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Research Article

On Two-Parameter Regularized Semigroups and the Cauchy Problem

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Suppose that X is a Banach space and C is an injective operator in $B(X)$, the space of all bounded linear operators on X. In this note, a two-parameter C-semigroup (regularized semigroup) of operators is introduced, and some of its properties are discussed. As an application we show that the existence and uniqueness of solution of the 2-abstract Cauchy problem $(\partial/(\partial t_i))u(t_1,t_2) = H_i u(t_1,t_2)$, $i = 1,2$, $t_i > 0$, $u(0,0) = x$, $x \in C(D(H_1) \cap D(H_2))$ is closely related to the two-parameter C-semigroups of operators.

1. Introduction and Preliminaries

Suppose that X is a Banach space and A is a linear operator in X with domain $D(A)$ and range $R(A)$. For a given $x \in D(A)$, the abstract Cauchy problem for A with the initial value x consists of finding a solution $u(t)$ to the initial value problem

$$\mathrm{ACP}(A;x)\begin{cases} \dfrac{du(t)}{dt} = Au(t), & t \in \mathbb{R}_+, \\ u(0) = x, \end{cases} \tag{1.1}$$

where by a solution we mean a function $u : \mathbb{R}_+ \to X$, which is continuous for $t \geq 0$, continuously differentiable for $t > 0$, $u(t) \in D(A)$ for $t \in \mathbb{R}_+$, and $ACP(A;x)$ is satisfied.

If $C \in B(X)$, the space of all bounded linear operators on X, is injective, then a one-parameter C-semigroup (regularized semigroup) of operators is a family $\{T(t)\}_{t\in\mathbb{R}_+} \subset B(X)$

for which $T(0) = C$, $T(s+t)C = T(s)T(t)$, and for each $x \in X$, the mapping $t \mapsto T(t)x$ is continuous. An operator $A : D(A) \to X$ with

$$D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - Cx}{t} \text{ exists in the range of } C \right\}, \tag{1.2}$$

and where, for $x \in D(A)$, $Ax := C^{-1}\lim_{t\to 0}((T(t)x - Cx)/t)$ is called the infinitesimal generator of $T(t)$.

Regularized semigroups and their connection with the $ACP(A;x)$ have been studied in [1–6] and some other papers. Also the concept of local C-semigroups and their relation with the $ACP(A;x)$ have been considered in [7–10].

In Section 2, we introduce the concept of two-parameter regularized semigroups of operators and their generator. Some basic properties of two-parameter regularized semigroups and their relation with the generators are studied in this section.

In Section 3, two-parameter abstract Cauchy problems are considered. It is proved that the existence and uniqueness of its solutions is closely related with two-parameter regularized semigroups of operators.

2. Two-Parameter Regularized Semigroups

In this section we introduce two-parameter regularized semigroup and its generator on Banach spaces. Then some properties of two-parameter regularized semigroups are studied.

Definition 2.1. Suppose that X is a Banach space and $C \in B(X)$ is an injective operator. A family $\{W(s,t)\}_{s,t\in\mathbb{R}_+} \subset B(X)$ is called a two-parameter regularized semigroup (or two parameter C-semigroup) if

(i) $W(0,0) = C$,

(ii) $W(s+s', t+t')C = W(s,t)W(s',t')$, for all $s, s', t, t' \in \mathbb{R}_+$,

(iii) $\lim_{(s',t')\to(s,t)} W(s',t')x = W(s,t)x$, for all $x \in X$.

It is called exponentially bounded if $\|W(s,t)\| \le Me^{(s+t)\omega}$, for some $M, \omega > 0$.

Suppose that $\{W(s,t)\}_{s,t\in\mathbb{R}_+}$ is a two-parameter C-semigroup. Put $u(s) := W(s,0)$ and $v(t) := W(0,t)$, then it is easy to see that these families are two commuting one-parameter C-semigroups such that $W(s,t)C = u(s)v(t)$. Also $u(s)$ and $v(t)$ commute with C. If H_1 and H_2 are their generators, respectively, then we will think of (H_1, H_2) as the generator of $W(s,t)$.

From the one-parameter case (see [8]), one can prove that $R(C) \subseteq \overline{D(H_1)} \cap \overline{D(H_2)}$, and $C^{-1}H_iC = H_i$, $i = 1,2$.

Also if $\{U(s)\}_{s\in\mathbb{R}_+}$ and $\{V(t)\}_{t\in\mathbb{R}_+}$ are two commuting one-parameter C-semigroups, then one can see that $W(s,t) := U(s)V(t)$ is a two-parameter C^2-semigroup of operators.

The following is an example of a two-parameter C-semigroup which is not exponentially bounded.

Example 2.2. Let $X = L^2(\mathbb{C})$, and $[W(s,t)f](z) := e^{-|z|^2+(s+t)z} f(z)$, $(Cf)(z) := e^{-|z|^2} f(z)$, then $W(s,t)$ is a two-parameter C-semigroup which is not exponentially bounded.

In the following theorem we can see some elementary properties of a two-parameter C-semigroup.

Theorem 2.3. *Suppose that $W(s,t)$ is a two-parameter C-semigroup with the infinitesimal generator (H_1, H_2). Then, one has the following.*

(i) *For each $x \in X$ and for every $s,t \geq 0$, $\int_0^t \int_0^s W(\mu,\nu)x\,d\mu\,d\nu$, is in $D(H_1) \cap D(H_2)$. Also*

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk} \int_t^{t+h} \int_s^{s+k} W(\mu,\nu)x\,d\mu\,d\nu = W(s,t)x. \tag{2.1}$$

(ii) *For each $x \in X$, and for every $s,t \in \mathbb{R}_+$, $\int_0^s W(\mu,t)x\,d\mu \in D(H_1)$ and $\int_0^t W(s,\nu)x\,d\nu \in D(H_2)$; furthermore*

$$\begin{aligned} H_1 \int_0^s W(\mu,t)x\,d\mu &= W(s,t)x - W(0,t)x, \\ H_2 \int_0^t W(s,\nu)x\,d\nu &= W(s,t)x - W(s,0)x. \end{aligned} \tag{2.2}$$

(iii) $\overline{R(C)} \subseteq \overline{D(H_1) \cap D(H_2)}$ *and H_1 and H_2 are closed.*

(iv) *For any $x \in D(H_1) \cap D(H_2)$, and each $s,t > 0$, $u(s)x$ and $v(t)x$ are in $D(H_1) \cap D(H_2)$. Also for this x, and $i = 1,2$,*

$$\frac{\partial}{\partial t_i} W(t_1,t_2)x = H_i W(t_1,t_2)x = W(t_1,t_2)H_i x. \tag{2.3}$$

(v) *For any $a,b > 0$, $T(t) := W(ta,tb)$ is a one-parameter C-semigroup whose generator is an extension of $aH_1 + bH_2$.*

Proof. To prove (i), suppose $x \in X$. First we note that for any $\nu \geq 0$,

$$\begin{aligned} \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} W(\mu,\nu)Cx\,d\mu &= W(0,\nu) \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} W(\mu,0)x\,d\mu \\ &= W(0,\nu)W(t,0)x \\ &= W(t,\nu)Cx. \end{aligned} \tag{2.4}$$

Thus

$$\begin{aligned} &\frac{1}{h}\left(W(h,0) \int_0^s \int_0^t W(\mu,\nu)x\,d\mu\,d\nu - C \int_0^s \int_0^t W(\mu,\nu)x\,d\mu\,d\nu \right) \\ &\quad = \frac{1}{h} C \left(\int_0^s \int_h^{t+h} W(\mu,\nu)x\,d\mu\,d\nu - \int_0^s \int_0^t W(\mu,\nu)x\,d\mu\,d\nu \right) \\ &\quad = \int_0^s \left(\frac{1}{h} \left[\int_t^{t+h} W(\mu,\nu)Cx\,d\mu - \int_0^h W(\mu,\nu)Cx\,d\mu \right] \right) d\nu, \end{aligned} \tag{2.5}$$

which tends to $C\int_0^s (W(t,\nu) - W(0,\nu))x\,d\nu$ as $h \to 0$. This implies that $\int_0^s\int_0^t W(\mu,\nu)x\,d\mu\,d\nu$ is in $D(H_1)$ and

$$H_1\int_0^s\int_0^t W(\mu,\nu)x\,d\mu\,d\nu = \int_0^s (W(t,\nu) - W(0,\nu))x\,d\nu. \tag{2.6}$$

A similar argument implies that it is in $D(H_2)$ and

$$H_2\int_0^s\int_0^t W(\mu,\nu)x\,d\mu\,d\nu = \int_0^t (W(\mu,s) - W(\mu,0))x\,d\nu. \tag{2.7}$$

For the second part, from the continuity of C we have

$$\begin{aligned} C &\lim_{(h,k)\to(0,0)} \frac{1}{hk}\int_t^{t+h}\int_s^{s+k} W(\mu,\nu)x\,d\mu\,d\nu \\ &= \lim_{(h,k)\to(0,0)} \frac{1}{hk}\int_t^{t+h}\int_s^{s+k} W(\mu,\nu)Cx\,d\mu\,d\nu \\ &= \lim_{(h,k)\to(0,0)} \frac{1}{h}\int_t^{t+h} W(0,\nu)\frac{1}{k}\int_s^{s+k} W(\mu,0)x\,d\mu\,d\nu \\ &= \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} W(0,\nu)\left(\lim_{k\to 0}\frac{1}{k}\int_s^{s+k} W(\mu,0)x\,d\mu\right)d\nu \\ &= W(0,t)W(s,0)x \\ &= W(s,t)Cx. \end{aligned} \tag{2.8}$$

Now the fact that C is injective completes the proof of this part.

The proof of (ii) has a process similar to the first part of (i).

To prove (iii), we first note that H_1 and H_2 are closed as a trivial consequence of the one-parameter case (see [2]). For any $x \in X$ we saw that

$$\frac{1}{h}\int_0^h\int_0^h W(\mu,\nu)x\,d\mu\,d\nu \in D(H_1)\cap D(H_2), \tag{2.9}$$

which tends to $W(0,0)x = Cx \in R(C)$, as $h \to 0$. This implies that $\overline{R(C)} \subseteq \overline{D(H_1)\cap D(H_2)}$.

To prove (iv), we let $x \in D(H_1)\cap D(H_2)$. If $u(s) = W(s,0)$ and $v(t) = W(s,t)$, there is $y \in X$ such that

$$\lim_{s\to 0}\frac{u(s)x - Cx}{s} = Cy. \tag{2.10}$$

Hence

$$\lim_{s \to 0} \frac{u(s)v(t)x - Cv(t)x}{s} = v(t)Cy = Cv(t)y, \tag{2.11}$$

which is in the $R(C)$, and this implies that $v(t)x$ is in $D(H_1)$, similarly it is in $D(H_2)$.

Now from [2, Theorem 2.4(b)], for $x \in D(H_1) \cap D(H_2)$, from the fact that $v(t)x$ is in $D(H_1)$,

$$\begin{aligned} \frac{\partial}{\partial s} W(s,t)Cx &= \frac{d}{ds}(u(s)(v(t)x)) \\ &= H_1 u(s)(v(t)x) \\ &= H_1 W(s,t)Cx \\ &= CH_1 W(s,t)x. \end{aligned} \tag{2.12}$$

On the other hand from the part (ii) and closedness of H_1,

$$\int_0^s H_1 W(\mu,t)x \, d\mu = H_1 \int_0^s W(\mu,t)x \, d\mu = W(s,t)x - W(0,t)x, \tag{2.13}$$

which implies that $(\partial/\partial s)W(s,t)x$ exists. Hence from the continuity of C

$$C\frac{\partial}{\partial s} W(s,t)x = \frac{\partial}{\partial s} W(s,t)Cx = CH_1 W(s,t)x. \tag{2.14}$$

But C is injective so

$$\frac{\partial}{\partial s} W(s,t)x = H_1 W(s,t)x = W(s,t)H_1 x. \tag{2.15}$$

The second one is similar.

To prove (v), first we note that $T(t)$ is a one-parameter C-semigroup. Now if $x \in D(aH_1 + bH_2) = D(H_1) \cap D(H_2)$,

$$\begin{aligned} C \lim_{t \to 0^+} \frac{T(t)x - Cx}{t} &= \lim_{t \to 0^+} \frac{W(ta,0)W(0,tb)x - W(ta,0)Cx + W(ta,0)Cx - C^2 x}{t} \\ &= b \lim_{t \to 0^+} W(ta,0) \frac{W(0,tb)x - Cx}{bt} + a \lim_{t \to 0^+} \frac{W(at,0)Cx - C^2 x}{t} \\ &= bC^2 H_2 x + aH_1 C^2 x. \end{aligned} \tag{2.16}$$

Now the fact that C is injective implies that

$$C^{-1}\lim_{t\to 0^+}\frac{T(t)x-Cx}{t}=aH_1x+bH_2x. \tag{2.17}$$

□

For an exponentially bounded one-parameter C-semigroup $T(t)$ with the generator A, from [1] the existence of $L_\lambda(A)x=\int_0^\infty e^{-\lambda t}T(t)x\,dt$ is guaranteed for sufficiently large $\lambda\in\mathbb{R}$. Now we have the following lemma for one-parameter C-semigroups of operators which is similar to the Yosida-approximation theorem for strongly continuous semigroups. This will be applied in our study of two-parameter regularized semigroups.

Lemma 2.4. *Let $\{T(t)\}_{t\in\mathbb{R}_+}$ be a one-parameter C-semigroup satisfying the condition $\|T(t)\|\le Me^{\omega t}$, for some $\omega>0$ and $M>0$, with the generator A. If for $\lambda>\omega$, $A_\lambda:=\lambda AL_\lambda(A)$, then one has the following.*

(i) *For any $x\in X$, $\|L_\lambda(A)x\|\le(M/(\lambda-\omega))\|x\|$, $A_\lambda=\lambda^2L_\lambda(A)-\lambda C$, and so A_λ is bounded. Also $S(t):=Ce^{tA_\lambda}$ is a one-parameter C-semigroup which is exponentially bounded.*

(ii) *For any $x\in\overline{D(A)}$, $\lim_{\lambda\to\infty}\lambda L_\lambda(A)x=Cx$ and for all $x\in D(A)$, $\lim_{\lambda\to\infty}A_\lambda x=CAx$. Also if $R(C)$ is dense in X, then the first equality holds on X.*

(iii) *For any $x\in\overline{D(A)}$, $T(t)x=\lim_{\lambda\to\infty}Ce^{tA_\lambda}x$.*

Proof. The first inequality of (i) is trivial. From [2, Lemma 2.8], we know that for any $x\in X$, $(\lambda-A)L_\lambda(A)x=Cx$; thus,

$$-\lambda(\lambda-A)L_\lambda(A)x=-\lambda Cx. \tag{2.18}$$

This implies our desired equality.

For the second part, first we show that $CA_\lambda=A_\lambda C$. For this we note that

$$\begin{aligned} CL_\lambda(A)&=C\int_0^\infty e^{-\lambda t}T(t)x\,dx\\ &=\int_0^\infty Ce^{-\lambda t}T(t)x\,dx\\ &=\int_0^\infty e^{-\lambda t}T(t)Cx\,dx\\ &=L_\lambda(A)Cx. \end{aligned} \tag{2.19}$$

This and the first part imply that $CA_\lambda=A_\lambda C$. Now we prove the C-semigroup properties of $S(t)$. Trivially $S(0)=C$. Also from the last equality,

$$S(s+t)C=Ce^{(s+t)A_\lambda}C=Ce^{sA_\lambda}Ce^{tA_\lambda}=S(s)S(t). \tag{2.20}$$

The fact that A_λ, $\lambda>\omega$, is a bounded operator trivially implies that $S(\cdot)$ is exponentially bounded. Now the continuity of the mapping $t\mapsto S(t)x$ at zero implies the strongly continuity of $S(t)$.

To prove (ii), for $x \in D(A)$, from (i) and the fact that A is closed, we have

$$\begin{aligned}\|\lambda L_\lambda(A)x - Cx\| &= \|AL_\lambda(A)x\| \\ &= \|L_\lambda(A)Ax\| \\ &\le \|L_\lambda(A)\|\|Ax\| \\ &\le \frac{M}{(\lambda-\omega)}\|Ax\| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty.\end{aligned} \tag{2.21}$$

The continuity of C and $L_\lambda(A)$ implies that for any $x \in \overline{D(A)}$, $\lim_{\lambda\to\infty}\lambda L_\lambda(A)x = Cx$.

Now for $x \in D(A)$,

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda L_\lambda(A)Ax = CAx = ACx. \tag{2.22}$$

For the last part of (ii), if C has a dense range, then by [8, Lemma 1.1.3], $R(C) \subseteq \overline{D(A)}$, and so $X = \overline{R(C)} \subseteq \overline{D(A)} \subseteq X$, which means that $\overline{D(A)} = X$.

To prove (iii), for any $x \in D(A)$, we have

$$\begin{aligned}\left\|Ce^{tA_\lambda}x - Ce^{tA_\mu}x\right\| &= \left\|\int_0^1 \frac{d}{ds}\left(Ce^{tsA_\lambda}e^{t(1-sA_\mu)}x\right)\right\| \\ &\le \int_0^1 t\left\|Ce^{tsA_\lambda}e^{t(1-sA_\mu)}(A_\lambda x - A_\mu x)\right\|ds \\ &\le t\|C\|\left\|A_\lambda x - A_\mu x\right\| \\ &\le t\|C\|\left(\|A_\lambda x - ACx\| + \left\|ACx - A_\mu x\right\|\right).\end{aligned} \tag{2.23}$$

This and the previous part prove the existence of $\lim_{\lambda\to\infty}Ce^{tA_\lambda}x$. □

Using this theorem we may find the following approximation theorem for two-parameter regularized semigroups.

Corollary 2.5. *Suppose that (H,K) is the infinitesimal generator of an exponentially bounded two-parameter C-semigroup $W(s,t)$, then for each $x \in D(H) \cap D(K)$,*

$$W(s,t)x = C\lim_{\lambda\to\infty} e^{sH_\lambda + tK_\lambda}x. \tag{2.24}$$

For exponentially bounded C-semigroup $W(s,t)$ satisfying $\|W(s,t)\| \le Me^{(s+t)\omega}$, with the infinitesimal generator (H,K), define $L_{\lambda_1}(H)x := \int_0^\infty e^{-\lambda_1 s}W(s,0)x\,ds$ and $L_{\lambda_2}(K)x := \int_0^\infty e^{-\lambda_2 t}W(0,t)x\,dt$, where $\mathrm{Re}(\lambda_i) > \omega$. From the previous Lemma $L_{\lambda_1}(H)$ and $L_{\lambda_2}(K)$ are bounded operators.

Theorem 2.6. (i) *Let (H, K) be the generator of an exponentially bounded two-parameter C-semigroup, then for large enough λ_1, λ_2*

$$L_{\lambda_1}(H)L_{\lambda_2}(K) = L_{\lambda_2}(K)L_{\lambda_1}(H). \tag{2.25}$$

(ii) *Let (H, K) be the generator of an exponentially bounded two-parameter C-semigroup, then $D(H) \cap D(HK) \subseteq D(KH)$, and for $x \in D(H) \cap D(HK)$,*

$$HKx = KHx. \tag{2.26}$$

(iii) *Suppose that H and K are the generators of two exponentially bounded one-parameter C-semigroups $\{u(s)\}_{s\in\mathbb{R}_+}$ and $\{v(t)\}_{t\in\mathbb{R}_+}$, respectively. If their resolvents commute and $R(C)$ is dense in X, then $W(s,t) := u(s)v(t)$ is a two-parameter C^2-semigroup.*

Proof. The proof of (i) follows trivially from the properties of two-parameter C-semigroups.

To prove (ii), we let $x \in D(H) \cap D(HK)$; from the strongly continuity of $W(s,t)$ and the fact that K is closed, we have

$$\begin{aligned}
C^2HKx &= C\lim_{s\to 0}\frac{W(s,0)Kx - CKx}{s} \\
&= \lim_{s\to 0}\frac{1}{s}\left(W(s,0)\left(\lim_{t\to 0}\frac{W(0,t)x - Cx}{t}\right) - \lim_{t\to 0}\frac{W(0,t)x - Cx}{t}\right) \\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}(W(s,0)W(0,t)x - W(s,0)Cx - W(0,t)x + Cx) \\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}(W(0,t)W(s,0)x - W(s,0)Cx - W(0,t)x + Cx) \\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{t}\left(W(0,t)\left(\frac{W(s,0)x - Cx}{s}\right) - \frac{W(s,0)x - Cx}{s}\right) \\
&= C\lim_{s\to 0}K\left(\frac{W(s,0)x - Cx}{s}\right) \\
&= C^2KHx.
\end{aligned} \tag{2.27}$$

However, C is injective, and this completes the proof of (i).

To prove (iii), from our hypothesis, for sufficiently large λ, λ', we know that

$$L_{\lambda}(H)L_{\lambda'}(K) = L_{\lambda'}(K)L_{\lambda}(H). \tag{2.28}$$

By Lemma 2.4, $H_\lambda = \lambda^2L_\lambda(H) - \lambda C$ and $K_{\lambda'} = \lambda'^2L_{\lambda'}(H) - \lambda' C$, thus $H_\lambda K_{\lambda'} = K_{\lambda'}H_\lambda$. From (iii) of Lemma 2.4, for each $x \in D(H) \cap D(K)$,

$$u(s)x = \lim_{\lambda\to\infty} Ce^{sH_\lambda}x, \qquad v(t) = \lim_{\lambda'\to\infty} Ce^{tK_{\lambda'}}x. \tag{2.29}$$

So

$$\begin{aligned}
u(s)v(t)x &= C\lim_{\lambda\to\infty} e^{sH_\lambda}v(t)x \\
&= C^2\lim_{\lambda\to\infty} e^{sH_\lambda}\left(\lim_{\lambda'\to\infty} e^{tK_{\lambda'}}x\right), \\
\left(e^{sH_\lambda}\ \text{is continuous}\right) &= C^2\lim_{\lambda\to\infty}\lim_{\lambda'\to\infty} e^{sH_\lambda}e^{tK_{\lambda'}}x \\
&= C^2\lim_{\lambda\to\infty}\lim_{\lambda'\to\infty} e^{tK_{\lambda'}}e^{sH_\lambda}x \\
&= C\lim_{\lambda\to\infty} v(t)e^{sH_\lambda}x \\
&= v(t)u(s)x.
\end{aligned} \tag{2.30}$$

Now the continuity of $u(s)$ and $v(t)$ and the fact that $\overline{D(H)\cap D(K)} = \overline{R(C)} = X$ imply that for each $x \in X$, $u(s)v(t)x = v(t)u(s)x$. Thus

$$\begin{aligned}
W(s,t)W(s',t') &= u(s)v(t)u(s')v(t') \\
&= u(s)u(s')v(t)v(t') \\
&= Cu(s+s')Cv(t+t') \\
&= W(s+s',t+t')C^2.
\end{aligned} \tag{2.31}$$

On the other hand $W(0,0) = C^2$, which completes the proof. □

If H and K are two closed operators on X, then $X_1 := D(H)\cap D(K)$ with $\|x\|_1 = \|x\| + \|Hx\| + \|Kx\|$, $x \in X_1$, is a Banach space.

Proposition 2.7. *Suppose that $C \in B(X)$ is injective and $\{W(s,t)\}$ is a two-parameter C-semigroup with the generator (H,K). Then $W_1(s,t) := W(s,t)|_{X_1}$ defines a two-parameter C_1-semigroup, with the generator (H_1,K_1), where $C_1 = C|_{X_1}$, and H_1, K_1 are the part of H and K on X_1, respectively.*

Proof. The C_1-semigroup properties of $W_1(s,t)$ are obvious. Let (A,B) be the generator of $W_1(s,t)$; we show that $A = H_1$ and $B = H_2$. First we note that

$$\begin{aligned}
D(H_1) &= \{x \in X_1 : Hx \in X_1\} \\
&= \left\{x \in D(H)\cap D(K) : x \in D\left(H^2\right)\cap D(KH)\right\} \\
&= D(K)\cap D\left(H^2\right)\cap D(KH).
\end{aligned} \tag{2.32}$$

Let $x \in D(H_1)$. So we have

$$\begin{aligned}
\frac{W_1(s,0)x - C_1x}{t} &= \frac{W(s,0)x - Cx}{t} \longrightarrow CHx = C_1H_1x, \\
H\frac{W_1(s,0)x - C_1x}{t} &= \frac{W(s,0)Hx - CHx}{t} \longrightarrow CH^2x = HC_1H_1x, \\
K\frac{W_1(s,0)x - C_1x}{t} &= \frac{W(s,0)Kx - CKx}{t} \longrightarrow CHKx \\
&= KCHx = KC_1H_1x.
\end{aligned} \tag{2.33}$$

These show that $(W_1(s,0)x - C_1x)/t \to C_1H_1x$ in $\|\cdot\|_1$, that is, $x \in D(A)$ and $Ax = H_1x$. Hence $H_1 \subseteq A$. Conversely, if $x \in D(A) \subseteq X_1$, then

$$\begin{aligned}
\|\cdot\|_1 - \lim_{t\to 0}\frac{W(s,0)x - Cx}{t} &= \|\cdot\|_1 - \lim_{t\to 0}\frac{W_1(s,0)x - C_1x}{t} \\
&= C_1Ax \\
&= CAx,
\end{aligned} \tag{2.34}$$

so $Hx = Ax \in X_1$. Hence $x \in D(K) \cap D(H^2) \cap D(KH) = D(H_1)$ and $H_1x = Hx = Ax$.

A similar argument shows that $K_1 = B$, which completes the proof. □

3. Two-Parameter Abstract Cauchy Problems

Suppose that $H_i : D(H_i) \subseteq X \to X$, $i = 1, 2$, is linear operator. Consider the following two-parameter Cauchy problem:

$$2\text{-}ACP(H_1, H_2; x)\begin{cases} \dfrac{\partial}{\partial t_i}u(t_1,t_2) = H_iu(t_1,t_2), & t_i > 0,\ i = 1,2, \\ u(0,0) = x, & x \in C(D(H_1) \cap D(H_2)). \end{cases} \tag{3.1}$$

We mean by a solution a continuous Banach-valued function $u(\cdot,\cdot) : [0,\infty) \times [0,\infty) \to X$ which has continuous partial derivative and satisfies 2-$ACP(H_1, H_2; x)$.

In this section first we prove that if (H_1, H_2) is the infinitesimal generator of a two-parameter C-semigroup of operators, then 2-$ACP(H_1, H_2; x)$ has a unique solution for any $x \in C(D(H_1) \cap D(H_2))$. Next it is proved that under some condition on C, existence and uniqueness of solutions of 2-$ACP(H_1, H_2; Cx)$, for every $x \in D(H_1) \cap D(H_2)$, imply that this unique solution is induced by a two-parameter regularized semigroup.

Theorem 3.1. *Suppose that an extension of (H_1, H_2) is the generator of a two-parameter C-semigroup $W(s,t)$, then 2-$ACP(H_1, H_2; x)$ has the unique solution $u(s,t;x) := W(s,t)C^{-1}x$, for all $x \in C(D(H_1) \cap D(H_2))$.*

Proof. The fact that $u(s,t;x) := W(s,t)C^{-1}x$ is a solution of 2-$ACP(H_1, H_2; x)$ is obvious from Theorem 2.3. It is enough to show that 2-$ACP(H_1, H_2; x)$ has the unique solution $u(s,t) = 0$, for the initial value $x = 0$. From one-parameter case (see [2]), we know that the systems

$$\frac{du(t)}{dt} = H_1 u(t), \quad t \in \mathbb{R}_+, \qquad u(0) = 0, \tag{3.2}$$

$$\frac{dv(t)}{dt} = H_2 v(t), \quad t \in \mathbb{R}_+, \qquad v(0) = 0 \tag{3.3}$$

have the unique solution zero. Now if $u(s,t;0)$ is a solution of 2-$ACP(H_1, H_2; 0)$, then

$$u_1(s) := W(s,0)C^{-1}u(0,t;0), \qquad u_2(s) := u(s,t;0) \tag{3.4}$$

are two solutions of (3.2), for the initial value $u(0,t;0)$, since

$$\begin{aligned} \frac{d}{ds}u_1(s) &= \frac{d}{ds}W(s,0)C^{-1}u(0,t;0) \\ &= H_1 W(s,0)C^{-1}u(0,t;0) \\ &= H_1 u_1(s), \\ \frac{d}{ds}u_2(s) &= \frac{\partial}{\partial s}u(s,t;0) \\ &= H_1 u(s,t;0) \\ &= H_1 u_2(s). \end{aligned} \tag{3.5}$$

The uniqueness of solution in one-parameter case implies that $u_1(s) = u_2(s)$. So

$$W(s,0)C^{-1}u(0,t;0) = u(s,t;0). \tag{3.6}$$

Also $v_1(t) := W(0,t)C^{-1}u(s,0;0)$ and $v_2(t) := u(s,t;0)$ are two solutions of (3.3) for the initial value $u(s,0;0)$. From the uniqueness of solution in (3.3), $W(0,t)C^{-1}u(s,0;0) = u(s,t;0)$, for all $s,t \geq 0$. Thus

$$u(s,t;0) = W(s,0)C^{-1}u(0,t;0) = W(s,0)C^{-1}W(0,t)u(0,0;0) = 0. \tag{3.7}$$

□

The uniqueness of solution 2-$ACP(H, K; Cx)$, for all $x \in D(H) \cap D(K)$, also leads us to a two-parameter C-semigroup. This will be shown in the following theorem.

In this theorem X_1 and C_1 have their meaning in Proposition 2.7.

Theorem 3.2. *Suppose that $C \in B(X)$ is injective and H, K are two closed operators satisfying*

$$Cx \in X_1, \quad KCx = CKx, \quad HCx = CHx, \quad \forall x \in X_1. \tag{3.8}$$

If, for each $x \in X_1$, the Cauchy problem 2-$ACP(H, K; Cx)$ has a unique solution $u(\cdot, \cdot; Cx)$, then there exists a two-parameter C_1-semigroup $W_1(\cdot, \cdot)$ on X_1 such that $u(\cdot, \cdot; Cx) = W_1(\cdot, \cdot)x$. Moreover, the infinitesimal generator of $W_1(\cdot, \cdot)$ is a restriction of (H_1, K_1), where H_1 and K_1 are the part of H and K on X_1, respectively.

Proof. Suppose that, for any $x \in X_1$, 2-$ACP(H, K; Cx)$ has a unique solution $u(\cdot, \cdot; Cx) \in C^1([0, \infty) \times [0, \infty), X)$. For $x \in X_1$ and $0 < s, t < \infty$, define $W_1(s, t)x := u(s, t; Cx)$.

From the uniqueness of solution $W_1(s, t)$ is a well-defined and linear operator on X_1 and

$$W_1(0, 0)x = u(0, 0; x) = Cx. \tag{3.9}$$

By uniqueness of solutions one can see that

$$W_1(s + s', t + t')C_1 = W_1(s, t)W_1(s', t'). \tag{3.10}$$

We are going to show that $W_1(s, t)$ is a bounded operator on $(X_1, \|\cdot\|_1)$. Let $0 < s, t < \infty$. Define the mapping $\phi_{s,t} : X_1 \to C([0, s] \times [0, t], X_1)$ by $\phi_{s,t}x = W_1(\cdot, \cdot)x = u(\cdot, \cdot; Cx)$. Obviously $\phi_{s,t}$ is linear. We claim that this mapping is closed. Suppose that $x_n \in X_1$, $x_n \to x$ and $u(\cdot, \cdot; Cx_n) = \phi_{s,t}(x_n) \to y$ in $C([0, s] \times [0, t], X_1)$ with its usual supremum norm. From the Cauchy problem we know that

$$\begin{aligned} u(\mu, \nu; Cx_n) &= Cx_n + \int_0^{\mu} Hu(\eta, \nu; Cx_n)\,d\eta, \\ u(\mu, \nu; Cx_n) &= Cx_n + \int_0^{\nu} Ku(\mu, \eta; Cx_n)\,d\eta. \end{aligned} \tag{3.11}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} y(\mu, \nu) &= Cx + \int_0^{\mu} Hy(\eta, \nu)\,d\eta, \\ y(\mu, \nu) &= Cx + \int_0^{\nu} Ky(\mu, \eta)\,d\eta \end{aligned} \tag{3.12}$$

for any $(\mu,\nu) \in [0,s] \times [0,t]$. Now define $\tilde{y}$ on $[0,\infty) \times [0,\infty)$ by

$$\tilde{y}(\mu,\nu) = \begin{cases} Cy(\mu,\nu), & 0 \le \mu \le s,\ 0 \le \nu \le t, \\ W_1(0,\nu-t)y(\mu,t), & 0 \le \mu \le s,\ t < \nu < \infty, \\ W_1(\mu-s,0)y(s,\nu), & s < \mu < \infty,\ 0 \le \nu \le t, \\ W_1(\mu-s,\nu-t)y(s,t), & s < \mu < \infty,\ t < \nu < \infty. \end{cases} \tag{3.13}$$

One can see that $\tilde{y}$ is a solution of 2-$ACP(H,K;C^2x)$. Indeed from (3.12)

$$\tilde{y}(0,0) = Cy(0,0) = C^2x. \tag{3.14}$$

Also (3.12) and the fact that C commutes with H and K imply that

$$\begin{aligned} \frac{\partial}{\partial\mu}\tilde{y}(\mu,\nu) &= \begin{cases} Hy(\mu,\nu), & 0 \le \mu \le s,\ 0 \le \nu \le t, \\ HW_1(0,\nu-t)y(\mu,t), & 0 \le \mu \le s,\ t < \nu < \infty, \\ HW_1(\mu-s,0)y(s,\nu), & 0 < \mu < \infty,\ 0 \le \nu \le t, \\ HW_1(\mu-s,\nu-t)y(s,t), & 0 < \mu < \infty,\ 0 < \nu < \infty, \end{cases} \\ &= H\tilde{y}(\mu,\nu). \end{aligned} \tag{3.15}$$

Similarly

$$\frac{\partial}{\partial\nu}\tilde{y}(\mu,\nu) = K\tilde{y}(\mu,\nu). \tag{3.16}$$

Uniqueness of the solution implies that

$$\tilde{y}(\cdot,\cdot) = u\left(\cdot,\cdot;Cx^2\right) = W_1(\cdot,\cdot)Cx = CW_1(\cdot,\cdot)x. \tag{3.17}$$

In particular for $0 \le \mu \le s$ and $0 \le \nu \le s$,

$$Cy(\mu,\nu) = \tilde{y}(\mu,\nu) = CW_1(\mu,\nu)x = C\phi_{s,t}(x)(\mu,\nu). \tag{3.18}$$

The fact that C is injective implies that $y = \phi_{s,t}(x)$, which shows that $\phi_{s,t}$ is closed operator.

By the Closed Graph Theorem $\phi_{s,t}$ is a continuous operator from Banach space X_1 into the Banach space $C([0,s] \times [0,t], X_1)$. So if $x_n \to x$ in X_1, then $\phi_{s,t}(x_n) \to \phi_{s,t}(x)$ in $C([0,s] \times [0,t], X_1)$; thus for each $(\mu,\nu) \in [0,s] \times [0,t]$,

$$W_1(s,t)x_n = \phi_{s,t}(x_n)(\mu,\nu) \longrightarrow \phi_{s,t}(x)(\mu,\nu) = W_1(\mu,\nu)x. \tag{3.19}$$

But s and t were arbitrary; hence $W_1(\mu,\nu)$ is continuous for any $\mu,\nu\in[0,\infty)$. Also for every $x\in X_1$, $W_1(\cdot,\cdot)x=\phi_{s,t}(x)$ is continuous on $[0,s]\times[0,t]$; that is, $W_1(\cdot,\cdot)$ is strongly continuous family of operators.

Now let (A,B) be its infinitesimal generator and $x\in D(A)$, then

$$\|\cdot\|_1-\lim_{s\to 0}\frac{W_1(s,0)x-C_1x}{s}=C_1Ax, \tag{3.20}$$

which implies that $\lim_{s\to 0}((W_1(s,0)x-Cx)/s)=CAx$, but $D(A)\subseteq D(H)$

$$\begin{aligned}\lim_{s\to 0}\frac{W_1(s,0)x-Cx}{s}&=\lim_{s\to 0}\frac{u(s,0;Cx)-Cx}{s}\\&=\frac{\partial}{\partial s}u(0,0;Cx)\\&=HCx\\&=CHx.\end{aligned} \tag{3.21}$$

Hence $CHx=CAx$. The injectivity of C implies that $Hx=Ax\in X_1=D(H)\cap D(K)$. Thus $x\in D(K)\cap D(H^2)\cap D(KH)=D(H_1)$ and $H_1x=Ax$. This shows that A is a restriction of H_1. Similarly one can see that B is a restriction of K_1, which completes the proof. □

We conclude this section with a simple example as an application of our discussion. Consider the following sequence of initial value problems:

$$\begin{aligned}\frac{\partial}{\partial s}u_n(s,t)&=nu_n(s,t),\\\frac{\partial}{\partial t}u_n(s,t)&=n^2u_n(s,t),\quad n\in\mathbb{N},\\u_n(0,0)&=e^{-n^2}q_n.\end{aligned} \tag{3.22}$$

Suppose that $X=c_0$, the space of all complex sequences in $\mathbb{C}$ which vanish at infinity. Now define linear operators H and K in X and operator C on X as follows:

$$H(x_n)_{n\in\mathbb{N}}=(nx_n)_{n\in\mathbb{N}},\qquad K(x_n)_{n\in\mathbb{N}}=\left(n^2x_n\right)_{n\in\mathbb{N}},\qquad C(x_n)_{n\in\mathbb{N}}=\left(e^{-n^2}x_n\right)_{n\in\mathbb{N}}. \tag{3.23}$$

Using these operators the initial value problem (3.22) can be rewrite as follows:

$$\begin{aligned}\frac{\partial}{\partial s}u(s,t)&=Hu(s,t),\\\frac{\partial}{\partial t}u(s,t)&=Ku(s,t),\\u(0,0)&=Cq,\end{aligned} \tag{3.24}$$

where $u(s,t) = (u_n(s,t))_{n\in\mathbb{N}}$ and $q = (q_n)_{n\in\mathbb{N}}$. One can easily see that (H, K) is the generator of the following two-parameter C-semigroup:

$$W(s,t)(x_n)_{n\in\mathbb{N}} = (e^{n^2(t-1)+sn}x_n)_{n\in\mathbb{N}} \tag{3.25}$$

on X. Hence for every $q = (q_n)_{n\in\mathbb{N}} \in D(H) \cap D(K)$, by Theorem 3.1, the abstract Cauchy problem (3.24) has the unique solution

$$u(s,t) = W(s,t)q = (e^{n^2(t-1)+sn}q_n)_{n\in\mathbb{N}}. \tag{3.26}$$

This implies that for each $n \in \mathbb{N}$, $u_n(s,t) = e^{n^2(t-1)+tn}q_n$ is a solution of (3.22).

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