

Research Article

Multipliers in Holomorphic Mean Lipschitz Spaces on the Unit Ball

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For $1 \leq p \leq \infty$ and $s > 0$, let Λ_s^p be holomorphic mean Lipschitz spaces on the unit ball in \mathbb{C}^n . It is shown that, if $s > n/p$, the space Λ_s^p is a multiplicative algebra. If $s > n/p$, then the space Λ_s^p is not a multiplicative algebra. We give some sufficient conditions for a holomorphic function to be a pointwise multiplier of $\Lambda_{n/p}^p$.

1. Introduction

Let X and Y be two function spaces. We call φ a pointwise multiplier from X to Y if $\varphi f \in Y$ for every $f \in X$. The collection of all pointwise multipliers from X to Y is denoted by $\mathcal{M}(X \rightarrow Y)$. When $X = Y$, we let $\mathcal{M}(X) = \mathcal{M}(X \rightarrow X)$.

Multipliers arise in the theory of differential equations. Coefficients of differential operators can be naturally considered as multipliers. The same is true for symbols of more general pseudodifferential operators.

To give some motivations for our study, we recall studies on multipliers of Sobolev spaces. Strichartz [1] was the first who studied on multipliers of Sobolev spaces. Let Ω be a bounded domain in \mathbb{R}^n with Lipschitz boundary. Let $1 < p < \infty$. For $s > 0$, let $W^{s,p}(\Omega)$ be the Sobolev spaces over Ω . Given f and g in $W^{s,p}(\Omega)$, one cannot in general expect that their product fg will belong to $W^{s,p}(\Omega)$. However, if $s > n/p$, then there exists a constant K depending on s, p, n , and Ω , such that [1–3]

$$\|fg\|_{W^{s,p}(\Omega)} \leq K \|f\|_{W^{s,p}(\Omega)} \|g\|_{W^{s,p}(\Omega)}. \quad (1.1)$$

This implies that $W^{s,p}(\Omega) \subset \mathcal{M}(W^{s,p}(\Omega))$. Since $W^{s,p}(\Omega)$ contains constant functions, $\mathcal{M}(W^{s,p}(\Omega)) \subset W^{s,p}(\Omega)$. Thus, we have

$$\mathcal{M}(W^{s,p}(\Omega)) = W^{s,p}(\Omega) \quad \text{if } s > \frac{n}{p}. \quad (1.2)$$

In the complex case, multipliers on Hardy-Sobolev spaces on the unit ball in \mathbb{C}^n were studied by [4, 5] for $n = 1$ and [6] for $n \geq 1$. Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball in \mathbb{C}^n . Let $1 \leq p < \infty$ and m be a positive integer. Let $H_m^p(\mathbb{B}^n)$ be the Hardy-Sobolev space of order m . In papers [4–6], it was proved that

$$\mathcal{M}(H_m^p(\mathbb{B}^n)) = H_m^p(\mathbb{B}^n) \quad \text{if } m > \frac{n}{p}. \quad (1.3)$$

Complete characterization of multipliers on other Hardy-Sobolev spaces of nonregular cases $m \leq n/p$ remains open, but Beatrous and Burbea [7] gave some sufficient conditions for functions to be pointwise multipliers in these nonregular cases. Ortega and Fabrega [8] introduced a family of nonisotropic tent-Sobolev spaces to characterize multipliers in some Hardy-Sobolev spaces of nonregular cases. Usually, characterization of multipliers of nonregular cases is difficult.

Many authors have studied properties of multipliers for several function spaces (see [9–12] for Dirichlet-type spaces, [13, 14] for Bloch-type spaces, [4] for Bergman-Sobolev spaces, [15] for mixed norm spaces, [16] for Q_s spaces, [17] for $F_s^{\infty,q}$ spaces, and [18] for the BMO space).

For points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in \mathbb{C}^n , we write

$$\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n, \quad |z| = \sqrt{|z_1|^2 + \dots + |z_n|^2}. \quad (1.4)$$

Let $\mathbb{S}^n = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$ denote the unit sphere in \mathbb{C}^n . The normalized Lebesgue measure on \mathbb{S}^n will be denoted by $d\sigma$. Let $H(\mathbb{B}^n)$ denote the space of all holomorphic functions in \mathbb{B}^n . Given $0 < r < 1$, $0 < p < \infty$, and $f \in H(\mathbb{B}^n)$, we define

$$M_p(r, f) = \left[\int_{\mathbb{S}^n} |f(r\zeta)|^p d\sigma(\zeta) \right]^{1/p}. \quad (1.5)$$

When $p = \infty$, we write

$$M_\infty(r, f) = \sup\{|f(r\zeta)| : \zeta \in \mathbb{S}^n\}. \quad (1.6)$$

For $0 < p \leq \infty$, the Hardy space H^p consists of all functions $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty. \quad (1.7)$$

See [19] for basic information about the Hardy spaces.

We denote by $\mathcal{R}f$ the radial derivative of f in $H(\mathbb{B}^n)$ defined by

$$\mathcal{R}f = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} f. \quad (1.8)$$

We consider the space Λ_s^p of holomorphic functions on \mathbb{B}^n such that

$$\left[\int_{\mathbb{S}^n} |\mathcal{R}^m f(r\zeta)|^p d\sigma \right]^{1/p} \lesssim \frac{1}{(1-r)^{m-s}} \quad (1.9)$$

for $m > s$. We define the norm of Λ_s^p as follows:

$$\|f\|_{\Lambda_s^p} = \|f\|_{H^p} + \sup_{0 < r < 1} (1-r)^{m-s} M_p(r, \mathcal{R}^m f). \quad (1.10)$$

It can be shown that the norm is independent of the choice of m ; see [20]. When $p = \infty$, this is exactly the classical holomorphic Lipschitz space Λ_s ; see [19].

We adapt the first order H^p mean variation defined as follows:

$$\omega_p(t, f) = \sup \left\{ \left(\int_{\mathbb{S}^n} |f(U\zeta) - f(\zeta)|^p d\sigma(\zeta) \right)^{1/p} : U \in \mathcal{U}, \|U - I\| \leq t \right\}, \quad (1.11)$$

where \mathcal{U} denotes the group of all unitary operators on \mathbb{C}^n , I denotes the identity of \mathcal{U} , and $\|U - I\| := \sup_{\zeta \in \mathbb{S}^n} |U\zeta - \zeta|$. Then, we have

$$\|f\|_{\Lambda_s^p} \approx \|f\|_{H^p} + \sup_{0 < t < 1} \frac{\omega_p(t, f)}{t^s} \quad (1.12)$$

for $0 < s < 1$ (see [20]). This justifies our usage of the term *holomorphic mean Lipschitz space* for Λ_s^p with $0 < s < 1$. Now, for $s \geq 1$, we consider the second-order H^p mean variation defined as follows:

$$\omega_p^*(t, f) = \sup \left\{ \left(\int_{\mathbb{S}^n} |f(U\zeta) - 2f(\zeta) + f(U^{-1}\zeta)|^p d\sigma(\zeta) \right)^{1/p} : U \in \mathcal{U}, \|U - I\| \leq t \right\}. \quad (1.13)$$

It was shown in [21] that, if $0 < s < 2$, $1 \leq p < \infty$, and $f \in H^p$, then

$$\|f\|_{\Lambda_s^p} \approx \|f\|_{H^p} + \sup_{0 < t < 1} \frac{\omega_p^*(t, f)}{t^s}. \quad (1.14)$$

Theorem 1.1. *Let $1 \leq p < \infty$ and $s > 0$.*

- (i) *If $s > n/p$ (regular), then Λ_s^p is a multiplicative algebra.*
- (ii) *If $s \leq n/p$ (nonregular), then Λ_s^p is not a multiplicative algebra.*

By (ii) of Theorem 1.1, the space $\Lambda_{n/p}^p$ is not a multiplicative algebra. We give some sufficient conditions for a holomorphic function to be a pointwise multiplier of $\Lambda_{n/p}^p$ as follows. We do not know if our sufficient condition is also necessary.

Theorem 1.2. *Let $1 \leq p < q < \infty$. Then, $\Lambda_{n/p}^q \subset \mathcal{M}(\Lambda_{n/p}^p)$.*

Throughout the paper, we write $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y whenever there is a constant $C > 0$ (independent of the parameters in X and Y) such that $X \leq CY$. Similarly, we write $X \approx Y$ if $X \lesssim Y$ and $Y \lesssim X$.

2. Auxiliary Embedding Results

The ball algebra $A(\mathbb{B}^n)$ is the class of all functions $f : \mathbb{B}^n \rightarrow \mathbb{C}$ that are continuous on the closed ball $\overline{\mathbb{B}^n}$ and that are holomorphic in its interior \mathbb{B}^n .

Proposition 2.1. *Let $1 \leq p < \infty$ and $s > 0$.*

- (i) *There is a function in the ball algebra $A(\mathbb{B}^n)$ that is not in Λ_s^p .*
- (ii) *If $s \leq n/p$, then there is a function in Λ_s^p that is not in H^∞ .*
- (iii) *If $s > n/p$, then $\Lambda_s^p \subset \Lambda_{s-n/p} \subset A(\mathbb{B}^n)$.*

Remark 2.2. By (ii) and (iii), we can see that $\Lambda_s^p \subset H^\infty$ if and only if $s > n/p$.

Proof. (i) Let (p_k) be a sequence of Ryll and Wojtaszczyk [22] homogeneous polynomials in the unit sphere of $H^\infty(\mathbb{B}^n)$ such that p_k has degree k and

$$\|p_k\|_{H^2} \geq \frac{\sqrt{\pi}}{2^n}. \quad (2.1)$$

Let

$$f = \sum \frac{1}{k^2} p_{2k}. \quad (2.2)$$

This function was constructed in [7]. In fact, it was shown in [7] that this function is not contained in any Hardy-Sobolev space. Thus, the result of (i) of Proposition 2.1 follows, since every mean Lipschitz space is contained in a Hardy-Sobolev space.

Since the series converges uniformly on $\overline{\mathbb{B}^n}$, its sum f is therefore in the ball algebra. It is enough to prove that $f \notin \Lambda_s^1$ for $0 < s < 1$. If $f \in \Lambda_s^1$, then

$$\begin{aligned} \left| \int_{\mathbb{S}^n} \mathcal{R}f(r\zeta) \overline{p_{2^k}(\zeta)} d\sigma(\zeta) \right| &\leq \int_{\mathbb{S}^n} |\mathcal{R}f(r\zeta)| d\sigma(\zeta) \\ &\lesssim \frac{1}{(1-r)^{1-s}}, \quad 0 < r < 1. \end{aligned} \tag{2.3}$$

However, since the polynomials p_k are orthogonal, for any $0 < r < 1$,

$$\begin{aligned} \left| \int_{\mathbb{S}^n} \mathcal{R}f(r\zeta) \overline{p_{2^k}(\zeta)} d\sigma(\zeta) \right| &= \left| \int_{\mathbb{S}^n} \frac{2^k}{k^2} p_{2^k}(r\zeta) \overline{p_{2^k}(\zeta)} d\sigma(\zeta) \right| \\ &= \int_{\mathbb{S}^n} \frac{2^k}{k^2} r^{2^k} |p_{2^k}(\zeta)|^2 d\sigma(\zeta) \\ &= \frac{2^k}{k^2} r^{2^k} \|p_{2^k}\|_{H^2}^2 \\ &\geq \frac{2^k}{k^2} r^{2^k} \frac{\pi}{4^n}. \end{aligned} \tag{2.4}$$

Take

$$r_k = 1 - \frac{1}{2^k}. \tag{2.5}$$

Then,

$$\begin{aligned} \left| \int_{\mathbb{S}^n} \mathcal{R}f(r_k\zeta) \overline{p_{2^k}(\zeta)} d\sigma(\zeta) \right| &\geq \frac{1}{1-r_k} \frac{1}{(\log_2(1/(1-r_k)))^2} r_k^{2^k} \frac{\pi}{4^n} \\ &= \frac{1}{(1-r_k)^{1-s}} c_k, \end{aligned} \tag{2.6}$$

where

$$c_k = \frac{1}{(1-r_k)^s} \frac{1}{(\log_2(1/(1-r_k)))^2} r_k^{2^k} \frac{\pi}{4^n} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{2.7}$$

By (2.3) and (2.6), it is a contradiction. Thus, $f \notin \Lambda_s^1$ for $0 < s < 1$.

(ii) It is clear, if we consider the function

$$f(z) = \log(1 - \langle z, \zeta \rangle) \tag{2.8}$$

with $\zeta \in \mathbb{S}^n$.

(iii) Let $f \in \Lambda_s^p$. Let m be the greatest integer less than s and $\alpha = s - m$. Let $0 < \rho < 1$. By the Cauchy's integral formula, we have

$$\mathcal{R}^{m+1}f(\rho z) = \int_{\mathbb{S}^n} \frac{\mathcal{R}^{m+1}f(\rho \zeta)}{(1 - \langle z, \zeta \rangle)^n} d\sigma(\zeta), \quad z \in \mathbb{B}^n. \quad (2.9)$$

Making a change of variables and replacing ρz by z , we get

$$\mathcal{R}^{m+1}f(z) = \rho \int_{|\zeta|=\rho} \frac{\mathcal{R}^{m+1}f(\zeta)}{(\rho^2 - \langle z, \zeta \rangle)^n} d\sigma(\zeta), \quad |z| < \rho. \quad (2.10)$$

By (2.10) and Hölder's inequality, we have

$$|\mathcal{R}^{m+1}f(z)| \lesssim M_p(\rho, \mathcal{R}^{m+1}f) \frac{1}{(\rho^2 - |z|^2)^{n/p}}. \quad (2.11)$$

Take $\rho = (1 + |z|)/2$. Then, we obtain

$$\begin{aligned} |\mathcal{R}^{m+1}f(z)| &\lesssim M_p(\rho, \mathcal{R}^{m+1}f) \frac{1}{(1 - |z|^2)^{n/p}} \\ &\lesssim \frac{1}{(1 - \rho)^{1-\alpha}} \frac{1}{(1 - |z|^2)^{n/p}} \\ &\lesssim \frac{1}{(1 - |z|^2)^{m+1-(s-n/p)}}. \end{aligned} \quad (2.12)$$

Therefore, $f \in \Lambda_{s-n/p} \subset A(\mathbb{B}^n)$ if $s > n/p$. □

Proposition 2.3. Let $s > 0$ and $1 \leq p, q < \infty$.

(i) $\Lambda_s^p \subset H^q$, $0 \leq n(1/p - 1/q) < s$.

(ii) $\Lambda_{n/p}^p \subset \bigcap_{0 < q < \infty} H^q$.

Proof. (i) Let $1/2 < r < 1$ and $\alpha = s - m$, where m is the greatest integer less than s . By the fundamental theorem of calculus and Minkowski's inequality, we have

$$M_q(r, f) \lesssim \sup_{|z| < 1/2} |f(z)| + \int_0^r M_q(t, \mathcal{R}f) dt. \quad (2.13)$$

Applying this repeatedly and using Fubini's theorem, we obtain

$$\begin{aligned} M_q(r, f) &\lesssim \sup_{|z|<1/2} |f(z)| + \int_0^r (r-t)^m M_q(t, \mathcal{R}^{m+1} f) dt \\ &\lesssim \sup_{|z|<1/2} |f(z)| + \int_0^1 (1-t)^m M_q(t, \mathcal{R}^{m+1} f) dt. \end{aligned} \tag{2.14}$$

Let $0 < \rho < 1$. By (2.11), we have

$$|\mathcal{R}^{m+1} f(z)| \lesssim M_p(\rho, \mathcal{R}^{m+1} f) (\rho^2 - |z|^2)^{-n/p}, \quad |z| < \rho. \tag{2.15}$$

For $0 < r < 1$, we take $\rho = (1+r)/2$. Then,

$$M_\infty(r, \mathcal{R}^{m+1} f) \lesssim M_p(\rho, \mathcal{R}^{m+1} f) (1-r)^{-n/p}. \tag{2.16}$$

For $q \geq p$, we have

$$\begin{aligned} M_q(r, \mathcal{R}^{m+1} f) &= \left(\int_{\mathbb{S}} |\mathcal{R}^{m+1} f(r\zeta)|^p |\mathcal{R}^{m+1} f(r\zeta)|^{q-p} d\sigma(\zeta) \right)^{1/q} \\ &\lesssim M_\infty(r, \mathcal{R}^{m+1} f)^{1-p/q} M_p(r, \mathcal{R}^{m+1} f)^{p/q} \\ &\lesssim M_p(\rho, \mathcal{R}^{m+1} f)^{1-p/q} (1-r)^{-n/p(1-p/q)} M_p(r, \mathcal{R}^{m+1} f)^{p/q} \\ &\lesssim \|f\|_{\Lambda_s^p} (1-r)^{n(1/q-1/p)-1+\alpha}. \end{aligned} \tag{2.17}$$

By (2.14) and (2.17), we have

$$M_q(r, f) \lesssim \|f\|_{\Lambda_s^p} \int_0^1 \frac{1}{(1-t)^{n(1/p-1/q)+1-\alpha-m}} dt < \infty, \tag{2.18}$$

since $n(1/p - 1/q) + 1 - \alpha - m < s + 1 - \alpha - m = 1$. Thus, we get the result.

(ii) If $s = n/p$ in (2.18), then, for any $p < q < \infty$, we have

$$\begin{aligned} M_q(r, f) &\lesssim \|f\|_{\Lambda_{n/p}^p} \int_0^r \frac{1}{(1-t)^{1-n/q}} dt \\ &\lesssim \|f\|_{\Lambda_{n/p}^p}. \end{aligned} \tag{2.19} \quad \square$$

Remark 2.4. The obvious question: Is $\Lambda_{n/p}^p$ contained in *BMOA*? In the case $n = 1$, this was proved by Bourdon et al. in [23]. However, their method does not work in higher dimensions.

We have some observations by a Carleson measure. There is a characterization for *BMOA* functions by a Carleson measure such that $f \in \text{BMOA}$ if and only if

$(1 - |z|^2)|\mathcal{R}f(z)|^2 dV(z)$ is a Carleson measure, where dV is the volume measure on \mathbb{B}^n (see [19]). Even though we cannot prove that $(1 - |z|^2)|\mathcal{R}f(z)|^2 dV(z)$ is a Carleson measure for $f \in \Lambda_{n/p}^p$, we have some weak results such that $(1 - |z|^2)^{q-1}|\mathcal{R}f(z)|^q dV(z)$ is a Carleson measure for q with $\max\{p, n\} < q$. For the proof, let $f \in \Lambda_{n/p}^p$ and $1 \leq p < \infty$. By (2.17), we have $\Lambda_{n/p}^p \subset \Lambda_{n/q}^q$. This means that

$$M_q(r, \mathcal{R}f) \lesssim \frac{1}{(1-r)^{1-n/q}}. \quad (2.20)$$

Hence, we have

$$\begin{aligned} & \int_{\mathbb{B}^n} (1 - |z|^2)^{q-1} |\mathcal{R}f(z)|^q \frac{(1 - |a|^2)^n}{|1 - \langle a, z \rangle|^{2n}} dV(z) \\ & \lesssim \int_0^1 (1-r)^{q-1} \frac{(1 - |a|^2)^n}{(1-r|a|)^{2n}} M_q^q(r, \mathcal{R}f) dr \\ & \lesssim (1 - |a|^2)^n \int_0^1 (1-r)^{n-1} \frac{1}{(1-r|a|)^{2n}} dr \\ & \lesssim 1. \end{aligned} \quad (2.21)$$

Thus, $(1 - |z|^2)^{q-1}|\mathcal{R}f(z)|^q dV(z)$ is a Carleson measure (see [19]). However, the embedding problem $\Lambda_{n/p}^p \subset BMOA$ is still open.

3. Regular Cases

We need an elementary variant of Hölder's inequality.

Lemma 3.1. *Let $0 < p, q, \delta < \infty$ with $1/p = 1/q + 1/\delta$. Then, for $f \in L^q$ and $g \in L^\delta$, the product fg is in L^p and*

$$\|fg\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^\delta}. \quad (3.1)$$

Theorem 3.2. *If $s > n/p$, then Λ_s^p is a multiplicative algebra.*

Proof. Let m be the greatest integer less than s and $\alpha = s - m$. Let $f, g \in \Lambda_s^p$. We will prove that

$$M_p(r, \mathcal{R}^{m+1}(fg)) \lesssim \frac{1}{(1-r)^{1-\alpha}}. \quad (3.2)$$

We note that

$$\mathcal{R}^{m+1}(fg) = \sum_{j=0}^{m+1} c_j \mathcal{R}^j f \mathcal{R}^{m+1-j} g. \quad (3.3)$$

Since $s > n/p$, by (iii) of Proposition 2.1, we have $\Lambda_s^p \subset A(\mathbb{B}^n) \subset H^\infty$.

Let $0 \leq k \leq m+1$. Then,

$$M_{(m+1)p/k}(r, \mathcal{R}^k f) \leq C \|f\|_{H^\infty}^{1-k/(m+1)} M_p(r, \mathcal{R}^{m+1} f)^{k/(m+1)}, \quad (3.4)$$

applied to the sphere of radius r (see [24], Theorem 1 on page 69, for such inequalities). Thus, for $0 \leq j \leq m+1$, we have

$$\begin{aligned} M_{(m+1)p/j}(r, \mathcal{R}^j f) &\leq C \|f\|_{H^\infty}^{1-j/(m+1)} M_p(r, \mathcal{R}^{m+1} f)^{j/(m+1)}, \\ M_{(m+1)p/(m+1-j)}(r, \mathcal{R}^{m+1-j} g) &\leq C \|g\|_{H^\infty}^{1-(m+1-j)/(m+1)} M_p(r, \mathcal{R}^{m+1} g)^{(m+1-j)/(m+1)}. \end{aligned} \quad (3.5)$$

Therefore, by a variant of Hölder's inequality,

$$\begin{aligned} M_p(r, \mathcal{R}^{m+1}(fg)) &\lesssim \sum_{j=0}^{m+1} M_{(m+1)p/j}(r, \mathcal{R}^j f) M_{(m+1)p/(m+1-j)}(r, \mathcal{R}^{m+1-j} g) \\ &\lesssim \|f\|_{H^\infty}^{1-j/(m+1)} \|g\|_{H^\infty}^{1-(m+1-j)/(m+1)} M_p(r, \mathcal{R}^{m+1} f)^{j/(m+1)} M_p(r, \mathcal{R}^{m+1} g)^{(m+1-j)/(m+1)} \\ &\lesssim \|f\|_{H^\infty}^{1-j/(m+1)} \|g\|_{H^\infty}^{1-(m+1-j)/(m+1)} \|f\|_{\Lambda_s^p}^{j/(m+1)} \|g\|_{\Lambda_s^p}^{(m+1-j)/(m+1)} \frac{1}{(1-r)^{1-\alpha}} \\ &\lesssim \|f\|_{\Lambda_s^p} \|g\|_{\Lambda_s^p} \frac{1}{(1-r)^{1-\alpha}}. \end{aligned} \quad (3.6)$$

That is,

$$\|fg\|_{\Lambda_s^p} \lesssim \|f\|_{\Lambda_s^p} \|g\|_{\Lambda_s^p}. \quad (3.7)$$

□

4. Nonregular Cases

A function φ is a multiplier for Λ_s^p if the multiplication operator $M_\varphi f = \varphi f$ is continuous from Λ_s^p to itself. The space of those multipliers will be denoted by $\mathcal{M}(\Lambda_s^p)$.

Let $\varphi \in \mathcal{M}(\Lambda_s^p)$. Since $1 \in \Lambda_s^p$, we have $\varphi = \varphi \cdot 1 \in \Lambda_s^p$. Thus,

$$\mathcal{M}(\Lambda_s^p) \subset \Lambda_s^p. \quad (4.1)$$

Hence,

$$\mathcal{M}(\Lambda_s^p) \subset H(\mathbb{B}^n). \quad (4.2)$$

The following is a special case of Lemma 5.1 in [7].

Lemma 4.1 (see [7]). *Let $s > 0$. Then, $\varphi \in \mathcal{M}(\Lambda_s^p)$ is bounded pointwise by its multiplier norm.*

Corollary 4.2. *Let $s > 0$. Then, we have*

$$\mathcal{M}(\Lambda_s^p) \subsetneq H^\infty. \quad (4.3)$$

Proof. Let $\varphi \in \mathcal{M}(\Lambda_s^p)$. Since $1 \in \Lambda_s^p$, we have $\varphi = \varphi \cdot 1 \in \Lambda_s^p$. Thus,

$$\mathcal{M}(\Lambda_s^p) \subset \Lambda_s^p. \quad (4.4)$$

Hence,

$$\mathcal{M}(\Lambda_s^p) \subset H(\mathbb{B}^n). \quad (4.5)$$

Thus, by Lemma 4.1, we get $\mathcal{M}(\Lambda_s^p) \subset H^\infty$. By (i) of Proposition 2.1, there is a function $\varphi \in A(\mathbb{B}^n) \setminus \Lambda_s^p \subset H^\infty \setminus \Lambda_s^p$. Since $\mathcal{M}(\Lambda_s^p) \subset \Lambda_s^p$, it follows that $\varphi \notin \mathcal{M}(\Lambda_s^p)$. Thus, we get the result. \square

Theorem 4.3. *Let $0 < s \leq n/p$. Then, Λ_s^p is not a multiplicative algebra.*

Proof. If Λ_s^p is multiplicative, then

$$\Lambda_s^p = \mathcal{M}(\Lambda_s^p) \subsetneq H^\infty. \quad (4.6)$$

By (ii) of Proposition 2.1, this is a contradiction. \square

5. Some Sufficient Conditions for a Nonregular Case

Lemma 5.1. *Let $f \in \Lambda_s^p$. Let m be the integer part of s and $\alpha = s - m$. Let $0 < r < 1$. Then, for $q > p$, we have*

$$M_q(r, \mathcal{R}^{m+1} f) \lesssim \|f\|_{\Lambda_s^p} (1-r)^{n(1/q-1/p)-1+\alpha}. \quad (5.1)$$

Proof. We proved this inequality at (2.17) in the proof of (i) of Proposition 2.3. \square

Lemma 5.2. *Let $0 \leq l < k$ and $1/2 < r < 1$. Then, we have*

$$M_p(r, \mathcal{R}^l f) \lesssim \sup_{|z| < 1/2} |f(z)| + \int_0^r (r-t)^{k-l-1} M_p(t, \mathcal{R}^k f) dt. \quad (5.2)$$

Proof. We proved this inequality at (2.14) in the proof of (i) of Proposition 2.3. \square

Theorem 5.3. *Let $1 \leq p < q < \infty$. Then, $\Lambda_{n/p}^q \subset \mathcal{M}(\Lambda_{n/p}^p)$.*

Proof. Let $\varphi \in \Lambda_{n/p}^q$ and $f \in \Lambda_{n/p}^p$. Let m be the integer part of n/p and $\alpha = n/p - m$. It is enough to prove that

$$M_p(r, \mathcal{R}^{m+1}(\varphi f)) \lesssim \frac{1}{(1-r)^{1-\alpha}}, \quad 0 < r < 1. \quad (5.3)$$

We note that

$$\mathcal{R}^{m+1}(\varphi f) = \sum_{j=0}^{m+1} c_j (\mathcal{R}^j \varphi) (\mathcal{R}^{m+1-j} f). \quad (5.4)$$

Thus,

$$M_p(r, \mathcal{R}^{m+1}(\varphi f)) \lesssim \sum_{j=0}^{m+1} M_p(r, (\mathcal{R}^j \varphi) (\mathcal{R}^{m+1-j} f)). \quad (5.5)$$

Let $m = 0$. Then,

$$M_p(r, \mathcal{R}(\varphi f)) \leq M_p(r, (\mathcal{R}\varphi) f) + M_p(r, \varphi (\mathcal{R}f)). \quad (5.6)$$

By Proposition 2.1, we have

$$\begin{aligned} \varphi &\in \Lambda_{n/p}^q \subset \Lambda_{n/p-n/q} \subset H^\infty, \\ f &\in \Lambda_{n/p}^p \subset \bigcap_{\delta > 0} H^\delta. \end{aligned} \quad (5.7)$$

Choose $\delta > 1$ with $1/p = 1/q + 1/\delta$. Then,

$$\begin{aligned} M_p(r, (\mathcal{R}\varphi)f) &\leq M_q(r, \mathcal{R}\varphi) M_\delta(r, f) \\ &\lesssim M_q(r, \mathcal{R}\varphi) \|f\|_{H^\delta} \\ &\lesssim M_q(r, \mathcal{R}\varphi) \|f\|_{\Lambda_{n/p}^p} \\ &\lesssim \|\varphi\|_{\Lambda_{n/p}^q} \|f\|_{\Lambda_{n/p}^p} \frac{1}{(1-r)^{1-\alpha}}. \end{aligned} \quad (5.8)$$

Also,

$$\begin{aligned} M_p(r, \varphi(\mathcal{R}f)) &\lesssim \|\varphi\|_{H^\infty} M_p(r, \mathcal{R}f) \\ &\lesssim \|\varphi\|_{H^\infty} \|f\|_{\Lambda_{n/p}^p} \frac{1}{(1-r)^{1-\alpha}}. \end{aligned} \quad (5.9)$$

Thus,

$$\|\varphi f\|_{\Lambda_{n/p}^p} \lesssim \|\varphi\|_{\Lambda_{n/p}^q} \|f\|_{\Lambda_{n/p}^p}. \quad (5.10)$$

Now, let $m \geq 1$.

If $j = 0$, then, since $\varphi \in H^\infty$ and $f \in \Lambda_{n/p}^p$,

$$\begin{aligned} M_p(r, \varphi \mathcal{R}^{m+1} f) &\lesssim \|\varphi\|_{H^\infty} M_p(r, \mathcal{R}^{m+1} f) \\ &\lesssim \|\varphi\|_{H^\infty} \|f\|_{\Lambda_{n/p}^p} \frac{1}{(1-r)^{1-\alpha}}. \end{aligned} \quad (5.11)$$

If $j = m + 1$, we choose $\delta > p$ such that

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{\delta}. \quad (5.12)$$

Then,

$$M_p(r, (\mathcal{R}^{m+1}\varphi)f) \leq M_q(r, \mathcal{R}^{m+1}\varphi) M_\delta(r, f). \quad (5.13)$$

By (ii) of Proposition 2.3, $\Lambda_{n/p}^p \subset \bigcap_{\delta>0} H^\delta$. By a variant of Hölder’s inequality, we have

$$\begin{aligned} M_p\left(r, \left(\mathcal{R}^{m+1}\varphi\right)f\right) &\lesssim M_q\left(r, \mathcal{R}^{m+1}\varphi\right)\|f\|_{H^\delta} \\ &\lesssim M_q\left(r, \mathcal{R}^{m+1}\varphi\right)\|f\|_{\Lambda_{n/p}^p} \\ &\lesssim \|\varphi\|_{\Lambda_{n/p}^q}\|f\|_{\Lambda_{n/p}^p}\frac{1}{(1-r)^{1-\alpha}}. \end{aligned} \tag{5.14}$$

Now, we consider the case $1 \leq j \leq m$. Since $\Lambda_{n/p}^q$ is decreasing in the parameter q , it is enough to consider for q sufficiently near p so that $j > n/p - n/q$. We choose $\mu_j > 0$ such that

$$\frac{n}{\mu_j} = j - \left(\frac{n}{p} - \frac{n}{q}\right). \tag{5.15}$$

Then,

$$n\left(\frac{1}{q} - \frac{1}{\mu_j}\right) = \frac{n}{p} - j > 0. \tag{5.16}$$

Since $p < q < \mu_j$, we can choose $\delta_j > 0$ such that

$$\frac{1}{p} = \frac{1}{\mu_j} + \frac{1}{\delta_j}, \quad j = 1, 2, \dots, m. \tag{5.17}$$

By a variant of Hölder’s inequality, we have

$$M_p\left(r, \left(\mathcal{R}^j\varphi\right)\left(\mathcal{R}^{m+1-j}f\right)\right) \leq M_{\mu_j}\left(r, \mathcal{R}^j\varphi\right)M_{\delta_j}\left(r, \mathcal{R}^{m+1-j}f\right). \tag{5.18}$$

By Lemmas 5.2 and 5.1,

$$\begin{aligned} M_{\mu_j}\left(r, \mathcal{R}^j\varphi\right) &\lesssim \sup_{|z|<1/2} |\varphi(z)| + \int_0^r (r-t)^{m-j} M_{\mu_j}\left(t, \mathcal{R}^{m+1}\varphi\right) dt \\ &\lesssim \|\varphi\|_{\Lambda_s^q} \int_0^r \frac{(r-t)^{m-j}}{(1-t)^{n(1/q-1/\mu_j)+1-\alpha}} dt. \end{aligned} \tag{5.19}$$

Here,

$$\int_0^r \frac{(r-t)^{m-j}}{(1-t)^{n(1/q-1/\mu_j)+1-\alpha}} dt \lesssim \int_0^r \frac{1}{1-t} dt \lesssim \log\left(\frac{1}{1-r}\right). \tag{5.20}$$

Moreover, by Lemmas 5.2 and 5.1 again, we have

$$\begin{aligned} M_{\delta_j}(r, \mathcal{R}^{m+1-j} f) &\lesssim \sup_{|z| < 1/2} |f(z)| + \int_0^r (r-t)^{j-1} M_{\delta_j}(t, \mathcal{R}^{m+1} f) dt \\ &\lesssim \|f\|_{\Lambda_{n/p}^p} \int_0^r \frac{(r-t)^{j-1}}{(1-t)^{n(1/p-1/\delta_j)+1-\alpha}} dt \\ &\lesssim \|f\|_{\Lambda_{n/p}^p} \frac{1}{(1-r)^{1-\alpha-(n/p-n/q)}}. \end{aligned} \quad (5.21)$$

Thus, it follows that

$$M_p(r, (\mathcal{R}^j \varphi)(\mathcal{R}^{m+1-j} f)) \lesssim \|\varphi\|_{\Lambda_{n/p}^p} \|f\|_{\Lambda_{n/p}^p} \frac{1}{(1-r)^{1-\alpha}} c_r, \quad (5.22)$$

where

$$c_r = \log\left(\frac{1}{1-r}\right) (1-r)^{n/p-n/q} \rightarrow 0 \quad \text{as } r \rightarrow 1. \quad (5.23)$$

Thus,

$$M_p(r, (\mathcal{R}^j \varphi)(\mathcal{R}^{m+1-j} f)) \lesssim \frac{1}{(1-r)^{1-\alpha}}. \quad (5.24)$$

This completes the proof. \square

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