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Research Article

The Ratio of Eigenvalues of the Dirichlet Eigenvalue Problem for Equations with One-Dimensional p-Laplacian

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We establish an estimate for the ratio of eigenvalues of the Dirichlet eigenvalue problem for the equation with one-dimensional p-Laplacian involving a nonnegative unimodal (single-well) potential.

1. Introduction

We consider the eigenvalue problem for the equation

$$-\left(\Phi\left(x'\right)\right)' + c(t)\Phi(x) = \lambda\Phi(x), \quad t \in \left[0, \pi_p\right], \tag{1.1}$$

with the one-dimensional p-Laplacian $(\Phi(x'))' = (|x'|^{p-2}x')'$, $p > 1$, a nonnegative differentiable function c, and the Dirichlet boundary condition

$$x(0) = 0 = x\left(\pi_p\right), \tag{1.2}$$

where $\pi_p := 2\pi/p\sin(\pi/p)$. Equation (1.1) is also frequently called *half-linear* equation, since its solution space is homogeneous but not additive, that is, it has just one half of the properties which characterize linearity. We refer to the books in [1, 2] for the presentation of the essentials of the qualitative theory of differential equations with the one-dimensional

p-Laplacian. Our research is motivated by [3], where the linear case $p = 2$ in (1.1), (1.2) is investigated under the assumption that c is a nonnegative *unimodal function* (an alternative terminology is the *single-well* potential). Concerning the history of the problem of the ratio of eigenvalues in the linear case, we refer to the papers [4–7] and the reference given therein. For estimates of the ratio of eigenvalues of BVP's involving p-Laplacian see, for example, [8, 9]. Similarly to the linear case treated in [3], throughout the paper we suppose that

$$\begin{gathered}\text{there exist } t^* \in [0, \pi_p] \quad \text{such that } c \text{ is} \\ \text{nonincreasing on } [0, t^*] \text{ and nondecreasing on } [t^*, \pi_p].\end{gathered} \tag{1.3}$$

Under this assumption it is shown in [3] that the eigenvalues of

$$-x'' + c(t)x = \lambda x, \qquad x(0) = 0 = x(\pi) \tag{1.4}$$

satisfy

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \quad n, m \in \mathbb{N}, \ n > m. \tag{1.5}$$

Moreover, if the equality holds in (1.5) for a pair of different integers, then $c(t) \equiv 0$ in $[0, \pi]$. In our paper we show that this statement can be extended in a natural way to (1.1), (1.2). We show that (1.5) holds true for the half-linear case if in (1.5) the power 2 by integers m, n is replaced by the power p. As we will see, some arguments used in [3] can be extended directly to (1.1), while others have to be "properly half linearized".

The investigation of BVP (1.1), (1.2) is closely related to the half-linear trigonometric functions and to the half-linear Prüfer transformation. Consider the equation

$$(\Phi(x'))' + (p-1)\Phi(x) = 0 \tag{1.6}$$

and its solution given by the initial condition $x(0) = 0$, $x'(0) = 1$. This solution is a $2\pi_p$ periodic odd function, we denote it by $\sin_p t$, see [2, 10, Section 1.1.2]. If $p = 2$, it reduces to the classical sine function. The derivative $(\sin_p t)' =: \cos_p t$ defines the half-linear cosine function and for these functions the Pythagorian identity can be formulated as the identity

$$|\sin_p t|^p + |\cos_p t|^p \equiv 1. \tag{1.7}$$

We will also use the half-linear tangent and cotangent functions

$$\tan_p t := \frac{\sin_p t}{\cos_p t}, \qquad \cot_p t := \frac{\cos_p t}{\sin_p t}. \tag{1.8}$$

By a direct computation, (1.6) can be written in the form

$$x'' + |x'|^{2-p}\Phi(x) = 0 \tag{1.9}$$

and using (1.9) we have

$$\left(\tan_p t\right)' = 1 - \frac{\sin_p t\left(\cos_p t\right)'}{\cos_p^2 t} = 1 + \left|\tan_p t\right|^p. \tag{1.10}$$

Like for $p = 2$, $\tan_p t > t$ for $t \in (0, \pi_p/2)$ and $\tan_p t < t$ for $t \in (-\pi_p/2, 0)$, which is equivalent to

$$\left|\sin_p\right|^p > t\Phi\left(\sin_p t\right)\cos_p t \tag{1.11}$$

for $t \in (-\pi_p/2, \pi_p/2)$, $t \neq 0$. A similar formula to (1.10) for $\cot_p$ is related to the Riccati equation associated with (1.1). Namely, if $x(t) \neq 0$ is a solution of (1.1) in some interval $I \subset \mathbb{R}$, then the function $w = \Phi(x'/x)$ solves the Riccati equation

$$w' - c(t) + \lambda + (p-1)|w|^q = 0, \qquad q := \frac{p}{p-1}. \tag{1.12}$$

In particular, from (1.6)

$$\left[\Phi\left(\cot_p t\right)\right]' = -(p-1)\left[1 + \left|\cot_p t\right|^p\right] = -\frac{p-1}{\left|\sin_p t\right|^p} < 0, \quad t \neq k\pi_p. \tag{1.13}$$

Let x be a nontrivial solution of (1.1) and consider the half-linear Prüfer transformation (see [2, 10, Section 1.1.3])

$$x(t) = r(t)\sin_p\varphi(t), \qquad x'(t) = r(t)\cos_p\varphi(t). \tag{1.14}$$

Then using the same procedure as in case of the classical linear Prüfer transformation one can verify that φ and r are solutions of

$$\varphi' = \left|\cos_p\varphi\right|^p - \frac{c(t)-\lambda}{p-1}\left|\sin_p\varphi\right|^p, \tag{1.15}$$

$$r' = \Phi\left(\sin_p\varphi\right)\cos_p\varphi\left[1 - \frac{c(t)-\lambda}{p-1}\right]r. \tag{1.16}$$

From (1.15), $\varphi' > 0$ at the points where $x(t) = 0$, that is, where $\varphi(t) = n\pi_p$, $n \in \mathbb{N}$. Also, solutions of (1.15) behave similarly as in the linear case which means that the eigenvalues of (1.1), (1.2) are simple, form an increasing sequence $\lambda_n \to \infty$ and the corresponding eigenfunction x_n has exactly $n-1$ zeros in $(\pi_p, 0)$. Moreover, if $c(t) \equiv 0$, then $\lambda_n = (p-1)n^p$ with the associated eigenfunction $x_n(t) = \sin_p nt$.

2. Preliminary Computations

To prove our main result, we will use the half-linear Prüfer transformation in a modified form. Therefore, we rewrite (1.1) into the form

$$-(\Phi(x'))' + c(t)\Phi(x) = (p-1)z^p\Phi(x) \tag{2.1}$$

with $z > 0$. Note that due to the fact that $c(t) \geq 0$, all eigenvalues of (1.1), (1.2) are positive. Let $x = x(t,z)$ be a nontrivial solution of (2.1) for which $x(0) = 0$. For this solution we introduce the Prüfer angle φ and radius r by

$$x(t) = \frac{r(t)}{z}\sin_p\varphi(t,z), \qquad x'(t) = r(t)\cos_p\varphi(t,z). \tag{2.2}$$

Differentiating the first equation and comparing it with the second one we obtain

$$\frac{r'}{z}\sin_p\varphi + \frac{r}{z}\varphi'\cos_p\varphi = r\cos_p\varphi. \tag{2.3}$$

Equation (2.1) can be written as

$$-x'' + \frac{c(t)}{p-1}\left|x'\right|^{2-p}\Phi(x) = z^p\left|x'\right|^{2-p}\Phi(x) \tag{2.4}$$

and similarly one can rewrite (1.6) as $(\sin_p t)'' = -|\cos_p t|^{2-p}\Phi(\sin_p t)$. Differentiating the second equation in (2.2) and substituting into (2.4) we have

$$r'\cos_p\varphi - r\left|\cos_p\varphi\right|^{2-p}\Phi(\sin_p\varphi)\varphi' = \frac{c(t)-(p-1)z^{p-1}}{(p-1)z^p}r\left|\cos_p\varphi\right|^{2-p}\Phi(\sin_p\varphi). \tag{2.5}$$

Multiplying (2.3) by $z\cos_p t$, (2.5) by $-\sin_p\varphi$, adding the resulting equations and dividing them by $\cos_p^2\varphi$ we get

$$\varphi' = z - \frac{c(t)}{(p-1)z^{p-1}}\left|\sin_p\varphi\right|^p. \tag{2.6}$$

By a similar computation, we get the equation for the radius r

$$\frac{r'}{r} = \frac{c(t)}{(p-1)z^{p-1}}\Phi(\sin_p\varphi)\cos_p\varphi. \tag{2.7}$$

Concerning the dependence of $\varphi = \varphi(t,z)$ on the eigenvalue parameter z, we have from (2.6)

$$\frac{d}{dz}\varphi'(t,z) =: \dot{\varphi}' = 1 + \frac{c(t)}{z^p}\left|\sin_p\varphi\right|^p - \frac{pc(t)}{(p-1)z^{p-1}}\dot{\varphi}\,\Phi(\sin_p\varphi)\cos_p\varphi. \tag{2.8}$$

Sometimes, we will skip the argument z of r and φ when its value is not important or it is clear what value we mean. The last equation can be regarded as a first-order linear (nonhomogeneous) differential equation for $\dot{\varphi}$. Multiplying this equation by the integration factor

$$\exp\left\{p\int_0^t \frac{c(s)}{(p-1)z^{p-1}}\Phi(\sin_p\varphi(s))\cos_p\varphi(s)ds\right\} = \exp\left\{p\int_0^t \frac{r'(s)}{r(s)}\,ds\right\} = \frac{r^p(t)}{r^p(0)}, \tag{2.9}$$

we have (since $\varphi(0,z)=0$ for $z>0$ and hence $\dot{\varphi}(0,z)=0$)

$$\dot{\varphi}(t,z) = \frac{1}{r^p(t)}\int_0^t r^p(s)\left(1+\frac{c(s)}{z^p}|\sin_p\varphi(s,z)|^p\right)ds. \tag{2.10}$$

The dependence of the function

$$\varsigma(t,z) := \frac{\varphi(t,z)}{z} \tag{2.11}$$

on z plays a crucial role in the proof of our main statement. Applying (2.10) and (2.6), we have

$$\begin{aligned}
\dot{\varsigma}(t,z) &= \frac{\dot{\varphi}(t,z)}{z} - \frac{\varphi(t,z)}{z^2}\\
&= \frac{1}{r^p(t)z^2}\left\{\int_0^t\left(z+\frac{c(s)}{z^{p-1}}|\sin_p\varphi(s)|^p\right)r^p(s)\,ds - r^p(t)\varphi(t)\right\}\\
&= \frac{1}{r^p(t)z^2}\left\{\int_0^t[z+(p-1)(z-\varphi'(s))]r^p(s)ds\right.\\
&\qquad\left.-\int_0^t\left(pr^{p-1}(s)r'(s)\varphi(s)+r^p(s)\varphi'(s)\right)ds\right\}\\
&= \frac{p}{r^p(t)z^2}\int_0^t\left[\frac{c(s)}{(p-1)z^{p-1}}|\sin_p\varphi(s)|^p\right.\\
&\qquad\left.-\frac{c(s)\varphi(s)}{(p-1)z^{p-1}}\Phi(\sin_p\varphi(s))\cos_p\varphi(s)\right]r^p(s)ds.
\end{aligned} \tag{2.12}$$

3. Ratio of Eigenvalues

In the previous section we have prepared computations which we now use in the proof of our main result which reads as follows.

Theorem 3.1. *Suppose that c is a nonnegative differentiable function such that* (1.3) *holds. Then one has for eigenvalues of* (1.1), (1.2)

$$\frac{\lambda_n}{\lambda_m} \le \frac{n^p}{m^p}, \quad n > m, \ n, m \in \mathbb{N}. \tag{3.1}$$

If for two different integers n, m the equality holds, then $c(t) \equiv 0$ on $[0, \pi_p]$.

Proof. Let $x = x(t, z)$ be a nontrivial solution of (2.1) for which $x(0, z) = 0$, $x'(0, z) > 0$ and let $r = r(t, z)$, $\varphi = \varphi(t, z)$ be its Prüfer radius and angle given by (2.2) with $\varphi(0, z) = 0$. A value $z_n > 0$ corresponds to an eigenvalue $\lambda_n = (p-1)z_n^p$ of (1.1), (1.2) if and only if $\sin_p \varphi(\pi_p, z_n) = 0$. As noted below (1.16), it follows from (2.2) that $\varphi'(t) > 0$ when $\varphi(t) = k\pi_p$, $k \in \mathbb{N}$. Using the same argument as in the linear case (see, e.g., [6]) $\varphi(\pi_p, z_n) = n\pi_p$ holds.

Let $\psi(t, z)$ be given by (2.11). Suppose that we have already proved that $\dot{\psi}(t^*, z) \ge 0$ for $z \ge 0$ and when the equality $\dot{\psi}(t^*, z) = 0$ happens for some $z > 0$, then $c(t) \equiv 0$ on $[0, t^*]$. Like in [3], we investigate (2.1) on the interval $[t^*, \pi_p]$ using the reflection argument. Let $\tilde{c}(t) = c(\pi_p - t)$. Then, for $\tilde{c}$ the value $\pi_p - t^*$ plays the same role as t^* for c, in particular, the function $\tilde{c}$ satisfies (1.3) when t^* is replaced by $\pi_p - t^*$. Further, let $x = x(t, z_n)$ be the eigenfunction of (2.1), (1.2) corresponding to the eigenvalue $\lambda_n = (p-1)z_n^p$, that is, $x(0) = 0 = x(\pi_p)$. Define

$$y(t, z_n) := (-1)^{n+1} x(\pi_p - t, z_n), \qquad \theta(t, z_n) := n\pi_p - \varphi(\pi_p - t, z_n), \tag{3.2}$$

where φ is the Prüfer angle of x for which $\varphi(0) = 0$. Then $y(0, z_n) = x(\pi_p, z_n) = 0$ and $y(\pi_p, z_n) = x(0, z_n) = 0$, hence y is an eigenfunction of (2.1), (1.2) when c is replaced by $\tilde{c}$. Moreover, we have $\theta(0, z_n) = 0$, $\theta(\pi_p, z_n) = n\pi_p$ and

$$\begin{aligned} \sin_p \theta(t, z_n) &= \sin_p\left(n\pi_p - \varphi(\pi_p - t, z_n)\right) = -\sin_p\left(\varphi(\pi_p - t, z_n) - n\pi_p\right) \\ &= (-1)^{n+1} \sin_p \varphi(\pi_p - t, z_n), \end{aligned} \tag{3.3}$$

and hence

$$\begin{aligned} y(t, z_n) &= (-1)^{n+1} x(\pi_p - t, z_n) = (-1)^{n+1} \frac{r(\pi_p - t, z_n)}{z_n} \sin_p \varphi(\pi_p - t, z_n) \\ &= \frac{r(\pi_p - t, z_n)}{z_n} \sin_p \theta(t, z_n). \end{aligned} \tag{3.4}$$

Similarly $y'(t, z_n) = r(\pi_p - t, z_n)\cos_p \theta(t, z_n)$, that is, θ is the Prüfer angle corresponding to y. Denote $\omega(t, z) = \theta(t, z)/z$. The function ω plays the same role for (2.1), (1.2) with c replaced by $\tilde{c}$, as ψ for the original eigenvalue problem. Hence let us also suppose that we have proved that $\dot{\omega}(\pi_p - t^*, z) \ge 0$ with the equality only if $\tilde{c}(t) \equiv 0$ on $[0, \pi_p - t^*]$. Now, consider the function

$$F(z) := \psi(t^*, z) + \omega(\pi_p - t^*, z). \tag{3.5}$$

Under the monotonicity assumptions on ψ and ω, the function F is nondecreasing and when $F(\tilde{z}) = F(\hat{z})$ for two different values $\tilde{z}, \hat{z} > 0$, then $c(t) \equiv 0$ on $[0, t^*]$ and $\tilde{c}(t) = c(\pi_p - t) \equiv 0$ on

$[0, \pi_p - t^*]$. Let $m < n$ and $z_m < z_n$ be the values of the eigenvalue parameter corresponding to the eigenvalues $\lambda_m = (p-1)z_m^p$, $\lambda_n = (p-1)z_n^p$. Then

$$F(z_m) = \psi(t^*, z_m) + \omega(\pi_p - t^*, z_m) = \frac{\varphi(t^*, z_m)}{z_m} + \frac{\theta(\pi_p - t^*, z_m)}{z_m}$$
$$= \frac{1}{z_m}\left[\varphi(t^*, z_m) + m\pi_p - \varphi(t^*, z_m)\right] = \frac{m\pi_p}{z_m}. \tag{3.6}$$

Similarly, $F(z_n) = n\pi_p / z_n$. Consequently,

$$\frac{m\pi_p}{z_m} = F(z_m) \leq F(z_n) = \frac{n\pi_p}{z_n} \tag{3.7}$$

and therefore

$$\frac{\lambda_n}{\lambda_m} \leq \frac{(p-1)n^p}{(p-1)m^p} = \frac{n^p}{m^p}. \tag{3.8}$$

In case of the equality in (3.8), we have $c(t) \equiv 0$ on $[0, t^*]$ and $\widetilde{c}(t) \equiv 0$ on $[0, \pi_p - t^*]$, altogether $c(t) \equiv 0$ on $[0, \pi_p]$.

Now let us turn our attention to the monotonicity property of $\psi(t^*, z) = \varphi(t^*, z)/z$. First consider the case $\varphi(t^*, z) < \pi_p/2$. In this case the inequality $\dot{\psi}(t^*, z) \geq 0$ (with equality implying $c(t) \equiv 0$ on $[0, t^*]$) follows immediately from (2.12) since the integrand in this expression is nonnegative by (1.11). So suppose that $\varphi(t^*, z) = \pi_p/2 + k\pi_p + \alpha$ for some $\alpha \in [0, \pi_p)$ and some nonnegative integer k. Then we need some preliminary computations. Suppose that we already know that the function $\varphi(t, z)$ is strictly increasing with respect to t for $t \in [0, t^*]$. In this case we may split the integral below as

$$\dot{\psi}(t^*, z) = \frac{p}{r^p(t^*)z^2}\int_0^{t^*} \frac{c(s)r^p(s)\Phi(\sin_p\varphi(s))\cos_p\varphi(s)}{(p-1)z^{p-1}}\left[\tan_p\varphi(s) - \varphi(s)\right]ds$$
$$= \frac{p}{r^p(t^*)z^2}\int_0^{\varphi^{-1}(k\pi_p + \pi_p/2 + \alpha)} \frac{c(s)r^p(s)\Phi(\sin_p\varphi(s))\cos_p\varphi(s)}{(p-1)z^{p-1}}\left[\tan_p\varphi(s) - \varphi(s)\right]ds \tag{3.9}$$
$$= \int_0^{\varphi^{-1}(\pi_p/2)}[\cdot] + \sum_{j=1}^{k}\int_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)}[\cdot] + \int_{\varphi^{-1}(k\pi_p + \pi_p/2)}^{\varphi^{-1}(k\pi_p + \pi_p/2 + \alpha)}[\cdot],$$

where we have denoted the integrand in (3.9) by $[\cdot]$. As soon as we show that each integral is nonnegative and equals zero only if $c(t) \equiv 0$ on the corresponding interval, the monotonicity of $\psi(t^*, z)$ will be proved.

First we will show the strict monotonicity of $\varphi(t,z)$ with respect to t. Fix $z > 0$ and suppose, by contradiction, that $\varphi'(\bar{t},z) \leq 0$ for some $\bar{t} \in (0,t^*]$. This implies by (2.6) that

$$(p-1)z^p \leq c\left(\bar{t}\right)\left|\sin_p\varphi\left(\bar{t}\right)\right|^p \leq c\left(\bar{t}\right) \leq c(t) \tag{3.10}$$

for $t \in [0,\bar{t}]$ using (1.3) in the last inequality. Hence

$$(\Phi(x'))' = (p-1)x''|x'|^{p-2} = \left[c(t) - (p-1)z^p\right]\Phi(x), \tag{3.11}$$

that is, x is convex and strictly increasing for $t \in [0,\bar{t}]$, that is, $x'(t) > 0$ and hence by (2.2) $\varphi(t,z) < \pi_p/2$. By (2.6) also $\varphi'(0,z) > 0$, and $\varphi'(\bar{t},z) \leq 0$ implies the existence of $t_1 \in (0,\bar{t}]$ such that $\varphi'(t_1,z) = 0$ and $\varphi'(t,z) > 0$ for $t \in [0,t_1)$. Fix any $t_2 \in (0,t_1)$ and consider the function $w = \Phi(z)\Phi(\cot_p\varphi) = \Phi(x'/x)$. This function is a solution of Riccati equation (1.12) and from (1.13)

$$\begin{aligned} w' &= \Phi(z)\left(\Phi(\cot_p\varphi(t))\right)' = -\Phi(z)\frac{(p-1)\varphi'(t)}{\left|\sin_p\varphi(t)\right|^p} \\ &= c(t) - (p-1)z^p - (p-1)z^p\left|\cot_p\varphi(t)\right|^p. \end{aligned} \tag{3.12}$$

Hence $c(t)/(p-1) - z^p - z^p|\cot_p\varphi(t)|^p < 0$ for $t \in (t_2,t_1)$, which means that

$$z\cot_p\varphi(t) = \Phi^{-1}(w(t)) > \left(\frac{c(t)}{p-1} - z^p\right)^{1/p} \tag{3.13}$$

in (t_2,t_1) and equality happens for $t = t_1$, that is

$$w(t_1) = \left(\frac{c(t_1)}{p-1} - z^p\right)^{1/q}. \tag{3.14}$$

Recall that $q = p/(p-1)$ is the conjugate pair of p and $\Phi^{-1}(w) = |w|^{q-2}w$ is the inverse function of Φ. Let $t_3 \in (t_2,t_1)$ and denote for a moment $\hat{c}(t) = c(t)/p-1$. We have (suppressing the integration argument)

$$\begin{aligned} \int_{t_2}^{t_3}\frac{\left[w-(\hat{c}-z^p)^{1/q}\right]'}{w-(\hat{c}-z^p)^{1/q}} &= \int_{t_2}^{t_3}\frac{w' - \left[(\hat{c}-z^p)^{1/q}\right]'}{w-(\hat{c}-z^p)^{1/q}} \\ &= (p-1)\int_{t_2}^{t_3}\frac{\hat{c}-z^p-|w|^q}{w-(\hat{c}-z^p)^{1/q}} - \int_{t_2}^{t_3}\frac{\hat{c}'}{q(\hat{c}-z^p)^{1/p}\left[w-(\hat{c}-z^p)^{1/q}\right]} \\ &\geq (p-1)\int_{t_2}^{t_3}\frac{\hat{c}-z^p-|w|^q}{w-(\hat{c}-z^p)^{1/q}}. \end{aligned} \tag{3.15}$$

In the last inequality we have used that $\widehat{c}'(t) \le 0$ for $t \in [t_2, t_3] \subset [0, t^*]$ by (1.3). Denote $A := (\widehat{c} - z^p)^{1/q}$ and consider the function

$$G(t, w) = \frac{A^q - |w|^q}{w - A}. \tag{3.16}$$

This function is bounded when its argument is bounded as it can be verified by computing its limit for $w \to A$. But $w = \Phi(z)\Phi(\cot_p \varphi(t))$ is bounded since $0 < \varphi(t) < \pi_p/2$ for $t \in [t_2, t_1]$. Consequently, the last integral is bounded below as $t_3 \to t_1-$, while the integral in (3.15) equals

$$\left[\log\left(w(t) - \left(\frac{c(t)}{p-1} - z^p\right)^{1/q}\right)\right]_{t_2}^{t_3} \longrightarrow -\infty \quad \text{as } t_3 \longrightarrow t_1- \tag{3.17}$$

since at t_1 (3.14) holds. This contradiction shows that $\varphi'(t, z) > 0$ for $t \in [0, t^*]$ and $z > 0$.

Now we will deal with integrals in (3.9). The first one over the interval $[0, \varphi^{-1}(\pi_p/2)]$ is nonnegative since its integrand is nonnegative in this interval by (1.11) and equals 0 only if $c(t) \equiv 0$. Concerning the integrals under the summation sign, first observe that the value of the functions $|\sin_p \varphi|^p$ and $\Phi(\sin_p \varphi)\cos_p \varphi$ does not change if we replace φ by $\varphi - j\pi_p$ with any integer j. Hence, using the substitution $\varphi \mapsto \varphi - j\pi_p$ which moves $\varphi \in [j\pi_p - \pi_p/2, j\pi_p + \pi_p/2]$ to $[-\pi_p/2, \pi_p/2]$ (where (1.11) holds) we have

$$\begin{aligned}
&\int_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)} \frac{c(t) r^p(t)}{(p-1) z^{p-1}} \left[|\sin_p \varphi(t)|^p - \varphi(t)\Phi(\sin_p \varphi(t))\cos_p \varphi(t)\right] dt \\
&= \int_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)} \frac{c(t) r^p(t)}{(p-1) z^{p-1}} \\
&\quad \times \left[|\sin_p(\varphi(t) - j\pi_p)|^p - (\varphi(t) - j\pi_p)\Phi(\sin_p(\varphi(t) - j\pi_p))\cos_p(\varphi(t) - j\pi_p)\right] dt \\
&\ge -j\pi_p \int_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)} \frac{c(t) r^p(t)}{(p-1) z^{p-1}} \Phi(\sin_p \varphi(t))\cos_p \varphi(t) dt \\
&= -j\pi_p \int_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)} r'(t) r^{p-1}(t) dt = -\frac{j\pi_p}{p} \left[r^p(t)\right]_{\varphi^{-1}(j\pi_p - \pi_p/2)}^{\varphi^{-1}(j\pi_p + \pi_p/2)}.
\end{aligned} \tag{3.18}$$

Here we have again used (1.11) since this inequality can be applied in view of the transformation $\varphi \mapsto \varphi - j\pi_p$. The last result leads to the investigation of the monotonicity properties (with respect to t) of the radius $r = r(t, z)$. We will use the fact that the function $\log r(t)$ has the same monotonicity as $r(t)$. From (2.7) it immediately follows that r is increasing for $\varphi(t) \in (j\pi_p, j\pi_p + \pi_p/2)$ while it is decreasing for $\varphi(t) \in (j\pi_p - \pi_p/2, j\pi_p)$. Taking the integral of (2.7) in view of (2.6) and substituting $\varphi(t) = s$, one gets

$$\begin{aligned}
\int_{\varphi^{-1}(j\pi_p-\pi_p/2)}^{\varphi^{-1}(j\pi_p+\pi_p/2)} \frac{r'(t)}{r(t)}\,dt &= \int_{\varphi^{-1}(j\pi_p-\pi_p/2)}^{\varphi^{-1}(j\pi_p+\pi_p/2)} \frac{c(t)}{(p-1)z^{p-1}}\Phi(\sin_p\varphi(t))\cos_p\varphi(t)dt \\
&= \int_{\varphi^{-1}(j\pi_p-\pi_p/2)}^{\varphi^{-1}(j\pi_p+\pi_p/2)} \frac{\varphi'(t)c(t)\Phi(\sin_p\varphi(t))\cos_p\varphi(t)}{(p-1)z^p - c(t)|\sin\varphi(t)|^p}dt \\
&= \int_{j\pi_p-\pi_p/2}^{j\pi_p+\pi_p/2} \frac{c(\varphi^{-1}(s))\Phi(\sin_p s)\cos_p s}{(p-1)z^p - c(\varphi^{-1}(s))|\sin_p s|^p}ds.
\end{aligned} \tag{3.19}$$

The function $\Phi(\sin_p s)\cos_p s$ is negative between $j\pi_p - \pi_p/2$ and $j\pi_p$, while the denominator of the last fraction is positive by (3.10). Consequently, if we replace c by its minimum in this interval, we obtain

$$\begin{aligned}
\int_{\varphi^{-1}(j\pi_p-\pi_p/2)}^{\varphi^{-1}(j\pi_p)} \frac{r'(t)}{r(t)}\,dt &\le \int_{j\pi_p-\pi_p/2}^{j\pi_p} \frac{c(\varphi^{-1}(j\pi_p))\Phi(\sin_p s)\cos_p s}{(p-1)z^p - c(\varphi^{-1}(j\pi_p)|\sin_p s|^p}ds \\
&= -\Big[\log((p-1)z^p \; - c\Big(\varphi^{-1}(j\pi_p)\Big)|\sin_p s|^p\Big]_{j\pi_p-\pi_p/2}^{j\pi_p}.
\end{aligned} \tag{3.20}$$

Using the same argument in the interval $(j\pi_p, j\pi_p + \pi_p/2)$ where the function $\Phi(\sin_p s)\cos_p s$ is positive, so if we replace c by its maximum, we have

$$\int_{\varphi^{-1}(j\pi_p)}^{\varphi^{-1}(j\pi_p+\pi_p/2)} \frac{r'(t)}{r(t)}dt \le -\Big[\log\Big((p-1)z^p - c\Big(\varphi^{-1}(j\pi_p)\Big)|\sin_p s|^p\Big)\Big]_{j\pi_p}^{j\pi_p+\pi_p/2}. \tag{3.21}$$

Summing the last two results

$$\int_{\varphi^{-1}(j\pi_p-\pi_p/2)}^{\varphi^{-1}(j\pi_p+\pi_p/2)} \frac{r'(t)}{r(t)}dt \le -\Big[\log\Big((p-1)z^p - c\Big(\varphi^{-1}(j\pi_p)\Big)|\sin_p s|^p\Big)\Big]_{j\pi_p-\pi_p/2}^{j\pi_p+\pi_p/2} = 0. \tag{3.22}$$

Consequently, we have

$$r\left(\varphi^{-1}\left(j\pi_p - \frac{\pi_p}{2}\right)\right) \ge r\left(\varphi^{-1}\left(j\pi_p + \frac{\pi_p}{2}\right)\right) \tag{3.23}$$

and this inequality shows that each integral in the sum in (3.9) is nonnegative and equals 0 only if $c(t) \equiv 0$. We handle the last integral in (3.9) over $[k\pi_p + \pi_p/2, k\pi_p + \pi_p/2 + \alpha]$ in a similar way (suppressing the integration variable t)

$$\begin{aligned}
&\int_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} \frac{cr^p}{(p-1)z^{p-1}} \left[|\sin_p\varphi|^p - \varphi\Phi(\sin_p\varphi)\cos_p\varphi\right] \\
&\quad = \int_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} \frac{cr^p}{(p-1)z^{p-1}} \\
&\qquad \times \left[|\sin_p(\varphi-(k+1)\pi_p)|^p - (\varphi-(k+1)\pi_p)\Phi(\sin_p(\varphi-(k+1)\pi_p))\cos_p(\varphi-(k+1)\pi_p)\right] \\
&\qquad - (k+1)\pi_p \int_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} \frac{cr^p}{(p-1)z^{p-1}} \Phi(\sin_p\varphi)\cos_p\varphi \\
&\quad \geq -(k+1)\pi_p \int_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} \frac{cr^p}{(p-1)z^{p-1}} \Phi(\sin_p\varphi)\cos\varphi \\
&\quad = -(k+1)\pi_p \int_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} r^{p-1}r' = -(k+1)\frac{\pi_p}{p}\left[r^p\right]_{\varphi^{-1}(k\pi_p+\pi_p/2)}^{\varphi^{-1}(k\pi_p+\pi_p/2+\alpha)} \geq 0
\end{aligned} \tag{3.24}$$

because of the monotonicity property of r and since $\varphi - (k+1)\pi_p \in [-\pi_p/2, \pi_p/2]$, so the integrand containing this argument is nonnegative by (1.11).

Therefore, each integral in (3.9) is nonnegative and we have proved the required statement concerning monotonicity (with respect to z) of the function $\varphi(t^*, z)$. Finally, since the function $\omega(\pi_p - t^*, z)$ plays the same role as $\varphi(t^*, z)$, the above used arguments prove also monotonicity with respect to z of ω. This means that the function F given in (3.5) is monotone and the proof is complete. □

Remark 3.2. The assumption on the differentiability of c has only been used in (3.15). When we take the integral

$$\int_{t_2}^{t_3} \frac{d\left[w - (\widehat{c} - z^p)^{1/q}\right]}{w - (\widehat{c} - z^p)^{1/q}}, \tag{3.25}$$

in (3.15) in a more general sense than in the proof of Theorem 3.1, then the assumption of the smoothness of c can be considerably weakened.

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