Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 363927, 11 pages
doi:10.1155/2009/363927

Research Article

Growth of Solutions of Nonhomogeneous Linear Differential Equations

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Received 5 November 2008; Accepted 17 February 2009

Recommended by Paul Eloe

This paper is devoted to studying growth of solutions of linear differential equations of type $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = H(z)$ where A_j $(j = 0, \ldots, k-1)$ and H are entire functions of finite order.

1. Introduction and Main Results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [1–3]. Let now $f(z)$ be a nonconstant meromorphic function in the complex plane. We remark that $\rho(f)$ will be used to denote the order of f, and

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{1.1}$$

We now recall some previous results concerning nonhomogeneous linear differential equations of type

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = H(z), \tag{1.2}$$

where A_j $(j = 0, 1, \ldots, k-1)$ and $A_0 \not\equiv 0$, $H \not\equiv 0$ are entire functions of finite-order, $k \geq 2$. In the case that the coefficients A_j $(j = 0, 1, \ldots, k-1)$ are polynomials, growth properties of solutions of (1.2) have been extensively studied, see, for example, [4]. In (1.2), if p is the largest integer such that A_p is transcendental, it is well known that there exist at most p linearly independent finite-order solutions of the corresponding homogeneous equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0. \tag{1.3}$$

Thus, when at least one of the coefficients A_j is transcendental, most of the solutions of (1.2) and (1.3) are of infinite-order. In the case when

$$\max_{j \neq d}\{\rho(A_j), \rho(H)\} < \rho(A_d) \leq \frac{1}{2}, \tag{1.4}$$

Hellerstein et al. [5] proved that every transcendental solution of (1.2) is of infinite-order. As for sectorial growth conditions on the coefficients of (1.2) that imply that all solutions are of infinite-order, see, for example, [6]. As for the special case of $k = 2$, Wang and Laine studied equations of type

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = H(z), \tag{1.5}$$

where $A_1 \not\equiv 0$, $A_0 \not\equiv 0$, H are entire functions of order less than one, and the complex numbers a, b satisfy $ab \not\equiv 0$. They proved that every nontrivial solution of (1.5) is of infinite-order if $a \not\equiv b$, see [7]. We remark that (1.2) may indeed have solutions of finite-order as soon as $\rho(H) \geq \max\{\rho(A_j)\ (j = 0, \ldots, k)\}$, as shown by the next examples.

Example 1.1. The exponential function $f(z) = e^z$ satisfies the equation

$$f^{(k)} + f^{(k-1)} + \cdots + f'' + e^{-z}f' + Q(z)f = (k - 1 + Q(z))e^z + 1, \tag{1.6}$$

where $Q(z)$ can be any entire function. Choosing $Q(z) = 1 - k$ shows that (1.2) may admit a solution of finite-order even if $\rho(H) < \max\{\rho(A_j)\ (j = 0, \ldots, k)\}$. On the other hand, taking $Q(z) = e^z$, we have the case that $\rho(H) = \max\{\rho(A_j)\ (j = 0, \ldots, k)\}$ in (1.2).

Example 1.2. The function $f(z) = e^{z^2}$ satisfies the equation

$$f''' + e^{-z}f'' + e^z f' + e^{2z}f = (8z^3 + 12z + 4z^2e^{-z} + 2e^{-z} + 2ze^z + e^{2z})e^{z^2}. \tag{1.7}$$

In this paper, we continue to consider (1.2) in the case when $\rho(H) < \max\{\rho(A_j)\ (j = 0, \ldots, k)\}$. Recently, Tu and Yi investigated the growth of solutions of (1.3) when most coefficients have the same order, see [8]. We next prove two results of (1.2), which generalize Theorems 2 and 4 in [8] and Theorem 1.1 in [7].

Theorem 1.3. *Suppose that $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, \ldots, k-1)$ where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ are polynomials with degree $n \geq 1$, $h_j(z)$ are entire functions of order less than n, not all vanishing, and $H(z) \not\equiv 0$ is an entire function of order less than n. If a_{jn} $(j = 0, \ldots, k-1)$ are distinct complex numbers, then every solution of* (1.2) *is of infinite-order.*

Theorem 1.4. *Suppose that* $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, \ldots, k-1)$ *where* $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ *are polynomials with degree* $n \geq 1$, $h_j(z)$ *and* $H(z) \not\equiv 0$ *are entire functions of order less than* n. *Moreover, suppose that there are two coefficients* A_s, A_l *so that for* $a_{sn} = |a_{sn}|e^{i\theta_s}$ *and* $a_{ln} = |a_{ln}|e^{i\theta_l}$, *where* $0 \leq s < l \leq k-1$, $\theta_s, \theta_l \in [0, 2\pi)$, $\theta_s \not\equiv \theta_l$, $h_s h_l \not\equiv 0$, *and for all* $j \not\equiv s, l$, a_{jn} *satisfies either* $a_{jn} = d_j a_{sn}$ $(0 < d_j < 1)$ *or* $a_{jn} = d_j a_{ln}$ $(0 < d_j < 1)$. *Then every transcendental solution of* (1.2) *is of infinite-order.*

In the case when $A_j(z) = h_j e^{a_j z} + g_j$ where h_j, g_j $(j = 0, \ldots, k-1)$ are polynomials, Chen considered the growth of solutions of (1.3) with some additional conditions imposed upon on a_j, see [9]. Our last results generalizes his result and [7, Theorem 1.3].

Theorem 1.5. *Suppose that* $A_j(z) = h_j(z)e^{P_j(z)} + g_j(z)$ $(j = 0, \ldots, k-1)$ *where* $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ *are polynomials with degree* $n \geq 1$, $h_j(z)$, $g_j(z)$ *and* $H(z) \not\equiv 0$ *are entire functions of order less than* n. *Moreover, suppose that there exist* $a_{sn} = d_s e^{i\varphi}$ *and* $a_{ln} = -d_l e^{i\varphi}$ *with* $d_s > 0$, $d_l > 0$ *and* $0 \leq s < l \leq k-1$ *such that for* $j \neq s, l$, $a_{jn} = d_j e^{i\varphi}$ $(d_j \geq 0)$ *or* $a_{jn} = -d_j e^{i\varphi}$ $(d_j \geq 0)$, *and* $\max\{d_j,\ j \not\equiv s, l\} = d < \min\{d_s, d_l\}$. *If* $h_s h_l \not\equiv 0$, *then every transcendental solution of* (1.2) *is of infinite-order.*

Remark 1.6. Under the assumptions of Theorem 1.4, respectively, of Theorem 1.5, polynomial solutions may exist. However, such possible polynomial solutions must be of degree less than s. If not, a contradiction immediately follows by combining (5.1) with Lemma 2.1, if $F \equiv 0$, respectively, with Lemma 2.2, if $F \not\equiv 0$.

Remark 1.7. In the preceding three theorems, if $\rho(f) = \infty$, then we also have $\lambda(f) = \infty$ for the exponent of convergence of the zero-sequence of f. Indeed, rewriting (1.2) in the form

$$\frac{1}{f} = \frac{1}{H}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right), \tag{1.8}$$

we have

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{H}\right) + \sum_{j=0}^{k-1} m(r, A_j) + \sum_{j=0}^{k-1} m\left(r, \frac{f^{(j)}}{f}\right) = O\left(r^{\beta}\right) + S(r, f), \tag{1.9}$$

for some finite β. Therefore, $N(r, 1/f)$ must be of infinite-order.

2. Preliminary Lemmas

Lemma 2.1 (see [10]). *Suppose that* $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \ldots, g_n(z)$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$,

(iii) *for* $1 \le j \le n, 1 \le h < k \le n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad (r \longrightarrow \infty,\ r \notin E), \tag{2.1}$$

where E is a set with finite linear measure.

Then $f_j \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.2 (see [10]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \ge 2)$ are linearly independent meromorphic functions satisfying the following identity:*

$$\sum_{j=1}^{n} f_j \equiv 1. \tag{2.2}$$

Then for $1 \le j \le n$, one has

$$T(r, f_j) \le \sum_{j=1}^{k} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) - \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r), \tag{2.3}$$

where D is the Wronskian determinant $W(f_1, f_2, \ldots, f_n)$,

$$S(r) = o\Big(\max_{1 \le k \le n}\{T(r, f_k)\}\Big), \quad (r \longrightarrow \infty, r \notin E), \tag{2.4}$$

E is a set with finite linear measure.

Lemma 2.3 (see [11, 12]). *Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \ne 0$) is a polynomial with degree $n \ge 1$, and that $A(z)(\not\equiv 0)$ is an entire function with $\rho(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos(n\theta) - \beta \sin(n\theta)$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ of finite linear measure such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, one has*

(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} < \left|g(re^{i\theta})\right| < \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\}; \tag{2.5}$$

(ii) *if* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} < \left|g(re^{i\theta})\right| < \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\}, \tag{2.6}$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$.

Lemma 2.4 (see [13]). *Let $f(z)$ be a transcendental meromorphic function of finite-order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1,\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0,1]$ and for all $k, j,\ 0 \le j < k$, one has*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.7}$$

Similarly, there exists a set $E \subset [0,2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0,2\pi) \setminus E$, and for all $k, j,\ 0 \le j < k$, the inequality (2.7) holds.

Lemma 2.5. *Let $f(z)$ be an entire function and suppose that*

$$G(z) := \frac{\log^+ \left|f^{(k)}(z)\right|}{|z|^{\rho}} \tag{2.8}$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1,2,\ldots)$, where $r_n \to \infty$, such that $G(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \le \frac{1}{(k-j)!}(1+o(1))r_n^{k-j}, \quad j = 0,\ldots,k-1, \tag{2.9}$$

as $n \to \infty$.

Proof. The first assertion is trivial. Denoting

$$M(r,G,\theta) = \max\left\{G(z) : 0 \le |z| \le r,\ \arg z = \theta\right\}, \tag{2.10}$$

we may take the sequence $\{z_n\}$ in the first assertion so that $G(z_n) = M(r_n,G,\theta)$. Since

$$G(z_n) \longrightarrow \infty \tag{2.11}$$

as $n \to \infty$, we immediately see that

$$\left|f^{(k)}(z_n)\right| = M\left(r_n, f^{(k)}, \theta\right) \longrightarrow \infty \tag{2.12}$$

as $n \to \infty$. Using now the same reasoning as in the proof of [14, Lemma 4], see also [15, Lemma 3.1], the second assertion (2.9) follows. □

Lemma 2.6. *Let $f(z)$ be an entire function with $\rho(f) = \rho < \infty$. Suppose that there exists a set $E \subset [0,2\pi)$ which has linear measure zero, such that $\log^+|f(re^{i\theta})| \le Mr^{\sigma}$ for any ray $\arg z = \theta \in [0,2\pi) \setminus E$, where M is a positive constant depending on θ, while σ is a positive constant independent of θ. Then $\rho(f) \le \sigma$.*

Proof. Clearly, we may assume that $\sigma < \rho$. Since E has linear measure zero, we may choose $\theta_j \in [0, 2\pi) \setminus E$ such that $0 \le \theta_1 < \theta_2 < \cdots < \theta_{n+1} = 2\pi$, and

$$\max\{\theta_{j+1} - \theta_j,\ 1 \le j \le n\} \le \frac{\pi}{\rho + 1}. \tag{2.13}$$

We first treat the sector

$$H_1 := \{z \mid \theta_1 \le \arg z \le \theta_2\}, \tag{2.14}$$

defining

$$\phi(z) = f(z) \exp\{-be^{-i\theta_0} z^\sigma\}, \tag{2.15}$$

where $\theta_0 = \sigma(\theta_1 + \theta_2)/2$ and b is a positive constant, to be determined in what follows. Then $\phi(z)$ is a holomorphic inside the sector H_1. By (2.13), we have $\rho \le \pi/(\theta_2 - \theta_1) - 1$. Therefore,

$$0 > \arg\left(e^{-i\theta_0} z^\sigma\right) = \arg\left(e^{-i\theta_0} r^\sigma e^{i\sigma\theta_1}\right) = \frac{\sigma(\theta_1 - \theta_2)}{2} \ge \frac{-\pi}{2} + \frac{(\theta_2 - \theta_1)}{2} \tag{2.16}$$

on the ray $\arg z = \theta_1$, and, respectively,

$$0 < \arg\left(e^{-i\theta_0} z^\sigma\right) = \arg\left(e^{-i\theta_0} r^\sigma e^{i\sigma\theta_2}\right) = \frac{\sigma(\theta_2 - \theta_1)}{2} \le \frac{\pi}{2} - \frac{(\theta_2 - \theta_1)}{2} \tag{2.17}$$

on the ray $\arg z = \theta_2$. Hence, we may now fix $b > 0$ so that

$$b \cos\left(\frac{\pi}{2} - \frac{(\theta_2 - \theta_1)}{2}\right) > M. \tag{2.18}$$

By elementary computation, $|\phi(z)| \le M$ on the boundary of H_1, where $M > 0$ is a bounded constant, not the same at each occurrence. By the definition of ϕ in (2.15), it is immediate to see that ϕ is of order at most ρ. By the Phragmén-Lindelöf theorem, we conclude that $|\phi(z)| \le M$ holds on the whole sector H_1. Hence

$$\left|f(z)\right| \le \left|\exp\left\{be^{-i\theta_0} z^\sigma\right\}\right| \le \exp\left\{br^\sigma\right\} \tag{2.19}$$

on H_1. Repeating the same reasoning for all the sectors $H_j = \{z \mid \theta_j \le \arg z \le \theta_{j+1}\}$ where θ_j are determined in (2.13), the assertion immediately follows. □

3. Proof of Theorem 1.3

Suppose, contrary to the assertion, that f is a solution of (1.2) with $\rho(f) = \rho < \infty$, then $n \le \rho$. Indeed, if $f^{(k)} = H$, we may apply Lemma 2.1 to conclude that $h_s f^{(s)} \equiv 0$ for some s,

$0 \le s \le k-1$ such that $h_s \neq 0$. Then f has to be a polynomial of degree less than s, so $H(z) \equiv 0$, a contradiction. Therefore, we may assume that $f^{(k)} \not\equiv H$. By Lemma 2.2, it is easy to see that $n \le \rho$ since the exponential functions e^{P_j} $(j = 0, 1, \ldots, k-1)$ are linearly independent.

By Lemma 2.3, there is a set $E \subset [0, 2\pi)$ of linear measure such that whenever $\theta \in [0, 2\pi) \setminus E$, then $\delta(P_j, \theta) \neq 0$ for all $0 \le j \le k-1$ and $\delta(P_j - P_i, \theta) \neq 0$ for all i, j with $0 \le i < j \le k-1$. If, moreover, $z = \mathrm{re}^{i\theta}$ has r large enough, then each $A_j(z)$ satisfies either (2.5) or (2.6). By Lemma 2.4, we may assume, at the same time, that

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \le |z|^{k\rho}, \quad 0 \le i < j \le k. \tag{3.1}$$

Since a_{jn} are distinct complex numbers, then for any fixed $\theta \in [0, 2\pi) \setminus E$, there exists exactly one $s \in \{0, \ldots, k-1\}$ such that

$$\delta(P_s, \theta) = \delta := \max\{\delta(P_j, \theta) \mid j = 0, \ldots, k-1\}. \tag{3.2}$$

Denoting $\delta_1 = \max\{\delta(P_j, \theta) \mid j \neq s\}$, then $\delta_1 < \delta$ and $\delta \neq 0$. We now discuss two cases separately.

Case 1. Assume first that $\delta > 0$. By Lemma 2.3, for any given ε with $0 < 3\varepsilon < \min\{(\delta - \delta_1)/\delta,\ n - \rho(H)\}$, we have

$$\begin{aligned} \left|A_s(\mathrm{re}^{i\theta})\right| &\ge \exp\{(1-\varepsilon)\delta r^n\}, \\ \left|A_j(\mathrm{re}^{i\theta})\right| &\le \exp\{(1+\varepsilon)\delta_1 r^n\}, \end{aligned} \tag{3.3}$$

for $j \neq s$, provided that r is sufficiently large. We now proceed to show that

$$\frac{\log^+\left|f^{(s)}(z)\right|}{|z|^{\rho(H)+\varepsilon}} \tag{3.4}$$

is bounded on the ray $\arg z = \theta$. Supposing that this is not the case, then by Lemma 2.5, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to \infty$, and that

$$\frac{\log^+\left|f^{(s)}(z_m)\right|}{r_m^{\rho(H)+\varepsilon}} \longrightarrow \infty, \tag{3.5}$$

$$\left| \frac{f^{(j)}(z_m)}{f^{(s)}(z_m)} \right| \le (1 + o(1)) r_m^{s-j}, \quad (j = 0, \ldots, s-1). \tag{3.6}$$

From (3.5) and the definition of order, it is easy to see that

$$\left| \frac{H(z_m)}{f^{(s)}(z_m)} \right| \longrightarrow 0, \tag{3.7}$$

for m is large enough. From (1.2), we obtain

$$\begin{aligned} \left|A_s(z_m)\right| \le& \left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right| + \cdots + \left|A_{s+1}(z_m)\right|\left|\frac{f^{(s+1)}(z_m)}{f^{(s)}(z_m)}\right| + \left|A_{s-1}(z_m)\right|\left|\frac{f^{(s-1)}(z_m)}{f^{(s)}(z_m)}\right| \\ &+ \cdots + \left|A_0(z_m)\right|\left|\frac{f(z_m)}{f^{(s)}(z_m)}\right| + \left|\frac{H(z_m)}{f^{(s)}(z_m)}\right|. \end{aligned} \tag{3.8}$$

Using inequalities (3.1), (3.3), (3.6), and the limit (3.7), we conclude from the preceding inequality that

$$\exp\left\{(1-\varepsilon_1)\delta r_m^n\right\} \le (k+1)\,\exp\left\{(1+\varepsilon_1)\delta_1 r_m^n\right\} r_m^M, \tag{3.9}$$

where $M > 0$ is a bounded constant, which is a contradiction. Therefore, $\log^+|f^{(s)}(z)|/|z|^{\rho(H)+\varepsilon}$ is bounded, and we have $|f^{(s)}(z)| \le M\exp\{r^{\rho(H)+\varepsilon}\}$ on the ray $\arg z = \theta$. By the same reasoning as in the proof of [15, Lemma 3.1], we immediately conclude that

$$\left|f(z)\right| \le (1+o(1))r^s\left|f^{(s)}(z)\right| \le (1+o(1))Mr^s e^{r^{\rho(H)+\varepsilon}} \le Me^{r^{\rho(H)+2\varepsilon}} \tag{3.10}$$

on the ray $\arg z = \theta$.

Case 2. Suppose now that $\delta < 0$. From (1.2), we get

$$-1 = A_{k-1}\frac{f^{(k-1)}}{f^{(k)}} + \cdots + A_j\frac{f^{(j)}}{f^{(k)}} + \cdots + A_0\frac{f}{f^{(k)}} - \frac{H}{f^{(k)}}. \tag{3.11}$$

Again by Lemma 2.3, for any given ε with $0 < 3\varepsilon < \min\{1, n-\rho(H)\}$, we have

$$\left|A_j(\mathrm{r}e^{i\theta})\right| \le \exp\left\{(1-\varepsilon)\delta r^n\right\}, \quad (j = 0, 1, \ldots, k-1), \tag{3.12}$$

for r sufficiently large. As in Case 1, we prove that

$$\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^{\rho(H)+\varepsilon}} \tag{3.13}$$

is bounded on the ray $\arg z = \theta$. If not, similarly as in Case 1, it follows from Lemma 2.5 that there is a sequence of points $z_m = r_m e^{i\theta}$, such that

$$\begin{gathered} \left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \le r_m^{k-j}(1+o(1)), \quad (j = 0, \ldots, k-1), \\ \left|\frac{H(z_m)}{f^{(k)}(z_m)}\right| \longrightarrow 0, \end{gathered} \tag{3.14}$$

for all m large enough. Substituting the inequalities (3.12) and (3.14) into (3.11), a contradiction immediately follows. Hence, we have $|f^{(k)}(z)| \le M\exp\{r^{\rho(H)+\varepsilon}\}$ on the ray $\arg z = \theta$. This implies, as in Case 1, that

$$|f(z)| \le M\exp\left\{r^{\rho(H)+2\varepsilon}\right\}. \tag{3.15}$$

Therefore, for any given $\theta \in [0, 2\pi) \setminus E$, E of linear measure zero, we have got (3.15) on the ray $\arg z = \theta$, provided that r is large enough. Then by Lemma 2.6, $\rho(f) \le \rho(H) + 2\varepsilon < n$, a contradiction. Hence, every transcendental solution of (1.2) must be of infinite-order.

4. Proof of Theorem 1.4

Suppose that f is a transcendental solution of (1.2) with $\rho(f) = \rho < \infty$.

If $f^{(k)} \equiv H$ and $\rho < n$, it follows from (1.2) that

$$f^{(l)}h_l e^{P_l(z)} + f^{(s)}h_s e^{P_s(z)} + \sum_{u=1}^{p} B_u(z)e^{d_{j_u}P_l(z)} + \sum_{v=1}^{q} C_v(z)e^{d_{j_v}P_s(z)} = 0, \tag{4.1}$$

where B_u $(u = 1, \dots, p)$, C_v $(v = 1, \dots, q)$ are entire functions of order less than n. Collecting terms of the same type together, if needed, we may assume that the coefficients d_{j_u} $(u = 1, \dots, p)$, respectively, d_{j_v} $(v = 1, \dots, q)$, are distinct. Since $\theta_s \ne \theta_l$ and $\theta_s, \theta_l \in [0, 2\pi)$, we conclude that $d_{j_u}P_l(z) - d_{j_v}P_s(z)$ are polynomials of degree n. Indeed, if $d_{j_u}a_{ln} = d_{j_v}a_{sn}$, we have

$$0 < \frac{d_{j_u}}{d_{j_v}}\left|\frac{a_{ln}}{a_{sn}}\right| = e^{i(\theta_s - \theta_l)} \tag{4.2}$$

which is impossible. Similarly, $P_l(z) - P_s(z), P_l(z) - d_{j_v}P_s(z)$, and $P_s(z) - d_{j_u}P_l(z)$ are also polynomials of degree n. Therefore, applying Lemma 2.1 to (4.1), we infer that $f^{(l)}h_l \equiv f^{(s)}h_s \equiv 0$. Since $h_s h_l \not\equiv 0$, f has to be a polynomial of degree less than s, then $H \equiv 0$, a contradiction.

Therefore, we may proceed under the assumption that $f^{(k)} \not\equiv H$. By Lemma 2.2, if $f^{(k)} \not\equiv H$, then $n \le \rho$ since the exponential functions $e^{P_l}, e^{P_s}, e^{d_{j_u}P_l}$ $(u = 1, 2, \dots, p)$ and $e^{d_{j_v}P_s}$ $(v = 1, 2, \dots, q)$ are linearly independent.

Since $\theta_s \ne \theta_l$, by Lemmas 2.3 and 2.4, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that whenever $\theta \in [0, 2\pi) \setminus E$ then $A_j(\mathrm{r}e^{i\theta})$ satisfies either (2.5) or (2.6), (3.1) holds, and

$$\delta(P_s, \theta) \ne \delta(P_l, \theta), \qquad \delta_2 := \max\{\delta(P_s, \theta), \delta(P_l, \theta)\} \ne 0. \tag{4.3}$$

In what follows, we apply the notations δ, δ_1 from the proof of Theorem 1.5 as well.

Case 1. Firstly assume that $\delta_2 > 0$. Without loss of generality, we may assume that $\delta_2 = \delta(P_s, \theta)$. From the hypothesis of a_{jn}, we know that $\delta_1 < \delta_2 = \delta$. Therefore, (3.3) holds by Lemma 2.3. Using the same reasoning as in Case 1 of the proof of Theorem 1.3, we obtain the inequality (3.15) on the ray $\arg z = \theta$.

Case 2. Finally, assume that $\delta_2 < 0$. Again by the condition on a_{jn}, we see that $\delta < 0$. Then the same argument as in Case 2 of the proof of Theorem 1.3 applies, and we again obtain (3.15).

Therefore, by Lemma 2.6, we obtain a contradiction, so $\rho(f) = \infty$.

5. Proof of Theorem 1.5

Contrary to the assertion, suppose that f is a transcendental solution of (1.2) of finite-order. If $\rho < n$, then it follows from (1.2) that

$$f^{(l)} h_l e^{P_l(z)} + f^{(s)} h_s e^{P_s(z)} + \sum_{u=1}^{p} B_u(z) e^{d_{j_u} P_l(z)} + \sum_{v=1}^{q} C_v(z) e^{d_{j_v} P_s(z)} = F(z), \tag{5.1}$$

where B_u $(u = 1, \dots, p)$, C_v $(v = 1, \dots, q)$, and $F(z)$ are entire functions of order less than n, $d_{j_u} \neq 0$ $(u = 1, \dots, p)$ are distinct, and $d_{j_v} \neq 0$ $(v = 1, \dots, q)$ are also distinct. Similarly as in the proof of Theorem 1.4, we may assume that $n \leq \rho$. Since $\sigma = \max\{\rho(g_j)\ (j = 0, \dots, k-1)\} < n$, we have

$$\max\left\{\left|g_j(z)\right|\ (j = 0, \dots, k-1),\ \left|H(z)\right|\right\} \leq \exp\left\{r^{\sigma+\varepsilon}\right\} \tag{5.2}$$

for any ε with $0 < 3\varepsilon < n - \sigma$, and for $|z|$ sufficiently large. Since d_s and d_l in $a_{sn} = d_s e^{i\varphi}$ and $a_{ln} = -d_l e^{i\varphi}$ are strictly positive, the set $\{\theta \in [0, 2\pi), \delta(P_s, \theta) = \delta(P_l, \theta)\}$ is of linear measure zero. Therefore, again by Lemmas 2.3 and 2.4, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for any given $\theta \in [0, 2\pi) \setminus E$, $h_j e^{P_j}$ satisfies either (2.5) or (2.6), and (3.1) holds. Moreover, $\delta(P_s, \theta) \neq \delta(P_l, \theta)$. Without loss of generality, we may assume that $\delta_2 := \max\{\delta(P_s, \theta), \delta(P_l, \theta)\} = \delta(P_l, \theta) = -d_l \cos(\varphi + n\theta)$, where $\cos(\varphi + n\theta) < 0$. Then from (2.5) and (5.2), for any ε also satisfying $0 < 3\varepsilon < (d_l - d) \setminus d_l$, we obtain for $|z|$ sufficiently large that

$$\left|A_l\left(re^{i\theta}\right)\right| \geq \exp\left\{-(1-\varepsilon) d_l \cos\left(\varphi + n\theta\right) r^n\right\}. \tag{5.3}$$

For all other coefficients A_j $(j \neq s)$, considering the hypothesis of a_{jn}, we have

$$\left|A_j\left(re^{i\theta}\right)\right| \leq \exp\left\{-(1+\varepsilon) d \cos(\varphi + n\theta) r^n\right\}, \tag{5.4}$$

when r is large enough. It follows from (1.2) that

$$-A_l = \frac{f^{(k)}}{f^{(s)}} + \cdots + A_{l+1} \frac{f^{(l+1)}}{f^{(l)}} + A_{l-1} \frac{f^{(l-1)}}{f^{(l)}} + \cdots + A_0 \frac{f}{f^{(l)}} - \frac{H}{f^{(l)}}. \tag{5.5}$$

Similarly as in Case 1 of the proof of Theorem 1.4, and using Lemma 2.5, we may prove that

$$\frac{\log^+\left|f^{(l)}(z)\right|}{|z|^{\rho(H)+\varepsilon}} \tag{5.6}$$

is bounded on the ray $\arg z = \theta$. Therefore, the inequality (3.15) always holds on the ray $\arg z = \theta$. Then, by Lemma 2.6, a contradiction follows, and so $\rho(f) = \infty$.

Acknowledgments

The authors wish to express their thanks to the referee for his valuable suggestions and comments. The authors have been partially supported by the National Natural Science Foundation of China (Grant No. 10871047) and the Academy of Finland (Grant no. 210245 and no. 124954).

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