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Research Article

Fixed Point Theorems of Single-Valued Mapping for c-Distance in Cone Metric Spaces

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A new concept of the c-distance in cone metric space has been introduced recently in 2011. The aim of this paper is to extend and generalize some fixed point theorems on c-distance in cone metric space.

1. Introduction

The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone, for new results on cone metric spaces see [1–6]. This cone naturally induces a partial order in the Banach spaces. The concept of cone metric space was introduced in the work of Huang and Zhang [7], where they also established the Banach contraction mapping principle in this space. Then, several authors have studied fixed point problems in cone metric spaces. Some of these works are noted in [8–13].

In [14], Cho et al. introduced a new concept of the c-distance in cone metric spaces and proved some fixed point theorems in ordered cone metric spaces. This is more general than the classical Banach contraction mapping principle.

In [15], Sintunavarat et al. extended and developed the Banach contraction theorem on c-distance of Cho et al. [14]. They gave some illustrative examples of the main results. Their results improve, generalize, and unify the results of Cho et al. [14] and some results of the fundamental metrical fixed point theorems in the literature.

In this paper we proved some fixed point theorems for c-distance in cone metric space. These theorems extend and develop some theorems in literature on c-distance of Cho et al. [14] in cone metric space.

The following theorems are the main results given in [7, 14, 16].

Theorem 1.1 (see [16]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$d(fx, fy) \preceq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.

Theorem 1.2 (see [7]). *Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$d(fx, fy) \preceq k(d(fx, x) + d(fy, y)), \tag{1.2}$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.

Theorem 1.3 (see [7]). *Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$d(fx, fy) \preceq k(d(fx, y) + d(fy, x)), \tag{1.3}$$

for all $x, y \in X$, where $k \in [0, 1/2)$ is a constant. Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.

Theorem 1.4 (see [14]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q is a c-distance on X and $f : X \to X$ be a continous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(1) *there exist $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$ such that*

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fx) + \gamma q(y, fy), \tag{1.4}$$

for all $x, y \in X$ with $y \sqsubseteq x$,

(2) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then f has a fixed point $x^* \in X$. If $v = fv$ then $q(v, v) = \theta$.

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that

(1) P is nonempty set closed and $P \neq \{\theta\}$,

(2) if a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,

(3) $x \in P$ and $-x \in P$ imply that $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|, \tag{2.1}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

Definition 2.1 (see [7]). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \prec d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, y) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 (see [7]). Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X, and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.3 (see [17]).

(1) *If E is a real Banach space with a cone P and $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.*

(2) *If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Next we give the notation of c-distance on a cone metric space which is a generalization of ω-distance of Kada et al. [18] with some properties.

Definition 2.4 (see [14]). Let (X, d) is a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q2) $q(x, y) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$, (q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.5 (see [14]). Let $E = \mathbb{R}$ and $P = (x \in E : x \geq 0)$. let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. then (X, d) is a cone metric space. define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Lemma 2.6 (see [14]). *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$,*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z,*

(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X,*

(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.7 (see [14]).

(1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.

(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main Results

In this section we prove some fixed point theorems using c-distance in cone metric space. In whole paper cone metric space is over nonnormal cone with nonempty interior.

Theorem 3.1. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$q(fx, fy) \preceq kq(x, y), \tag{3.1}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v, v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2 x_0, \ldots, x_{n+1} = fx_n = f^{n+1} x_0$. We have:

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq kq(x_{n-1}, x_n) \\ &\preceq k^2 q(x_{n-2}, x_{n-1}) \preceq \cdots \preceq k^n q(x_0, x_1). \end{aligned} \tag{3.2}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{k^n}{1-k} q(x_0, x_1). \end{aligned} \tag{3.3}$$

Thus, Lemma 2.6 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By (q3) we have:

$$q(x_n, x^*) \preceq \frac{k^n}{1-k} q(x_0, x_1). \tag{3.4}$$

On the other hand,

$$\begin{aligned} q(x_n, fx^*) &= q(fx_{n-1}, fx^*) \\ &\preceq kq(x_{n-1}, x^*) \\ &\preceq k\frac{k^{n-1}}{1-k} q(x_0, x_1) \\ &= \frac{k^n}{1-k} q(x_0, x_1). \end{aligned} \tag{3.5}$$

By Lemma 2.6 part 1, (3.4) and (3.5), we have $x^* = fx^*$. Thus, x^* is a fixed point of f.

Suppose that $v = fv$, then we have the following: $q(v, v) = q(fv, fv) \preceq kq(v, v)$. Since $k < 1$, Lemma 2.3 show that $q(v, v) = \theta$.

Finally suppose there is another fixed point y^* of f, then we have the following: $q(x^*, y^*) = q(fx^*, fy^*) \preceq kq(x^*, y^*)$. Since $k < 1$, Lemma 2.3 show that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$. Hence by Lemma 2.6 part 1, $x^* = y^*$. Therefore the fixed point is unique. □

Corollary 3.2. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$q(f^n x, f^n y) \preceq kd(x, y), \tag{3.6}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f has a unique fixed point $x^* \in X$. If $v = fv$ then $q(v, v) = \theta$.

Proof. From Theorem 3.1 f^n has a unique fixed point x^*. But $f^n(fx^*) = f(f^n x^*) = f(x^*)$, so $f(x^*)$ is also a fixed point of f^n. Hence $x^* = fx^*$. Thus, x^* is a fixed point of f. Since the fixed point of f is also fixed point of f^n, the fixed point of f is unique.

Suppose that $v = fv$. From above the fixed point of f is also fixed point of f^n, then we have the following: $q(v, v) = q(fv, fv) = q(f^n v, f^n v) \preceq kq(v, v)$. Since $k < 1$, Lemma 2.3 show that $q(v, v) = \theta$. □

The following result is generalized from Theorem 1.4. We prove a fixed point theorem and we do not require that X is a partially ordered set.

Theorem 3.3. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:*

$$q(fx, fy) \preceq kq(x, y) + lq(x, fx) + rq(y, fy), \tag{3.7}$$

for all $x, y \in X$, where k, l, r are nonnegative real numbers such that $k + l + r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v, v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0$. We have the following:

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq kq(x_{n-1}, x_n) + lq(x_{n-1}, fx_{n-1}) + rq(x_n, fx_n) \\ &= kq(x_{n-1}, x_n) + lq(x_{n-1}, x_n) + rq(x_n, x_{n+1}). \end{aligned} \tag{3.8}$$

So

$$q(x_n, x_{n+1}) \preceq \frac{k+l}{1-r} q(x_{n-1}, x_n) = hq(x_{n-1}, x_n), \tag{3.9}$$

where $h = (k + l)/(1 - r) < 1$.

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{h^n}{1-h} q(x_0, x_1). \end{aligned} \tag{3.10}$$

Thus, Lemma 2.6 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, then $x^* = \lim x_{n+1} = \lim f(x_n) = f(\lim x_n) = f(x^*)$. Therefore x^* is a fixed point of f.

Suppose that $v = fv$, then we have

$$q(v, v) = q(fv, fv) \preceq kq(v, v) + lq(v, fv) + rq(v, fv) = (k + l + r)q(v, v), \tag{3.11}$$

since $k + l + r < 1$, Lemma 2.3 show that $q(v, v) = \theta$.

Finally, suppose that, there is another fixed point y^* of f, then we have the following:

$$\begin{aligned} q(x^*, y^*) &= q(fx^*, fy^*) \\ &\preceq kq(x^*, y^*) + lq(x^*, fx^*) + rq(y^*, fy^*) \\ &= kq(x^*, y^*) + lq(x^*, x^*) + rq(y^*, y^*) \\ &= kq(x^*, y^*) \\ &\preceq kq(x^*, y^*) + lq(x^*, y^*) + rq(x^*, y^*) \\ &= (k + l + r)q(x^*, y^*). \end{aligned} \tag{3.12}$$

Since $k + l + r < 1 < 1$, Lemma 2.3 shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$. Hence by Lemma 2.6 part 1, $x^* = y^*$. Therefore the fixed point is unique. □

If $k = 0$ and $r = l$, then we have the following result.

Corollary 3.4. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ is continuous and satisfies the contractive condition:*

$$q(fx, fy) \preceq l(q(x, fx) + q(y, fy)), \tag{3.13}$$

for all $x, y \in X$,where $l \in [0, 1/2)$ is a constant. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v, v) = \theta$. The fixed point is unique.

Finally, we provide another result and we do not require that f is continuous.

Theorem 3.5. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $f : X \to X$ satisfies the contractive condition:*

$$(1 - r)q(fx, fy) \preceq kq(x, fy) + lq(x, fx), \tag{3.14}$$

for all $x, y \in X$, where k, l, r are nonnegative real numbers such that $2k + l + r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$ then $q(v, v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2 x_0, \ldots, x_{n+1} = fx_n = f^{n+1} x_0$. Observe that

$$(1 - r)q(fx, fy) \preceq kq(x, fy) + lq(x, fx), \tag{3.15}$$

equivalently

$$q(fx, fy) \preceq kq(x, fy) + lq(x, fx) + rq(fx, fy). \tag{3.16}$$

Then we have:

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq kq(x_{n-1}, fx_n) + lq(x_{n-1}, fx_{n-1}) + rq(fx_{n-1}, fx_n) \\ &= kq(x_{n-1}, x_{n+1}) + lq(x_{n-1}, x_n) + rq(x_n, x_{n+1}) \\ &\preceq kq(x_{n-1}, x_n) + kq(x_n, x_{n+1}) + lq(x_{n-1}, x_n) + rq(x_n, x_{n+1}). \end{aligned} \tag{3.17}$$

So,

$$q(x_n, x_{n+1}) \preceq \frac{k+l}{1-k-r} q(x_{n-1}, x_n) = hq(x_{n-1}, x_n), \tag{3.18}$$

where $h = (k+l)/(1-k-r) < 1$.

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{h^n}{1-h} q(x_0, x_1). \end{aligned} \tag{3.19}$$

Thus, Lemma 2.6 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

By (q3) we have:

$$q(x_n, x^*) \preceq \frac{h^n}{1-h} q(x_0, x_1). \tag{3.20}$$

On the other hand,

$$\begin{aligned} q(x_n, fx^*) &= q(fx_{n-1}, fx^*) \\ &\preceq kq(x_{n-1}, fx^*) + lq(x_{n-1}, fx_{n-1}) + rq(fx_{n-1}, fx^*) \\ &= kq(x_{n-1}, fx^*) + lq(x_{n-1}, x_n) + rq(x_n, x_{n+1}) \\ &\preceq kq(x_{n-1}, x_n) + kq(x_n, fx^*) + lq(x_{n-1}, x_n) + rq(x_n, fx^*). \end{aligned} \tag{3.21}$$

So,

$$\begin{aligned} q(x_n, fx^*) &\preceq \frac{k+l}{1-k-r} q(x_{n-1}, x_n) \\ &\preceq \frac{k+l}{1-k-r} h^{n-1} q(x_0, x_1) \\ &= hh^{n-1} q(x_0, x_1) \end{aligned}$$

$$
\begin{aligned}
&= h^n q(x_0, x_1) \\
&\preceq \frac{h^n}{1-h} q(x_0, x_1).
\end{aligned} \tag{3.22}
$$

By Lemma 2.6 part 1, (3.20) and (3.22), we have $x^* = fx^*$. Thus, x^* is a fixed point of f. Suppose that $v = fv$, then we have

$$
\begin{aligned}
q(v, v) &= q(fv, fv) \\
&\preceq kq(v, fv) + lq(v, fv) + rq(v, fv) \\
&= kq(v, v) + lq(v, v) + rq(v, v) \\
&\preceq kq(v, v) + kq(v, v) + lq(v, v) + rq(v, v) \\
&= (2k + l + r)q(v, v).
\end{aligned} \tag{3.23}
$$

Since $2k + l + r < 1$, Lemma 2.3 shows that $q(v, v) = \theta$.

Finally, suppose that, there is another fixed point y^* of f, then we have

$$
\begin{aligned}
q(x^*, y^*) &= q(fx^*, fy^*) \\
&\preceq kq(x^*, fy^*) + lq(x^*, fx^*) + rq(fx^*, fy^*) \\
&\preceq kq(x^*, fy^*) + kq(x^*, fy^*) + lq(x^*, fx^*) + rq(fx^*, fy^*) \\
&= kq(x^*, y^*) + kq(x^*, y^*) + lq(x^*, x^*) + rq(x^*, y^*) \\
&= kq(x^*, y^*) + kq(x^*, y^*) + rq(x^*, y^*) \\
&\preceq kq(x^*, y^*) + kq(x^*, y^*) + lq(x^*, y^*) + rq(x^*, y^*) \\
&= (2k + l + r)q(x^*, y^*).
\end{aligned} \tag{3.24}
$$

Since $2k + l + r < 1$, Lemma 2.3 shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$. Hence by Lemma 2.6 part 1, $x^* = y^*$. Therefore the fixed point is unique. □

Example 3.6. Consider Example 2.5. Define the mapping $f : X \to X$ by $f(3/4) = 1/4$ and $fx = x/2$ for all $x \in X$ with $x \neq 3/4$. Since $d(f(1), f(3/4)) = d(1, 3/4)$, there is not $k \in [0, 1)$ such that $d(fx, fy) \preceq kd(x, y)$. Since Theorem 2.3 of Rezapour and Hamlbarani [16] cannot be applied to this example on cone metric space. To check this example on c-distance we have:

(1) If $x = y = 3/4$, then we have the following.

$$
q\left(f\left(\frac{3}{4}\right), f\left(\frac{3}{4}\right)\right) = f\left(\frac{3}{4}\right) = \frac{1}{4} \preceq k\frac{3}{4} = kq\left(\frac{3}{4}, \frac{3}{4}\right) \quad \text{with } k = \frac{2}{3}. \tag{3.25}
$$

(2) If $x \neq y \neq 3/4$, then we have

$$
q(fx, fy) = \frac{y}{2} \preceq kq(x, y) \quad \text{with } k = \frac{2}{3}. \tag{3.26}
$$

(3) If $x = 3/4, y \neq 3/4$, then we have

$$q\left(f\left(\frac{3}{4}\right), fy\right) = \frac{y}{2} \preceq kq\left(\frac{3}{4}, y\right) \quad \text{with } k = \frac{2}{3}. \tag{3.27}$$

(4) If $x \neq 3/4, y = 3/4$, then we have

$$q\left(fx, f\left(\frac{3}{4}\right)\right) = f\left(\frac{3}{4}\right) = \frac{1}{4} \preceq k\frac{3}{4} = kq\left(x, \frac{3}{4}\right) \quad \text{with } k = \frac{2}{3}. \tag{3.28}$$

Hence $q(fx, fy) \preceq kq(x, y)$ for all $x \in X$. Therefore, the condition of Theorem 3.1 are satisfied and then f has a unique fixed point $x = 0$, $f(0) = 0$ with $q(0, 0) = 0$.

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