frac{1}{p^+}\|u\|_{1,p}^{p^-} - \frac{h^0}{p_s^{*-}}\theta_k(r_k) - \beta_k(r_k)\\ &\geq \frac{1}{p^+}\|u\|_{1,p}^{p^-} - \frac{h^0}{p_s^{*-}}\left(\sum_{j\in J} v_j + v_0 + 1\right) - 1,\end{aligned} \tag{4.36}$$

where h^0 is as defined in Lemma 4.2. So

$$\inf_{u\in Z_k, \|u\|_{1,p}=r_k} \widetilde{I}(u) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{4.37}$$

Note that for $\varepsilon \in (0,1)$, $|F(x,u)| \leq C\varepsilon|u|^{p_s^*(x)} + C(\varepsilon)$, then

$$\int_\Omega F(x,u)dx \leq C\varepsilon\int_\Omega |u|^{p_s^*(x)}dx + C(\varepsilon)|\Omega|. \tag{4.38}$$

We have

$$\widetilde{I}(u) \leq \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx - \int_\Omega \frac{\overline{H}|u|^{p_s^*(x)}}{p_s^{*+}}dx + C\varepsilon\int_\Omega |u|^{p_s^*(x)}dx + C(\varepsilon)|\Omega|. \tag{4.39}$$

Take ε sufficiently small so that $C\varepsilon \leq \overline{H}/2p_s^{*+}$, then

$$\widetilde{I}(u) \leq \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}dx - m\int_\Omega |u|^{p_s^*(x)}dx + C, \tag{4.40}$$

where $m = \overline{H}/2p_s^{*+}$. Since the dimension of Y_k is finite, any two norms on Y_k are equivalent, then $k_1\|u\|_{1,p} \leq \|u\|_{p_s^*} \leq k_2\|u\|_{1,p}$, $k_1, k_2 > 0$. As in the proof of Theorem 4.2 in [13], we can find hypercubes $\{Q_i\}_{i=1}^{Q}$ which mutually have no common points such that $\overline{\Omega} \subseteq \bigcup_{i=1}^{Q}\overline{Q_i}$ and $p_i^+ = \sup_{y\in\Omega_i} p(y) < \inf_{y\in\Omega_i} p_s^*(y) = p_{si}^{*-}$, where $\Omega_i = Q_i \bigcap \Omega$. Then we have

$$\begin{aligned}\widetilde{I}(u) &\leq \sum_{\|u\|_{1,p,\Omega_i}>1}\left(\|u\|_{1,p,\Omega_i}^{p_i^+} - mk_2^{p_{si}^{*-}}\|u\|_{1,p,\Omega_i}^{p_{si}^{*-}}\right)\\ &\quad + \sum_{\|u\|_{1,p,\Omega_i}\leq 1}\left(\|u\|_{1,p,\Omega_i}^{p_i^-} - mk_2^{p_{si}^{*+}}\|u\|_{1,p,\Omega_i}^{p_{si}^{*+}}\right) + C\\ &\leq \sum_{\|u\|_{1,p,\Omega_i}>1}\left(\|u\|_{1,p,\Omega_i}^{p_i^+} - mk_2^{p_{si}^{*-}}\|u\|_{1,p,\Omega_i}^{p_{si}^{*-}}\right) + Q + C.\end{aligned} \tag{4.41}$$

Let $f_i(t) = t^{p_i^+} - mk_2^{p_{si}^{*-}}t^{p_{si}^{*-}}$, for $i = 1,\ldots,Q$. Take $s_i > 0$ such that $f_i(s_i) = \max_{t\geq 0} f_i(t) \geq f_i(0) = 0$. Denote $g_i(t) = t^{p_i^+} - mk_2^{p_{si}^{*-}}t^{p_{si}^{*-}} + \sum_{j=1}^{Q} f_j(s_j) + Q + C$, for $i = 1,\ldots,Q$. By $\lim_{t\to\infty} g_i(t) = -\infty$, there exists $t_0 > 0$ such that $g_i(t) \leq 0$ for $t \in [t_0, +\infty)$, for all $i = 1, \ldots, Q$. For any $k = 1, 2, \ldots$, take $\|u\|_{1,p} = \rho_k = \max\{Qt_0, r_k + 1\}$. Note that $\exists i_0$ such that

$$\|u\|_{1,p,\Omega_{i_0}} \geq \frac{1}{Q}\sum_{i=1}^{Q}\|u\|_{1,p,\Omega_i} \geq \frac{\rho_k}{Q} \geq t_0. \tag{4.42}$$

Then we have $g_{i_0}(\|u\|_{1,p,\Omega_{i_0}}) \leq 0$. Thus,

$$\widetilde{I}(u) \leq g_{i_0}\left(\|u\|_{1,p,\Omega_{i_0}}\right) = \sum_{i=1}^{Q} f_i(s_i) + f_{i_0}\left(\|u\|_{1,p,\Omega_{i_0}}\right) + Q + C \leq 0. \tag{4.43}$$

Therefore, $\widetilde{I}(u) \leq 0$ for $u \in Y_K \bigcap S_{\rho_k}$, where $S_{\rho_k} = \{u : \|u\|_{1,p} = \rho_k\}$. From Lemma 4.2 we have that $\widetilde{I}(u)$ satisfies $(PS)_c$ condition. In view of (F-4), by Fountain Theorem [21], we conclude the result. □

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