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Research Article

Some Inequalities Concerning the Weakly Convergent Sequence Coefficient in Banach Spaces

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We establish two inequalities concerning the weakly convergent sequence coefficient and other parameters, which enable us to obtain some sufficient conditions for normal structure.

1. Introduction

Throughout this paper, we denote by B_X and S_X the unit ball and the unit sphere of a Banach space X, respectively. Let C be a nonempty bounded subset of X. The numbers $\operatorname{diam}(C) = \sup\{\|x-y\| : x, y \in C\}$ and $r(C) = \inf\{\sup\{\|x-y\| : y \in C\} : x \in C\}$ are, respectively, called the diameter and the Chebyshev radius of C. A Banach space X is said to have normal structure (resp., weak normal structure) if

$$\frac{\operatorname{diam}(C)}{r(C)} > 1 \tag{1.1}$$

for every closed bounded (resp., weakly compact) convex subset C of X with $\operatorname{diam}(C) > 0$. A Banach space X is said to have uniform normal structure (resp., weak uniform normal structure) if

$$\inf\left\{\frac{\operatorname{diam}(C)}{r(C)}\right\} > 1, \tag{1.2}$$

where the infimum is taken over all bounded closed (resp., weakly compact) convex subsets C of X with $\operatorname{diam}(C) > 0$. It is clear that normal structure and weak normal structure coincide when X is reflexive.

Throughout this paper, we assume that X does not have the Schur property. The weakly convergent sequence coefficient WCS(X), introduced by Bynum [1], is reformulated by Sims and Smyth [2] as the following equivalent form:

$$\mathrm{WCS}(X) = \inf\left\{ \lim_{n,m,n\neq m} \|x_n - x_m\| : x_n \xrightarrow{w} 0,\ \|x_n\| = 1,\ \lim_{n,m,n\neq m} \|x_n - x_m\| \text{ exists} \right\}. \tag{1.3}$$

Obviously $1 \le \mathrm{WCS}(X) \le 2$ and it is known that $\mathrm{WCS}(X) > 1$ implies that X has weak normal structure.

There are many geometric conditions implying a Banach space to have normal structure (see, e.g., [2–8]), among them

$$C_{\mathrm{NJ}}(X) < 1 + \frac{1}{(\mu(X))^2} \quad \text{or} \quad J(X) < 1 + \frac{1}{\mu(X)}. \tag{1.4}$$

Here, the coefficient $\mu(X)$ [9] is defined as the infimum of the set of real numbers $r > 0$ such that

$$\limsup_{n\to\infty} \|x + x_n\| \le r \limsup_{n\to\infty} \|x - x_n\| \tag{1.5}$$

for all $x \in X$ and all weakly null sequence (x_n) in X. The aim of this paper is to state some estimates concerning the weakly convergent sequence coefficient. By these estimates, we get sufficient conditions for normal structure in terms of the generalized James and von Neumann-Jordan constants and thus generalize the above results.

2. Generalized James constant

The James constant, or the nonsquare constant

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in S_X\} \tag{2.1}$$

is introduced by Casini [10] and Gao and Lau [11] and generalized by Dhompongsa et al. [4] in the following sense:

$$J(a, X) = \sup\{\|x + y\| \wedge \|x - z\| : x, y, z \in B_X,\ \|y - z\| \le a\|x\|\}, \tag{2.2}$$

where a is a nonnegative parameter. Obviously, $J(0, X) = J(X)$, since it is known that in the definition of $J(X)$, S_X can be replaced by B_X.

Theorem 2.1. *Let $0 \le a \le 1$ and let X be a Banach space without the Schur Property. Then*

$$\mathrm{WCS}(X) \ge \frac{1 + (1 + a)/\min(2, \mu(X) + a)}{J(a, X)}. \tag{2.3}$$

Proof. If $J(a,X)=2$, our estimate is trivial since $\mathrm{WCS}(X)\geq 1$ and $\mu(X)\geq 1$.

Suppose that $J(a,X)<2$. Then X is uniformly nonsquare and therefore reflexive (see [4]). Let $\{x_n\}$ be a weakly null sequence in S_X. Assume that $d=\lim_{n,m,n\neq m}\|x_n-x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n)=1$. Note that the reflexivity of X guarantees, by passing to a subsequence, if necessary, that there exists $x^*\in X^*$ such that $x_n^*\xrightarrow{w}x^*$. Let $0<\epsilon<1$ and choose N large enough so that $|x^*(x_N)|<\epsilon/2$ and

$$d-\epsilon<\|x_N-x_m\|<d+\epsilon \tag{2.4}$$

for all $m>N$. By the definition of $\mu(\mathrm{X})$,

$$\limsup_{m\to\infty}\left\|\frac{x_m+x_N}{d+\epsilon}\right\|\leq\mu(\mathrm{X})\limsup_{m\to\infty}\left\|\frac{x_N-x_m}{d+\epsilon}\right\|\leq\mu(\mathrm{X}). \tag{2.5}$$

Then we can choose $M>N$ large enough such that

(1) $|x_N^*(x_M)|<\epsilon$;

(2) $|(x_M^*-x^*)(x_N)|<\epsilon/2$;

(3) $\|(x_N+x_M)/(d+\epsilon)\|\leq\mu(\mathrm{X})+\epsilon$.

Hence

$$\left|x_M^*(x_N)\right|\leq\left|(x_M^*-x^*)(x_N)\right|+\left|x^*(x_N)\right|<\epsilon. \tag{2.6}$$

Let us put $x=(x_N-x_M)/(d+\epsilon)$,

$$y=\frac{(1+a)x_N+x_M}{(d+\epsilon)(\mu+a+\epsilon)},\qquad z=\frac{x_N+(1+a)x_M}{(d+\epsilon)(\mu+a+\epsilon)} \tag{2.7}$$

(for short $\mu=\mu(\mathrm{X})$). It follows that $x,y,z\in B_X$, $\|y-z\|\leq a\|x\|$ and

$$\begin{aligned}(d+\epsilon)\|x+y\|&=\left\|\left(1+\frac{1+a}{\mu+a+\epsilon}\right)x_N-\left(1-\frac{1}{\mu+a+\epsilon}\right)x_M\right\|\\&\geq\left(1+\frac{1+a}{\mu+a+\epsilon}\right)x_N^*(x_N)-\left(1-\frac{1}{\mu+a+\epsilon}\right)x_N^*(x_M)\\&\geq 1+\frac{1+a}{\mu+a+\epsilon}-\epsilon.\end{aligned} \tag{2.8}$$

Also

$$\begin{aligned}(d+\epsilon)\|x-z\|&=\left\|\left(1+\frac{1+a}{\mu+a+\epsilon}\right)x_M-\left(1-\frac{1}{\mu+a+\epsilon}\right)x_N\right\|\\&\geq\left(1+\frac{1+a}{\mu+a+\epsilon}\right)x_M^*(x_M)-\left(1-\frac{1}{\mu+a+\epsilon}\right)x_M^*(x_N)\\&\geq 1+\frac{1+a}{\mu+a+\epsilon}-\epsilon.\end{aligned} \tag{2.9}$$

This together with the definition of $J(a,X)$ gives that

$$(d+\epsilon)J(a,X) \geq 1 + \frac{1+a}{\mu+a+\epsilon} - \epsilon. \tag{2.10}$$

Since the sequence $\{x_n\}$ and ϵ are arbitrary, we get

$$\mathrm{WCS}(X) \geq \frac{\mu+1+2a}{J(a,X)(\mu+a)}. \tag{2.11}$$

Moreover, if we put $x = (x_N - x_M)/(d+\epsilon)$,

$$y = \frac{(1+a)x_N + (1-a)x_M}{2(d+\epsilon)}, \qquad z = \frac{(1-a)x_N + (1+a)x_M}{2(d+\epsilon)}. \tag{2.12}$$

It follows that $x, y, z \in B_X$, $\|y-z\| = a\|x\|$ and

$$\begin{aligned}(d+\epsilon)\|x+y\| &= \left\|\left(1+\frac{1+a}{2}\right)x_N - \left(1-\frac{1-a}{2}\right)x_M\right\| \\ &\geq \left(1+\frac{1+a}{2}\right)x_N^*(x_N) - \left(1-\frac{1-a}{2}\right)x_N^*(x_M) \\ &\geq 1+\frac{1+a}{2}-\epsilon.\end{aligned} \tag{2.13}$$

Also

$$\begin{aligned}(d+\epsilon)\|x-z\| &= \left\|\left(1+\frac{1+a}{2}\right)x_M - \left(1-\frac{1-a}{2}\right)x_N\right\| \\ &\geq \left(1+\frac{1+a}{2}\right)x_M^*(x_M) - \left(1-\frac{1-a}{2}\right)x_M^*(x_N) \\ &\geq 1+\frac{1+a}{2}-\epsilon.\end{aligned} \tag{2.14}$$

This together with the definition of $J(a,X)$ gives that

$$(d+\epsilon)J(a,X) \geq 1 + \frac{1+a}{2} - \epsilon. \tag{2.15}$$

Since the sequence $\{x_n\}$ and ϵ are arbitrary, we get

$$\mathrm{WCS}(X) \geq \frac{3+a}{2J(a,X)}. \tag{2.16}$$

Adding up (2.11) and (2.16) yields (2.3) as desired. □

Corollary 2.2. *Let X be a Banach space with*

$$J(a,X) < 1 + \frac{1+a}{\min(2,\, \mu(X)+a)}, \tag{2.17}$$

for some $0 \le a \le 1$. *Then X has normal structure.*

Corollary 2.3. *Let X be a Banach space with*

$$J(X) < 1 + \frac{1}{\min(2,\, \mu(X))}. \tag{2.18}$$

Then X has normal structure.

Remark 2.4. Corollary 2.3 includes [5, Theorem 2].

3. Generalized von Neumann-Jordan constant

The von Neumann-Jordan constant is introduced by Clarkson [12] and reformulated by Kato et al. [6] in the following way:

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x, y \in X,\ \|x\| + \|y\| \neq 0 \right\}. \tag{3.1}$$

The generalized version of this constant is given by Dhompongsa et al. [3] as

$$C_{\mathrm{NJ}}(a,X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} : \|x\| + \|y\| + \|z\| \neq 0,\ \|y-z\| \le a\|x\| \right\}, \tag{3.2}$$

where a is a nonnegative parameter. Obviously, $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(0,X)$.

Theorem 3.1. *Let* $0 \le a \le 1$ *and let X be a Banach space without the Schur Property. Then*

$$[\mathrm{WCS}(X)]^2 \ge \frac{1 + (1+a)^2/\min(\mu(X)+a, 2)^2}{C_{\mathrm{NJ}}(a,X)}. \tag{3.3}$$

Proof. If $C_{\mathrm{NJ}}(a,X) = 2$, then (3.3) is trivial.

Suppose that $C_{\mathrm{NJ}}(a,X) < 2$. Then X is uniformly nonsquare and therefore reflexive (see [3]). Let $\{x_n\}$ be a weakly null sequence in S_X and assume that $d = \lim_{n,m,n\neq m}\|x_n - x_m\|$ exists and let x_N, x_M, x_N^*, x_M^* be chosen as in Theorem 2.1.

Now let us put $x = (x_N - x_M)/(d+\epsilon)$,

$$y = \frac{(1+a)((1+a)x_N + x_M)}{(d+\epsilon)(\mu+a+\epsilon)^2}, \qquad z = \frac{(1+a)(x_N + (1+a)x_M)}{(d+\epsilon)(\mu+a+\epsilon)^2} \tag{3.4}$$

(for short $\mu = \mu(X)$). It is easy to check that $\|y - z\| \le a\|x\|$, $x \in B_X$:

$$\|y\| \le \frac{(1+a)}{(\mu + a + \epsilon)}, \qquad \|z\| \le \frac{(1+a)}{(\mu + a + \epsilon)}, \tag{3.5}$$

and also that

$$\begin{aligned}
(d+\epsilon)\|x+y\| &= \left\| \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right) x_N - \left(1 - \frac{1+a}{(\mu+a+\epsilon)^2}\right) x_M \right\| \\
&\ge \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right) x_N^*(x_N) - \left(1 - \frac{1+a}{(\mu+a+\epsilon)^2}\right) x_N^*(x_M) \\
&\ge \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right)(1-\epsilon), \\
(d+\epsilon)\|x-z\| &= \left\| \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right) x_M - \left(1 - \frac{1+a}{(\mu+a+\epsilon)^2}\right) x_N \right\| \\
&\ge \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right) x_M^*(x_M) - \left(1 - \frac{1+a}{(\mu+a+\epsilon)^2}\right) x_M^*(x_N) \\
&\ge \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right)(1-\epsilon).
\end{aligned} \tag{3.6}$$

By the definition of $C_{\mathrm{NJ}}(a, X)$,

$$C_{\mathrm{NJ}}(a, X) \ge \left(\frac{1-\epsilon}{d+\epsilon}\right)^2 \left(1 + \frac{(1+a)^2}{(\mu+a+\epsilon)^2}\right). \tag{3.7}$$

Since the sequence $\{x_n\}$ and ϵ are arbitrary, we get

$$[\mathrm{WCS}(X)]^2 C_{\mathrm{NJ}}(a, X) \ge 1 + \left(\frac{1+a}{\mu+a}\right)^2. \tag{3.8}$$

Moreover, if we put $x = (x_N - x_M)/(d+\epsilon)$,

$$y = \frac{(1+a)\big((1+a)x_N + (1-a)x_M\big)}{4(d+\epsilon)}, \qquad z = \frac{(1+a)\big((1-a)x_N + (1+a)x_M\big)}{4(d+\epsilon)}, \tag{3.9}$$

it follows that $x \in B_X$, $\|y - z\| \le a\|x\|$ and

$$\|y\| \le \frac{1+a}{2(d+\epsilon)} \le \frac{1+a}{2}, \qquad \|z\| \le \frac{1+a}{2(d+\epsilon)} \le \frac{1+a}{2}, \tag{3.10}$$

and also that

$$
\begin{aligned}
(d+\epsilon)\|x+y\| &= \left\|\left(1+\frac{(1+a)^2}{4}\right)x_N-\left(1-\frac{1-a^2}{4}\right)x_M\right\| \\
&\geq \left(1+\frac{(1+a)^2}{4}\right)x_N^*(x_N)-\left(1-\frac{1-a^2}{4}\right)x_N^*(x_M) \\
&\geq \left(1+\frac{(1+a)^2}{4}\right)(1-\epsilon), \\
(d+\epsilon)\|x-z\| &= \left\|\left(1+\frac{(1+a)^2}{4}\right)x_M-\left(1-\frac{1-a^2}{4}\right)x_N\right\| \\
&\geq \left(1+\frac{(1+a)^2}{4}\right)x_M^*(x_M)-\left(1-\frac{1-a^2}{4}\right)x_M^*(x_N) \\
&\geq \left(1+\frac{(1+a)^2}{4}\right)(1-\epsilon).
\end{aligned}
\tag{3.11}
$$

This together with the definition of $C_{\mathrm{NJ}}(a,X)$ gives that

$$
C_{\mathrm{NJ}}(a,X) \geq \left(1+\frac{(1+a)^2}{4}\right)\left(\frac{1-\epsilon}{d+\epsilon}\right)^2. \tag{3.12}
$$

Since the sequence $\{x_n\}$ and ϵ are arbitrary, we get

$$
[\mathrm{WCS}(X)]^2 C_{\mathrm{NJ}}(a,X) \geq 1+\left(\frac{1+a}{2}\right)^2. \tag{3.13}
$$

Adding up (3.8) and (3.13) yields the inequality (3.3) as desired. □

Corollary 3.2. *Let* X *be a Banach space with*

$$
C_{\mathrm{NJ}}(a,X) < 1+\frac{(1+a)^2}{\min\left(\mu(X)+a,2\right)^2} \tag{3.14}
$$

for some $a \in [0,1]$. *Then* X *has normal structure.*

Corollary 3.3. *Let* X *be a Banach space with*

$$
C_{\mathrm{NJ}}(X) < 1+\frac{1}{\min\left(2,\mu(X)\right)^2}. \tag{3.15}
$$

Then X *has normal structure.*

Remark 3.4. Corollary 3.3 includes [5, Theorem 1].

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