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Research Article

AQCQ-Functional Equation in Non-Archimedean Normed Spaces

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We prove the generalized Hyers-Ulam stability of generalized mixed type of quartic, cubic, quadratic and additive functional equation in non-Archimedean spaces.

1. Introduction and Preliminaries

In 1897, Hensel [1] has introduced a normed space which does not have the Archimedean property.

During the last three decades, theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings, and superstrings [2]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition [3–10].

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have that

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq \max\{|a|, |b|\}$.

Condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, that is, there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$,

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,

(NA3) the strong triangle inequality (ultrametric), namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X). \tag{1.1}$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

It follows from (NA3) that

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\} \quad (m > l), \tag{1.2}$$

therefore a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1940 and affirmatively solved by Hyers [12]. Perhaps Aoki was the first author who has generalized the theorem of Hyers (see [13]).

Theorem 1.1 (Aoki [13]). *If a mapping $f : X \to Y$ between two Banach spaces satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y) \tag{1.3}$$

for all $x, y \in X$, where $\varphi(x, y) = K(\|x\|^p + \|y\|^p)$ with $(K \geq 0, 0 \leq p < 1)$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{K}{1 - 2^{p-1}} \|x\|^p (x \in X). \tag{1.4}$$

Moreover, Bourgin [14], Rassias [15], and Găvruta [16] have considered the stability problem with unbounded Cauchy differences (see also [17]). On the other hand, Rassias [18–23] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruta [24]. This stability phenomenon is called the Ulam-Găvruta-Rassias stability (see also [25]).

Theorem 1.2 (Rassias [18]). *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.5}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r \tag{1.6}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Very recently, Rassias [26] in inequality (1.5) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{\|x\|^p\|y\|^p + (\|x\|^{2p} + \|y\|^{2p})\}$.

For more details about the results concerning such problems and mixed product-sum stability (Rassias Stability) the reader is referred to [27–42].

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.7}$$

is related to a symmetric biadditive function [43, 44]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.7) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B_1 such that $f(x) = B_1(x, x)$ for all x. The biadditive function B_1 is given by

$$B_1(x, y) = \frac{1}{4}(f(x+y) - f(x-y)). \tag{1.8}$$

The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [45]. In [46], Czerwik proved the Hyers-Ulam-Rassias stability of (1.7). Later, Jung [47] has generalized the results obtained by Skof and Czerwik.

Jun and Kim [48] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.9}$$

and they established the general solution and the generalized Hyers-Ulam stability for the functional equation (1.9). They proved that a function f between two real vector spaces X and Y is a solution of (1.9) if and only if there exists a unique function $C : X \times X \times X \to Y$

such that $f(x) = C(x,x,x)$ for all $x \in X$; moreover, C is symmetric for each fixed variable and is additive for fixed two variables. The function C is given by

$$C(x,y,z) = \frac{1}{24}(f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)) \tag{1.10}$$

for all $x, y, z \in X$ (see also [47, 49–55]).

Lee et al. [56] considered the following functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.11}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.11) if and only if there exists a unique symmetric biquadratic function $B_2 : X \times X \to Y$ such that $f(x) = B_2(x,x)$ for all x. The biquadratic function B_2 is given by

$$B_2(x,y) = \frac{1}{12}(f(x+y) + f(x-y) - 2f(x) - 2f(y)). \tag{1.12}$$

Obviously, the function $f(x) = cx^4$ satisfies the functional equation (1.11), which is called the quartic functional equation.

Eshaghi Gordji and Khodaei [49] have established the general solution and investigated the Hyers-Ulam-Rassias stability for a mixed type of cubic, quadratic, and additive functional equation (briefly, AQC-functional equation) with $f(0) = 0$,

$$f(x+ky) + f(x-ky) = k^2 f(x+y) + k^2 f(x-y) + 2\left(1-k^2\right)f(x) \tag{1.13}$$

in quasi-Banach spaces, where k is nonzero integer with $k \notin \{0, \pm 1\}$. Obviously, the function $f(x) = ax + bx^2 + cx^3$ is a solution of the functional equation (1.13). Interesting new results concerning mixed functional equations have recently been obtained by Najati et al. [57–59] and Jun and Kim [60, 61] as well as for the fuzzy stability of a mixed type of additive and quadratic functional equation by Park [62]. The stability of generalized mixed type functional equations of the form

$$f(x+ky) + f(x-ky) = k^2(f(x+y) + f(x-y)) + \left(k^2-1\right)\left(\frac{k^2}{12}\left(\tilde{f}(2y) - 4\tilde{f}(y)\right) - 2f(x)\right) \tag{1.14}$$

for fixed integers $k \notin \{0, \pm 1\}$, where $\tilde{f}(y) := f(y) + f(-y)$, in quasi-Banach spaces was investigated by Eshaghi Gordji et al. [63]. The mixed type functional equation (1.14) is additive, quadratic, cubic, and quartic (briefly, AQCQ-functional equation).

This paper is organized as follows. In Section 2, we prove the generalized Hyers-Ulam stability of the functional equation (1.14) in non-Archimedean normed spaces, for an odd case. The generalized Hyers-Ulam stability of the functional equation (1.14) in non-Archimedean normed spaces, for an even case, is discussed in Section 3. Finally, in Section 4, we show the generalized Hyers-Ulam stability of the AQCQ-functional equation (1.14) in non-Archimedean normed spaces.

Throughout this paper, assume that G is an additive group, X is a complete non-Archimedean spaces, and V_1, V_2 are vector spaces. Before taking up the main subject, given $f : G \times G \to X$, we define the difference operator

$$\begin{aligned} Df(x,y) &= f(x+ky) + f(x-ky) - k^2 f(x+y) - k^2 f(x-y) \\ &\quad - \left(k^2-1\right)\left(\frac{k^2}{12}\left(\tilde{f}(2y) - 4\tilde{f}(y)\right) - 2f(x)\right), \end{aligned} \tag{1.15}$$

where $\tilde{f}(y) := f(y) + f(-y)$ and $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ for all $x, y \in G$.

2. Stability of the AQCQ-Functional Equation (1.14): For an Odd Case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete non-Archimedean spaces: an odd case.

Lemma 2.1 (see [49, 59, 63]). *If an odd function $f : V_1 \to V_2$ satisfies (1.14), then the function $g_1 : V_1 \to V_2$ defined by $g_1(x) = f(2x) - 8f(x)$ is additive.*

Theorem 2.2. *Let $\ell \in \{1, -1\}$ be fixed, and let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^{n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) = 0 = \lim_{n\to\infty} |2|^{n\ell} \tilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) \tag{2.1}$$

for all $x, y \in G$. Suppose that an odd function $f : G \to X$ satisfies the inequality

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.2}$$

for all $x, y \in G$. Then there exists a unique additive function $A : G \to X$ such that

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{1}{|2|}\varphi_a(x) \tag{2.3}$$

for all $x \in G$, *where*

$$\varphi_a(x) = \lim_{n\to\infty} \max\left\{ |2|^{\ell(j+((1+\ell)/2))} \widetilde{\varphi}\left(\frac{x}{2^{\ell(j+((1+\ell)/2))}}\right) : 0 \le j < n \right\}, \tag{2.4}$$

$$\widetilde{\varphi}(x) := \frac{1}{|k^2(k^2-1)|} \max\{|2|\varphi_1(x), \varphi_2(x)\}, \tag{2.5}$$

$$\begin{aligned}\varphi_1(x) := \max\Big\{ &\left|2\left(k^2-1\right)\right|\varphi(x,x),\ \max\left\{\left|k^2\right|\varphi(2x,x), \varphi(x,2x)\right\}, \\ &\max\{\varphi((k+1)x,x), \varphi((k-1)x,x)\}\Big\}\end{aligned} \tag{2.6}$$

$$\begin{aligned}\varphi_2(x) := \max\Big\{ &\left\{\varphi(x,x), \left|k^2\right|\varphi(2x,2x)\right\},\ \max\left\{\left|2\left(k^2-1\right)\right|\varphi(x,2x), \varphi(x,3x)\right\}, \\ &\max\{\varphi((2k+1)x,x), \varphi((2k-1)x,x)\}\Big\},\end{aligned} \tag{2.7}$$

for all $x \in G$.

Proof. Let $\ell = 1$. It follows from (2.2) and using oddness of f that

$$\left\| f(ky+x) - f(ky-x) - k^2 f(x+y) - k^2 f(x-y) + 2\left(k^2-1\right) f(x) \right\| \le \varphi(x,y) \tag{2.8}$$

for all $x, y \in G$. Putting $y = x$ in (2.8), we have

$$\left\| f((k+1)x) - f((k-1)x) - k^2 f(2x) + 2\left(k^2-1\right) f(x) \right\| \le \varphi(x,x) \tag{2.9}$$

for all $x \in G$. It follows from (2.9) that

$$\left\| f(2(k+1)x) - f(2(k-1)x) - k^2 f(4x) + 2\left(k^2-1\right) f(2x) \right\| \le \varphi(2x,2x) \tag{2.10}$$

for all $x \in G$. Replacing x and y by $2x$ and x in (2.8), respectively, we get

$$\left\| f((k+2)x) - f((k-2)x) - k^2 f(3x) - k^2 f(x) + 2\left(k^2-1\right) f(2x) \right\| \le \varphi(2x,x) \tag{2.11}$$

for all $x \in G$. Setting $y = 2x$ in (2.8), one obtains

$$\left\| f((2k+1)x) - f((2k-1)x) - k^2 f(3x) - k^2 f(-x) + 2\left(k^2-1\right) f(x) \right\| \le \varphi(x,2x) \tag{2.12}$$

for all $x \in G$. Putting $y = 3x$ in (2.8), we obtain

$$\left\| f((3k+1)x) - f((3k-1)x) - k^2 f(4x) - k^2 f(-2x) + 2\left(k^2-1\right) f(x) \right\| \le \varphi(x,3x) \tag{2.13}$$

for all $x \in G$. Replacing x and y by $(k+1)x$ and x in (2.8), respectively, we get

$$\left\| f((2k+1)x) - f(-x) - k^2 f((k+2)x) - k^2 f(kx) + 2\left(k^2-1\right) f((k+1)x) \right\| \leq \varphi((k+1)x, x) \tag{2.14}$$

for all $x \in G$. Replacing x and y by $(k-1)x$ and x in (2.8), respectively, one gets

$$\left\| f((2k-1)x) - f(x) - k^2 f((k-2)x) - k^2 f(kx) + 2\left(k^2-1\right) f((k-1)x) \right\| \leq \varphi((k-1)x, x) \tag{2.15}$$

for all $x \in G$. Replacing x and y by $(2k+1)x$ and x in (2.8), respectively, we obtain

$$\begin{aligned} &\left\| f((3k+1)x) - f(-(k+1)x) - k^2 f(2(k+1)x) - k^2 f(2kx) + 2\left(k^2-1\right) f((2k+1)x) \right\| \\ &\quad \leq \varphi((2k+1)x, x) \end{aligned} \tag{2.16}$$

for all $x \in G$. Replacing x and y by $(2k-1)x$ and x in (2.8), respectively, we have

$$\begin{aligned} &\left\| f((3k-1)x) - f(-(k-1)x) - k^2 f(2(k-1)x) - k^2 f(2kx) + 2\left(k^2-1\right) f((2k-1)x) \right\| \\ &\quad \leq \varphi((2k-1)x, x) \end{aligned} \tag{2.17}$$

for all $x \in G$. It follows from (2.9), (2.11), (2.12), (2.14), and (2.15) that

$$\| f(3x) - 4f(2x) + 5f(x) \| \leq \frac{1}{|k^2(k^2-1)|} \varphi_1(x) \tag{2.18}$$

for all $x \in G$. Also, from (2.9), (2.10), (2.12), (2.13), (2.16), and (2.17), we conclude that

$$\| f(4x) - 2f(3x) - 2f(2x) + 6f(x) \| \leq \frac{1}{|k^2(k^2-1)|} \varphi_2(x) \tag{2.19}$$

for all $x \in G$. Finally, by using (2.18) and (2.19), we obtain that

$$\| f(4x) - 10f(2x) + 16f(x) \| \leq \widetilde{\varphi}(x) \tag{2.20}$$

for all $x \in G$. Let $g_1 : G \to X$ be a function defined by $g_1(x) := f(2x) - 8f(x)$ for all $x \in G$. From (2.20), we conclude that

$$\| g_1(2x) - 2g_1(x) \| \leq \widetilde{\varphi}(x) \tag{2.21}$$

for all $x \in G$. If we replace x in (2.21) by $x/2^{n+1}$, we get

$$\left\| 2^{n+1} g_1\left(\frac{x}{2^{n+1}}\right) - 2^n g_1\left(\frac{x}{2^n}\right) \right\| \leq |2|^n \widetilde{\varphi}\left(\frac{x}{2^{n+1}}\right) \tag{2.22}$$

for all $x \in G$. It follows from (2.1) and (2.22) that the sequence $\{2^n g_1(x/2^n)\}$ is Cauchy. Since X is complete, we conclude that $\{2^n g_1(x/2^n)\}$ is convergent. So one can define the function $A : G \to X$ by

$$A(x) := \lim_{n\to\infty} 2^n g_1\left(\frac{x}{2^n}\right) \tag{2.23}$$

for all $x \in G$. By using induction, it follows from (2.21) and (2.22) that

$$\left\| g_1(x) - 2^n g_1\left(\frac{x}{2^n}\right) \right\| \leq \frac{1}{|2|} \max\left\{ |2|^{j+1} \widetilde{\varphi}\left(\frac{x}{2^{j+1}}\right) : 0 \leq j < n \right\} \tag{2.24}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (2.24) and using (2.4) one gets (2.3). Now we show that A is additive. It follows from (2.1), (2.22), and (2.23) that

$$\begin{aligned} \|A(2x) - 2A(x)\| &= \lim_{n\to\infty} \left\| 2^n g_1\left(\frac{x}{2^{n-1}}\right) - 2^{n+1} g_1\left(\frac{x}{2^n}\right) \right\| \\ &= |2| \lim_{n\to\infty} \left\| 2^{n-1} g_1\left(\frac{x}{2^{n-1}}\right) - 2^n g_1\left(\frac{x}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} |2|^n \widetilde{\varphi}\left(\frac{x}{2^n}\right) = 0 \end{aligned} \tag{2.25}$$

for all $x \in G$. So

$$A(2x) = 2A(x) \tag{2.26}$$

for all $x \in G$. On the other hand it follows from (2.1), (2.2), and (2.23) that

$$\begin{aligned} \|DA(x,y)\| &= \lim_{n\to\infty} |2|^n \left\| Dg_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &= \lim_{n\to\infty} |2|^n \left\| Df\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 8Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} |2|^n \max\left\{ \varphi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right), |8|\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\} = 0 \end{aligned} \tag{2.27}$$

for all $x, y \in G$. Hence the function A satisfies (1.14). Thus by Lemma 2.1, the function $x \rightsquigarrow A(2x) - 8A(x)$ is cubic-additive. Therefore (2.26) implies that the function A is additive. If A' is another additive function satisfying (2.3), by using (2.1), we have

$$\begin{aligned}\|A(x) - A'(x)\| &= \lim_{i\to\infty} |2|^i \left\| A\left(\frac{x}{2^i}\right) - A'\left(\frac{x}{2^i}\right)\right\| \\ &\le \lim_{i\to\infty} |2|^i \max\left\{ \left\| A\left(\frac{x}{2^i}\right) - g_1\left(\frac{x}{2^i}\right)\right\|, \left\| g_1\left(\frac{x}{2^i}\right) - A'\left(\frac{x}{2^i}\right)\right\|\right\} \\ &\le \frac{1}{|2|} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{ |2|^{j+1}\widetilde{\varphi}\left(\frac{x}{2^{j+1}}\right) : i \le j < n+i\right\} = 0\end{aligned} \tag{2.28}$$

for all $x \in G$. Therefore $A = A'$. For $\ell = -1$, we can prove the theorem by a similar technique. □

Lemma 2.3 (see [49, 59, 63]). *If an odd function $f : V_1 \to V_2$ satisfies* (1.14)*, then the function $g_2 : V_1 \to V_2$ defined by $g_2(x) = f(2x) - 2f(x)$ is cubic.*

Theorem 2.4. *Let $\ell \in \{1, -1\}$ be fixed and let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^{3n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) = 0 = \lim_{n\to\infty} |2|^{3n\ell}\widetilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) \tag{2.29}$$

for all $x, y \in G$. Suppose that an odd function $f : G \to X$ satisfies inequality (2.2) *for all $x, y \in G$. Then there exists a unique cubic function $C : G \to X$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{1}{|2|^3}\varphi_c(x) \tag{2.30}$$

for all $x \in G$, where

$$\varphi_c(x) = \lim_{n\to\infty} \max\left\{ |2|^{3\ell(j+((1+\ell)/2))}\widetilde{\varphi}\left(\frac{x}{2^{\ell(j+((1+\ell)/2))}}\right) : 0 \le j < n\right\} \tag{2.31}$$

and $\widetilde{\varphi}(x)$ is defined as in (2.5) *for all $x \in G$.*

Proof. Let $\ell = -1$. Similar to the proof of Theorem 2.2, we have

$$\|f(4x) - 10f(2x) + 16f(x)\| \le \widetilde{\varphi}(x) \tag{2.32}$$

for all $x \in G$, where $\widetilde{\varphi}(x)$ is defined as in (2.5) for all $x \in G$. Let $g_2 : G \to X$ be a function defined by $g_2(x) := f(2x) - 2f(x)$ for all $x \in G$. From (2.32), we conclude that

$$\|g_2(2x) - 8g_2(x)\| \le \widetilde{\varphi}(x) \tag{2.33}$$

for all $x \in G$. If we replace x in (2.33) by $2^{n-1}x$, we get

$$\left\| \frac{g_2(2^n x)}{2^{3n}} - \frac{g_2(2^{n-1}x)}{2^{3(n-1)}} \right\| \leq \frac{1}{|2|^{3n}} \widetilde{\varphi}\left(2^{n-1}x\right) \tag{2.34}$$

for all $x \in G$. It follows from (2.29) and (2.34) that the sequence $\{g_2(2^n x)/2^{3n}\}$ is Cauchy. Since X is complete, we conclude that $\{g_2(2^n x)/2^{3n}\}$ is convergent. So one can define the function $C : G \to X$ by

$$C(x) := \lim_{n\to\infty} \frac{g_2(2^n x)}{2^{3n}} \tag{2.35}$$

for all $x \in G$. It follows from (2.33) and (2.34) by using induction that

$$\left\| g_2(x) - \frac{g_2(2^n x)}{2^{3n}} \right\| \leq \frac{1}{|2|^3} \max\left\{ \frac{1}{|2|^{3j}} \widetilde{\varphi}\left(2^j x\right) : 0 \leq j < n \right\} \tag{2.36}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (2.36) and using (2.29), one gets (2.30). Now we show that C is cubic. It follows from (2.29), (2.34), and (2.35) that

$$\begin{aligned}
\|C(2x) - 8C(x)\| &= \lim_{n\to\infty} \left\| \frac{g_2(2^{n+1}x)}{2^{3n}} - \frac{2^3 g_2(2^n x)}{2^{3n}} \right\| \\
&= |2|^3 \lim_{n\to\infty} \left\| \frac{g_2(2^{n+1}x)}{2^{3(n+1)}} - \frac{g_2(2^n x)}{2^{3n}} \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{|2|^{3n}} \widetilde{\varphi}(2^n x) = 0
\end{aligned} \tag{2.37}$$

for all $x \in G$. So

$$C(2x) = 8C(x) \tag{2.38}$$

for all $x \in G$. On the other hand it follows from (2.2), (2.29), and (2.35) that

$$\begin{aligned}
\|DC(x,y)\| &= \lim_{n\to\infty} \frac{1}{|2|^{3n}} \|Dg_2(2^n x, 2^n y)\| \\
&= \lim_{n\to\infty} \frac{1}{|2|^{3n}} \left\| Df\left(2^{n+1}x, 2^{n+1}y\right) - 2Df(2^n x, 2^n y) \right\| \\
&\leq \lim_{n\to\infty} \frac{1}{|2|^{3n}} \max\left\{ \varphi\left(2^{n+1}x, 2^{n+1}y\right), |2|\varphi(2^n x, 2^n y) \right\} = 0
\end{aligned} \tag{2.39}$$

for all $x, y \in G$. Hence the function C satisfies (1.14). Thus by Lemma 2.1, the function $x \rightsquigarrow C(2x) - 2C(x)$ is cubic-additive. Therefore (2.38) implies that the function C is cubic. The rest of the proof is similar to the proof of Theorem 2.2. For $\ell = 1$, we can prove the theorem by a similar technique. □

Lemma 2.5 (see [49, 63]). *If an odd function* $f : V_1 \to V_2$ *satisfies* (1.14)*, then* f *is cubic-additive function.*

Theorem 2.6. *Let* $\ell \in \{1, -1\}$ *be fixed, and let* $\varphi : G \times G \to [0, \infty)$ *be a function such that*

$$\lim_{n\to\infty}\left\{\frac{1-\ell}{2}|2|^{n\ell}\varphi\left(\frac{x}{2^{n\ell}},\frac{y}{2^{n\ell}}\right)+\frac{1+\ell}{2}|2|^{3n\ell}\varphi\left(\frac{x}{2^{n\ell}},\frac{y}{2^{n\ell}}\right)\right\}$$
$$=0=\lim_{n\to\infty}\left\{\frac{1-\ell}{2}|2|^{n\ell}\tilde{\varphi}\left(\frac{x}{2^{n\ell}}\right)+\frac{1+\ell}{2}|2|^{3n\ell}\tilde{\varphi}\left(\frac{x}{2^{n\ell}}\right)\right\} \tag{2.40}$$

for all $x, y \in G$*. Suppose that an odd function* $f : G \to X$ *satisfies inequality* (2.2) *for all* $x, y \in G$*. Then there exist a unique additive function* $A : G \to X$ *and a unique cubic function* $C : G \to X$ *such that*

$$\|f(x)-A(x)-C(x)\| \le \frac{1}{|12|}\max\left\{\psi_a(x),\frac{1}{|4|}\psi_c(x)\right\} \tag{2.41}$$

for all $x \in G$*, where* $\psi_a(x)$ *and* $\psi_c(x)$ *are defined as in Theorems 2.2 and 2.4.*

Proof. Let $\ell = 1$. By Theorems 2.2 and 2.4, there exists a additive function $A_0 : G \to X$ and a cubic function $C_0 : G \to X$ such that

$$\|f(2x)-8f(x)-A_0(x)\| \le \frac{1}{|2|}\psi_a(x),$$
$$\|f(2x)-2f(x)-C_0(x)\| \le \frac{1}{|2|^3}\psi_c(x) \tag{2.42}$$

for all $x \in G$. So we obtain (2.41) by letting $A(x) = -1/6A_0(x)$ and $C(x) = 1/6C_0(x)$ for all $x \in G$.

To prove the uniqueness property of A and C, let $C', A' : G \to X$ be other additive and cubic functions satisfying (2.41). Let $\overline{A} = A - A'$ and $\overline{C} = C - C'$. Hence

$$\left\|\overline{A}(x)+\overline{C}(x)\right\| \le \max\{\|f(x)-A(x)-C(x)\|,\|f(x)-A'(x)-C'(x)\|\}$$
$$\le \frac{1}{|12|}\max\left\{\psi_a(x),\frac{1}{|4|}\psi_c(x)\right\} \tag{2.43}$$

for all $x \in G$. Since

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{|2|^{j+1}\tilde{\varphi}\left(\frac{x}{2^{j+1}}\right):i\le j<n+i\right\}$$
$$=0=\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{|2|^{3(j+1)}\tilde{\varphi}\left(\frac{x}{2^{j+1}}\right):i\le j<n+i\right\} \tag{2.44}$$

for all $x \in G$,

$$\lim_{n\to\infty} |2|^{3n} \left\| \overline{A}\left(\frac{x}{2^n}\right) + \overline{C}\left(\frac{x}{2^n}\right) \right\| = 0 \tag{2.45}$$

for all $x \in X$. Therefore, we get $\overline{C} = 0$ and then $\overline{A} = 0$, and the proof is complete. For $\ell = -1$, we can prove the theorem by a similar technique. □

Theorem 2.7. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 = \lim_{n\to\infty} |2|^n \widetilde{\varphi}\left(\frac{x}{2^n}\right), \qquad \lim_{n\to\infty} \frac{1}{|2|^{3n}} \varphi(2^n x, 2^n y) = 0 = \lim_{n\to\infty} \frac{1}{|2|^{3n}} \widetilde{\varphi}(2^n x) \tag{2.46}$$

for all $x, y \in G$. Suppose that an odd function $f : G \to X$ satisfies inequality (2.2) *for all $x, y \in G$. Then there exist a unique additive function $A : G \to X$ and a unique cubic function $C : G \to X$ such that*

$$\| f(x) - A(x) - C(x) \| \le \frac{1}{|12|} \max\left\{ \psi_a(x), \frac{1}{|4|} \psi_c(x) \right\} \tag{2.47}$$

for all $x \in G$, where $\psi_a(x)$ and $\psi_c(x)$ are defined as in Theorems 2.2 and 2.4.

Proof. The proof is similar to the proof of Theorem 2.6, and the result follows from Theorems 2.2 and 2.4. □

3. Stability of the AQCQ-Functional Equation (1.14): For an Even Case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in complete non-Archimedean spaces: an even case.

Lemma 3.1 (see [63]). *If an even function $f : V_1 \to V_2$ satisfies* (1.14)*, then the function $h_1 : V_1 \to V_2$ defined by $h_1(x) = f(2x) - 16f(x)$ is quadratic.*

Theorem 3.2. *Let $\ell \in \{1, -1\}$ be fixed, and let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^{2n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) = 0 = \lim_{n\to\infty} |2|^{2n\ell} \widetilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) \tag{3.1}$$

for all $x, y \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies inequality (2.2) *for all $x, y \in G$. Then there exists a unique quadratic function $Q : G \to X$ such that*

$$\| f(2x) - 16f(x) - Q(x) \| \le \frac{1}{|2|^2} \psi_q(x) \tag{3.2}$$

for all $x \in G$, where

$$\psi_q(x) = \lim_{n\to\infty} \max\left\{ |2|^{2\ell(j+((1+\ell)/2))} \widetilde{\varphi}\left(\frac{x}{2^{\ell(jj+((1+\ell)/2)}} \right) : 0 \le j < n \right\} \tag{3.3}$$

$$\begin{aligned} \widetilde{\varphi}(x) \ :=\ & \frac{1}{|k^2(k^2-1)|} \\ & \times \max\left\{ \max\left\{ \left|12k^2\right| \varphi(x,x), \left|12\left(k^2-1\right)\right| \varphi(0,x) \right\}, \max\left\{ |6|\varphi(0,2x), |12|\varphi(kx,x) \right\} \right\} \end{aligned} \tag{3.4}$$

exists for all $x \in G$.

Proof. Let $\ell = 1$. It follows from (2.2) and using the evenness of f that

$$\begin{aligned} &\Big\| f(x+ky) + f(x-ky) - k^2 f(x+y) - k^2 f(x-y) - 2\left(1-k^2\right) f(x) \\ &\quad - \frac{k^2(k^2-1)}{6}(f(2y) - 4f(y)) \Big\| \\ &\quad \le \varphi(x,y) \end{aligned} \tag{3.5}$$

for all $x, y \in G$. Interchanging x with y in (3.5), we get by the evenness of f:

$$\begin{aligned} &\Big\| f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y) + 2\left(k^2-1\right) f(y) \\ &\quad - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) \Big\| \\ &\quad \le \varphi(y,x) \end{aligned} \tag{3.6}$$

for all $x, y \in G$. Setting $y = 0$ in (3.6), we have

$$\left\| 2f(kx) - 2k^2 f(x) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) \right\| \le \varphi(0,x) \tag{3.7}$$

for all $x \in G$. Putting $y = x$ in (3.6), we obtain

$$\begin{aligned} &\left\| f((k+1)x) + f((k-1)x) - k^2 f(2x) + 2\left(k^2-1\right) f(x) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x)) \right\| \\ &\quad \le \varphi(x,x) \end{aligned} \tag{3.8}$$

for all $x \in G$. Replacing x and y by $2x$ and 0 in (3.6), respectively, we see that

$$\left\|2f(2kx) - 2k^2 f(2x) - \frac{k^2(k^2-1)}{6}(f(4x) - 4f(2x))\right\| \le \varphi(0, 2x) \tag{3.9}$$

for all $x \in G$. Setting $y = kx$ in (3.6) and using the evenness of f, we get

$$\begin{aligned}&\left\|f(2kx) - k^2 f((k+1)x) - k^2 f((k-1)x) + 2\left(k^2 - 1\right) f(kx) - \frac{k^2(k^2-1)}{6}(f(2x) - 4f(x))\right\| \\ &\quad \le \varphi(kx, x)\end{aligned} \tag{3.10}$$

for all $x \in G$. It follows from (3.7), (3.8), (3.9), and (3.10) that

$$\|f(4x) - 20f(2x) + 64f(x)\| \le \widetilde{\varphi}(x) \tag{3.11}$$

for all $x \in G$. Let $h_1 : G \to X$ be a function defined by $h_1(x) := f(2x) - 16f(x)$ for all $x \in G$. From (3.11), we conclude that

$$\|h_1(2x) - 4h_1(x)\| \le \widetilde{\varphi}(x) \tag{3.12}$$

for all $x \in G$. Replacing x by $x/2^{n+1}$ in (3.12), we have

$$\left\|2^{2(n+1)} h_1\left(\frac{x}{2^{n+1}}\right) - 2^{2n} h_1\left(\frac{x}{2^n}\right)\right\| \le |2|^{2n} \widetilde{\varphi}\left(\frac{x}{2^{n+1}}\right) \tag{3.13}$$

for all $x \in G$. It follows from (3.1) and (3.13) that the sequence $\{2^{2n} h_1(x/2^n)\}$ is Cauchy. Since X is complete, we conclude that $\{2^{2n} h_1(x/2^n)\}$ is convergent. So one can define the function $Q : G \to X$ by

$$Q(x) := \lim_{n\to\infty} 2^{2n} h_1\left(\frac{x}{2^n}\right) \tag{3.14}$$

for all $x \in G$. It follows from (3.12) and (3.13) by using induction that

$$\left\|h_1(x) - 2^{2n} h_1\left(\frac{x}{2^n}\right)\right\| \le \frac{1}{|2|^2} \max\left\{|2|^{2(j+1)} \widetilde{\varphi}\left(\frac{x}{2^{j+1}}\right) : 0 \le j < n\right\} \tag{3.15}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.15) and using (3.3), one gets (3.2). Now we show that Q is quadratic. It follows from (3.1), (3.13), and (3.14) that

$$\begin{aligned} \|Q(2x) - 4Q(x)\| &= \lim_{n\to\infty} \left\| 2^{2n} h_1\left(\frac{x}{2^{n-1}}\right) - 2^{2(n+1)} h_1\left(\frac{x}{2^n}\right) \right\| \\ &= \lim_{n\to\infty} |2|^2 \left\| 2^{2(n-1)} h_1\left(\frac{x}{2^{n-1}}\right) - 2^{2n} h_1\left(\frac{x}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} |2|^{2(n+1)} \widetilde{\varphi}\left(\frac{x}{2^{n+1}}\right) = 0 \end{aligned} \tag{3.16}$$

for all $x \in G$. So

$$Q(2x) = 4Q(x) \tag{3.17}$$

for all $x \in G$. On the other hand it follows from (2.2), (3.1), and (3.14) that

$$\begin{aligned} \|DQ(x,y)\| &= \lim_{n\to\infty} |2|^{2n} \left\| Dh_1\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &= \lim_{n\to\infty} |2|^{2n} \left\| Df\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 16Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\ &\le \lim_{n\to\infty} |2|^{2n} \max\left\{ \varphi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right), |16|\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\} = 0 \end{aligned} \tag{3.18}$$

for all $x, y \in G$. Hence the function Q satisfies (1.14). Thus by Lemma 3.1, the function $x \rightsquigarrow Q(2x) - 16Q(x)$ is quartic-quadratic. Therefore (3.17) implies that the function Q is quadratic. The rest of the proof is similar to the proof of Theorem 2.2. For $\ell = -1$, we can prove the theorem by a similar technique. □

Lemma 3.3 (see [63]). *If an even function $f : V_1 \to V_2$ satisfies* (1.14), *then the function $h_2 : V_1 \to V_2$ defined by $h_2(x) = f(2x) - 4f(x)$ is quartic.*

Theorem 3.4. *Let $\ell \in \{1, -1\}$ be fixed, and let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^{4n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) = 0 = \lim_{n\to\infty} |2|^{4n\ell} \widetilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) \tag{3.19}$$

for all $x, y \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies inequality (2.2) *for all $x, y \in G$. Then there exists a unique quartic function $V : G \to X$ such that*

$$\|f(2x) - 4f(x) - V(x)\| \le \frac{1}{|2|^4} \psi_v(x) \tag{3.20}$$

for all $x \in G$, where

$$\psi_v(x) = \lim_{n\to\infty} \max\left\{ |2|^{4\ell(j+((1+\ell)/2))} \tilde{\varphi}\left(\frac{x}{2^{\ell(j+((1+\ell)/2))}}\right) : 0 \le j < n \right\} \tag{3.21}$$

and $\tilde{\varphi}(x)$ is defined as in (3.4) for all $x \in G$.

Lemma 3.5 (see [63]). *If an even function $f : V_1 \to V_2$ satisfies (1.14), then f is quartic-quadratic function.*

Theorem 3.6. *Let $\ell \in \{1, -1\}$ be fixed, and let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \frac{1-\ell}{2} |2|^{2n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) + \frac{1+\ell}{2} |2|^{4n\ell} \varphi\left(\frac{x}{2^{n\ell}}, \frac{y}{2^{n\ell}}\right) \right\} \\ &\quad = 0 = \lim_{n\to\infty} \left\{ \frac{1-\ell}{2} |2|^{2n\ell} \tilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) + \frac{1+\ell}{2} |2|^{4n\ell} \tilde{\varphi}\left(\frac{x}{2^{n\ell}}\right) \right\} \end{aligned} \tag{3.22}$$

for all $x, y \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies inequality (2.2) for all $x, y \in G$. Then there exist a unique quadratic function $Q : G \to X$ and a unique quartic function $V : G \to X$ such that

$$\|f(x) - Q(x) - V(x)\| \le \frac{1}{|48|} \max\left\{ \psi_q(x), \frac{1}{|4|}\psi_v(x) \right\} \tag{3.23}$$

for all $x \in G$, where $\psi_q(x)$ and $\psi_v(x)$ are defined as in Theorems 3.2 and 3.4.

Proof. The proof is similar to the proof of Theorem 2.6 and the result follows from Theorems 3.2 and 3.4. □

Theorem 3.7. *Let $\varphi : G \times G \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 = \lim_{n\to\infty} |2|^{2n} \tilde{\varphi}\left(\frac{x}{2^n}\right), \lim_{n\to\infty} \frac{1}{|2|^{4n}} \varphi(2^n x, 2^n y) = 0 = \lim_{n\to\infty} \frac{1}{|2|^{4n}} \tilde{\varphi}(2^n x) \tag{3.24}$$

for all $x, y \in G$. Suppose that an even function $f : G \to X$ with $f(0) = 0$ satisfies inequality (2.2) for all $x, y \in G$. Then there exist a unique quadratic function $Q : G \to X$ and a unique quartic function $V : G \to X$ such that

$$\|f(x) - Q(x) - V(x)\| \le \frac{1}{|48|} \max\left\{ \psi_q(x), \frac{1}{|4|}\psi_v(x) \right\} \tag{3.25}$$

for all $x \in G$, where $\psi_q(x)$ and $\psi_v(x)$ are defined as in Theorems 3.2 and 3.4.

4. AQCQ-Functional Equation in Non-Archimedean Normed Spaces

Now, we are ready to prove the main theorems concerning the generalized Hyers-Ulam stability problem for (1.14) in non-Archimedean spaces.

Lemma 4.1 (see [63]). *A function $f : V_1 \to V_2$ satisfies* (1.14) *for all $x, y \in V_1$ if and only if there exist a unique symmetric biquadratic function $B_2 : V_1 \times V_1 \to V_2$, a unique function $C : V_1 \times V_1 \times V_1 \to V_2$, a unique symmetric biadditive function $B_1 : V_1 \times V_1 \to V_2$, and a unique additive function $A : V_1 \to V_2$, such that $f(x) = B_2(x,x) + C(x,x,x) + B_1(x,x) + A(x)$ for all $x \in V_1$, where the function C is symmetric for each fixed variable and is additive for fixed two variables.*

Theorem 4.2. *Let $\ell \in \{1, -1\}$ be fixed, and let $\varphi : G \times G \to [0, \infty)$ be a function satisfying* (2.41) *and* (3.22) *for all $x, y \in G$. Then*

$$\lim_{n\to\infty} \max\left\{ \left[\frac{1-\ell}{2}|2|^{\ell(j+((1+\ell)/2))} + \frac{1+\ell}{2}|2|^{3\ell(j+((1+\ell)/2))}\right] \widetilde{\varphi}\left(\frac{x}{2^{\ell(j+((1+\ell)/2))}}\right) : 0 \le j < n \right\},$$
$$\lim_{n\to\infty} \max\left\{ \left[\frac{1-\ell}{2}|2|^{2\ell(j+((1+\ell)/2))} + \frac{1+\ell}{2}|2|^{4\ell(jj+((1+\ell)/2))}\right] \widetilde{\varphi}\left(\frac{x}{2^{\ell(j+((1+\ell)/2))}}\right) : 0 \le j < n \right\} \tag{4.1}$$

exist for all $x \in G$, where $\widetilde{\varphi}(x)$ and $\widetilde{\varphi}(x)$ are defined as in (2.3) *and* (3.3) *for all $x \in G$. Suppose that a function $f : G \to X$ with $f(0) = 0$ satisfies inequality* (2.2) *for all $x, y \in G$. Then there exist a unique additive function $A : G \to X$, a unique quadratic function $Q : G \to X$, a unique cubic function $C : G \to X$, and a unique quartic function $V : G \to X$ such that*

$$\| f(x) - A(x) - Q(x) - C(x) - V(x) \| \le \widetilde{\Phi}(x) \tag{4.2}$$

for all $x \in G$, where

$$\widetilde{\Phi}(x) := \frac{1}{|24|} \max\left\{ \varphi_3(x), \frac{1}{|4|}\varphi_4(x), \right\}, \tag{4.3}$$

$$\varphi_3(x) := \max\left\{ \max\left\{ \varphi_a(x), \frac{1}{|4|}\varphi_c(x) \right\}, \max\left\{ \varphi_a(-x), \frac{1}{|4|}\varphi_c(-x) \right\} \right\}, \tag{4.4}$$

$$\varphi_4(x) := \max\left\{ \max\left\{ \varphi_q(x), \frac{1}{|4|}\varphi_v(x) \right\}, \max\left\{ \varphi_q(-x), \frac{1}{|4|}\varphi_v(-x) \right\} \right\}, \tag{4.5}$$

for all $x \in G$, and $\varphi_a(x)$, $\varphi_c(x)$, $\varphi_q(x)$ and $\varphi_v(x)$ are defined as in Theorems 2.2, 2.4, 3.2, and 3.4.

Proof. Let $\ell = 1$ and $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in G$. Then

$$\| Df_o(x, y) \| \le \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\} \tag{4.6}$$

for all $x, y \in G$. From Theorem 2.6, it follows that there exist a unique additive function $A : G \to X$ and a unique cubic function $C : G \to X$ satisfying

$$\|f_o(x) - A(x) - C(x)\| \leq \frac{1}{|24|}\varphi_3(x) \tag{4.7}$$

for all $x \in G$. Also, let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in G$. Then

$$\|Df_e(x,y)\| \leq \frac{1}{|2|}\max\{\varphi(x,y), \varphi(-x,-y)\} \tag{4.8}$$

for all $x, y \in G$. From Theorem 3.6, it follows that there exist a quadratic function $Q : G \to X$ and a quartic function $V : G \to X$ satisfying

$$\|f_e(x) - Q(x) - V(x)\| \leq \frac{1}{|96|}\varphi_4(x) \tag{4.9}$$

for all $x \in G$. Hence, (4.2) follows from (4.7) and (4.9). To prove the uniqueness property of A, Q, C, and V, let $A', Q', C', V' : G \to X$ be other additive, quadratic, cubic, and quartic functions satisfying (4.2). Let $\overline{A} = A - A'$, $\overline{Q} = Q - Q'$, $\overline{C} = C - C'$, and $\overline{V} = V - V'$. So

$$\begin{aligned}
&\left\|\overline{A}(x) + \overline{Q}(x) + \overline{C}(x) + \overline{V}(x)\right\| \\
&\quad\leq \max\{\|f(x) - A(x) - Q(x) - C(x) - V(x)\|, \|f(x) - A'(x) - Q'(x) - C'(x) - V'(x)\|\} \\
&\quad\leq \widetilde{\Phi}(x)
\end{aligned} \tag{4.10}$$

for all $x \in G$. Since

$$\lim_{n\to\infty} |2|^{4n}\widetilde{\varphi}\left(\frac{x}{2^n}\right) = 0 = \lim_{n\to\infty} |2|^{3n}\widetilde{\varphi}\left(\frac{x}{2^n}\right) \tag{4.11}$$

for all $x \in G$, if we replace x in (4.10) by $x/2^n$ and multiply both sides of (4.10) by $|2|^{4n}$, we get

$$\lim_{n\to\infty} |2|^{4n}\left\|\overline{A}\left(\frac{x}{2^n}\right) + \overline{Q}\left(\frac{x}{2^n}\right) + \overline{C}\left(\frac{x}{2^n}\right) + \overline{V}\left(\frac{x}{2^n}\right)\right\| = 0 \tag{4.12}$$

for all $x \in G$. Therefore $\overline{V} = 0$. Putting $x = x/2^n$ and $\overline{V} = 0$ in (4.10), we obtain

$$\lim_{n\to\infty} |2|^{3n}\left\|\overline{A}\left(\frac{x}{2^n}\right) + \overline{Q}\left(\frac{x}{2^n}\right) + \overline{C}\left(\frac{x}{2^n}\right)\right\| = 0 \tag{4.13}$$

for all $x \in G$. Therefore $\overline{C} = 0$. Also by putting $\overline{V} = \overline{C} = 0$ and $x = x/2^n$ in (4.10), we have

$$\lim_{n\to\infty} |2|^{2n} \left\| \overline{A}\left(\frac{x}{2^n}\right) + \overline{Q}\left(\frac{x}{2^n}\right) \right\| = 0 \tag{4.14}$$

for all $x \in G$. Therefore $\overline{Q} = 0$, and then $\overline{A} = 0$.

For $\ell = -1$, we can prove the theorem by a similar technique. □

Theorem 4.3. *Let $\varphi : G \times G \to [0, \infty)$ be a function satisfying* (2.47) *and* (3.24) *for all $x, y \in G$. Suppose that a function $f : G \to X$ with $f(0) = 0$ satisfies inequality* (2.2) *for all $x, y \in G$. Then there exist a unique additive function $A : G \to X$, a unique quadratic function $Q : G \to X$, a unique cubic function $C : G \to X$, and a unique quartic function $V : G \to X$ such that*

$$\| f(x) - A(x) - Q(x) - C(x) - V(x) \| \le \widetilde{\Phi}(x) \tag{4.15}$$

for all $x \in G$, where $\widetilde{\Phi}(x)$ is defined as in Theorem 4.2.

Proof. The proof is similar to the proof of Theorem 4.2, and the result follows from Theorems 2.7 and 3.7. To prove the uniqueness property of A, Q, C, and V, let $A', Q', C', V' : G \to X$ be other additive, quadratic, cubic and quartic functions satisfying (4.15). Let $\overline{A} = A - A', \overline{Q} = Q - Q', \overline{C} = C - C'$, and $\overline{V} = V - V'$. So

$$\begin{aligned} &\left\| \overline{A}(x) + \overline{Q}(x) + \overline{C}(x) + \overline{V}(x) \right\| \\ &\quad \le \max\{ \| f(x) - A(x) - Q(x) - C(x) - V(x) \|, \| f(x) - A'(x) - Q'(x) - C'(x) - V'(x) \| \} \\ &\quad \le \widetilde{\Phi}(x) \end{aligned} \tag{4.16}$$

for all $x \in G$. Since

$$\lim_{n\to\infty} |2|^{2n} \widetilde{\varphi}\left(\frac{x}{2^n}\right) = 0 = \lim_{n\to\infty} \frac{1}{|2|^{3n}} \widetilde{\varphi}(2^n x) \tag{4.17}$$

for all $x \in G$, if we replace x in (4.16) by $2^n x$ and divide both sides of (4.16) by $|2|^{4n}$, we get

$$\lim_{n\to\infty} \frac{1}{|2|^{4n}} \left\| \overline{A}(2^n x) + \overline{Q}(2^n x) + \overline{C}(2^n x) + \overline{V}(2^n x) \right\| = 0 \tag{4.18}$$

for all $x \in G$. Therefore $\overline{V} = 0$. It follows that

$$\lim_{n\to\infty} \frac{1}{|2|^{3n}} \left\| \overline{A}(2^n x) + \overline{Q}(2^n x) + \overline{C}(2^n x) \right\| = 0 \tag{4.19}$$

for all $x \in G$. Therefore $\overline{C} = 0$. Also by putting $\overline{V} = \overline{C} = 0$ and $x = x/2^n$ in (4.16), we have

$$\lim_{n\to\infty} |2|^{2n} \left\| \overline{A}\left(\frac{x}{2^n}\right) + \overline{Q}\left(\frac{x}{2^n}\right) \right\| = 0 \tag{4.20}$$

for all $x \in G$. Therefore $\overline{Q} = 0$, and then $\overline{A} = 0$. □

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