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Research Article

Some Sufficient Conditions for Analytic Functions to Belong to $\mathcal{Q}_{K,0}(p,q)$ Space

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This paper gives some sufficient conditions for an analytic function to belong to the space consisting of all analytic functions f on the unit disk such $\lim_{|a|\to 1}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q K(g(z,a))\,dA(z)=0$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For $a \in \mathbb{D}$, let

$$\varphi_a(z) = \frac{a-z}{1-\overline{a}z} \tag{1.1}$$

be the Möbius transformation of $\mathbb{D}$ and let

$$g(z,a) = \log\left|\frac{1-\overline{a}z}{a-z}\right| \tag{1.2}$$

be the Green's function on $\mathbb{D}$. Let $\mathbb{D}(a,r)$ denote the pseudo-hyperbolic metric disk centered at $a \in \mathbb{D}$ with radius $r \in (0,1)$, that is, $\mathbb{D}(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$.

It is said that an analytic function $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is defined by a lacunary series if

$$\lambda = \inf_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} > 1. \tag{1.3}$$

For some results in the topic, see, for example, [1–6] and the references therein.

Given a function $K : (0,\infty) \to [0,\infty)$, we consider the space $Q_K(p,q)$ of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{Q_K(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K\left(g(z,a)\right) dA(z) < \infty. \tag{1.4}$$

By $Q_{K,0}(p,q)$ we denote the space consisting of all $f \in Q_K(p,q)$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K\left(g(z,a)\right) dA(z) = 0, \tag{1.5}$$

where $0 < p < \infty$, $-2 < q < \infty$, and $dA(z) = (1/\pi)dx\,dy = (1/\pi)r\,dr\,d\theta$.

For $p = 2$, $q = 0$, the space $Q_K(p,q)$ is reduced to Q_K (see, e.g., [7]). If $K(g(z,a)) = (g(z,a))^s$, $0 \le s < \infty$, then $Q_K(p,q) = F(p,q,s)$ (see, e.g., [8, 9]).

Throughout the paper, we assume that the condition holds (see [7])

$$\int_0^1 \left(1-r^2\right)^q K\left(\log\frac{1}{r}\right) < \infty, \tag{1.6}$$

so that the space $Q_K(p,q)$ we study is nontrivial. We also assume that K as a nondecreasing function. An important tool in the study of Q_K spaces is the auxiliary function φ_K defined by (see [10])

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty. \tag{1.7}$$

The following condition

$$\int_1^\infty \varphi_K(s) \frac{ds}{s^2} < \infty \tag{1.8}$$

is crucial in this paper. It has played an important role in the study of Q_K spaces during the last few years.

In this paper, we give some sufficient conditions for an analytic function f to belong to the space $Q_{K,0}(p,q)$.

The followings are our main results in this paper.

Theorem 1.1. *Let $f \in H(\mathbb{D})$, $0 < p < \infty$, $-2 < q < \infty$, and let φ be a monotone increasing function in r on $(0,1)$ such that $|f'(z)| \le \varphi(r)$, for $|z| = r$. If*

$$\int_0^1 \varphi^p(r)\left(1-r^2\right)^q K\left(1-r^2\right) r\,dr < \infty, \tag{1.9}$$

then $f \in Q_{K,0}(p,q)$.

Theorem 1.2. *For $1 \le p < 2$, $0 \le q < \infty$, and $1 \le p - 2q < 3$. If K satisfies condition (1.7) and is a function with the property that $K(t) = K(1)$ for $t \ge 1$, then a lacunary series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ belongs to $Q_{K,0}(p,q)$ if*

$$\sum_{k=1}^{\infty} n_k^{p-q-1} \left|a_k\right|^p K\left(\frac{1}{n_k}\right) < \infty. \tag{1.10}$$

Throughout this paper, C stands for a positive constant, whose value may differ from one occurrence to the other. The expression $a \approx b$ means that there is a positive constant C such that $C^{-1}a \le b \le Ca$.

2. Main results and proofs

In this section, we give the proofs of Theorems 1.1 and 1.2. Before formulating the main results, we give some lemmas which are used in the proofs.

Lemma 2.1 (see [7]). *Let $0 < p < \infty$, $-2 < q < \infty$, $f \in H(\mathbb{D})$. Then, $f \in Q_{K,0}(p,q)$ if and only if*

$$\lim_{|a|\to 1}\int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K\left(1-\left|\varphi_a(z)\right|^2\right) dA(z) = 0. \tag{2.1}$$

Lemma 2.2 (see [5]). *Let K be a function with the property that $K(t) = K(1)$ for $t \geq 1$. If K satisfies condition* (1.8), *then there exists a constant $C > 0$ such that $K(2t) \approx K(t)$ for $t > 0$.*

Lemma 2.3 (see [5]). *If K satisfies condition* (1.8), *then we can find another nonnegative function K^* such that $Q_K = Q_{K^*}$ and the new function K^* has the following properties:*

(a) *K^* is nondecreasing on $(0,\infty)$;*

(b) *K^* satisfies condition $(*)$;*

(c) *$K^*(2t) \approx K^*(t)$ on $(0,\infty)$;*

(d) *K^* is differentiable (up to any given order) on $(0,\infty)$;*

(e) *K^* is concave on $(0,\infty)$;*

(f) *$K^*(t) = K^*(1)$ for $t \geq 1$;*

(g) *$K^*(t) \approx K(1)$ on $(0,1]$.*

Lemma 2.4 (see [5]). *If K satisfies condition* (1.8), *then for any $\alpha \geq 1$ and $0 \leq \beta < 1$, one has*

$$\int_0^1 r^{\alpha-1}\left(\log\frac{1}{r}\right)^{-\beta} K(\log\frac{1}{r})dr \approx C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right), \tag{2.2}$$

where $C(\beta)$ is a constant depending on β alone.

Lemma 2.5 (see [11]). *For $0 < p \leq 1$, $a \in \mathbb{D}$, and $z = re^{i\theta} \in \mathbb{D}$,*

$$\int_0^{2\pi} \frac{d\theta}{\left|1-\bar{a}re^{i\theta}\right|^{2p}} \leq \frac{C}{(1-|a|r)^p}, \tag{2.3}$$

where $C > 0$ is a constant.

Proof of Theorem 1.1. Let $z = re^{i\theta}$. By Lemma 2.3, we may also assume that K is concave, so that the following inequality true holds

$$\frac{1}{2\pi}\int_0^{2\pi} K\left(1-\left|\varphi_a(re^{i\theta})\right|^2\right)d\theta \leq K\left(\frac{1}{2\pi}\int_0^{2\pi}\left(1-\left|\varphi_a(re^{i\theta})\right|^2\right)d\theta\right). \tag{2.4}$$

From the definition of φ_K for $0 < s, t < 1$, we have that $K(st) \leq \varphi_K(s)K(t)$. Using these facts and polar coordinates, it follows that

$$\begin{aligned}
I(a) &= \int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^q K\left(1-\left|\varphi_a(z)\right|^2\right) dA(z) \\
&= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \left|f'(re^{i\theta})\right|^p (1-r^2)^q K\left(1-\left|\varphi_a(re^{i\theta})\right|^2\right) r\,dr\,d\theta \\
&\leq 2\int_0^1 \varphi^p(r)(1-r^2)^q\left(\frac{1}{2\pi}\int_0^{2\pi} K\left(1-\left|\varphi_a(re^{i\theta})\right|^2\right)d\theta\right) r\,dr \\
&\leq 2\int_0^1 \varphi^p(r)(1-r^2)^q K\left(\frac{1}{2\pi}\int_0^{2\pi} \left(1-\left|\varphi_a(re^{i\theta})\right|^2\right)d\theta\right) r\,dr \\
&= 2\int_0^1 \varphi^p(r)(1-r^2)^q K\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{\left|1-\overline{a}re^{i\theta}\right|^2}d\theta(1-r^2)\right) r\,dr \\
&= 2\int_0^1 \varphi^p(r)(1-r^2)^q K\left(\frac{1-|a|^2}{1-|a|^2r^2}(1-r^2)\right) r\,dr \\
&\leq 2\int_0^1 \varphi^p(r)(1-r^2)^q K(1-r^2)\varphi_K\left(\frac{1-|a|^2}{1-|a|^2r^2}\right) r\,dr.
\end{aligned} \tag{2.5}$$

The last integral exists for every $a \in \mathbb{D}$ in view of (1.9), and $\varphi_K((1-|a|^2)/(1-|a|^2r^2)) \leq 1$. Further, since

$$\lim_{|a|\to 1} \varphi_K\left(\frac{1-|a|^2}{1-|a|^2r^2}\right) = \varphi_K(0) = 0, \tag{2.6}$$

then the last integral tends to zero for every $r \in (0,1)$ as $|a| \to 1$. By Lebesgue's dominated convergence theorem, we obtain $\lim_{|a|\to 1} I(a) = 0$. By Lemma 2.1, we get $f \in Q_{K,0}(p,q)$. □

From Theorem 1.1, we have the following corollary. Here, we give a different and technical proof.

Corollary 2.6. *Let $f \in H(\mathbb{D})$, $0 < p < \infty$, $-2 < q < \infty$, $0 < s \leq 1$, and let φ be a monotone increasing function of r in $(0,1)$ such that $|f'(z)| \leq \varphi(r)$, for $|z| = r$. If*

$$\int_0^1 \varphi^p(r)(1-r^2)^{q+s} r\,dr < \infty, \tag{2.7}$$

then $f \in F_0(p,q,s)$.

Proof. Let $a \in \mathbb{D}$. We have

$$\begin{aligned}
J(a) &= \int_{\mathbb{D}} \left|f'(z)\right|^p (1-|z|^2)^q (g(z,a))^s dA(z) \\
&= \left(\int_{\mathbb{D}\setminus\mathbb{D}(a,1/2)} + \int_{\mathbb{D}(a,1/2)}\right) |f'(z)|^p (1-|z|^2)^q \left(\log\frac{1}{|\varphi_a(z)|}\right)^s dA(z) = I_1(a) + I_2(a).
\end{aligned} \tag{2.8}$$

For $z \in \mathbb{D} \setminus \mathbb{D}(a, 1/2)$, $1/|\varphi_a(z)| \le 2$, hence

$$\log\left|\frac{1-\overline{a}z}{a-z}\right| < \left|\frac{1-\overline{a}z}{a-z}\right| - 1 \le 4\left(1 - \left|\frac{a-z}{1-\overline{a}z}\right|^2\right) = 4\frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}. \tag{2.9}$$

For $0 < s \le 1$, by Lemma 2.5, we have

$$\begin{aligned} I_1(a) &= \int_{\mathbb{D}\setminus\mathbb{D}(a,1/2)} |f'(z)|^p (1-|z|^2)^q \left(\log\frac{1}{|\varphi_a(z)|}\right)^s dA(z) \\ &\le 4^s \int_{\mathbb{D}\setminus\mathbb{D}(a,1/2)} |f'(z)|^p (1-|z|^2)^{q+s} \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} dA(z) \\ &\le \frac{4^s}{\pi} \int_0^1 |f'(re^{i\theta})|^p (1-r^2)^{q+s} \left[\int_0^{2\pi} \frac{(1-|a|^2)^s d\theta}{|1-\overline{a}re^{i\theta}|^{2s}}\right] r\,dr \\ &\le \frac{2^s 4^s}{\pi} C \int_0^1 \varphi^p(r)(1-r^2)^{q+s} \frac{(1-|a|)^s}{(1-|a|r)^s} r\,dr. \end{aligned} \tag{2.10}$$

The last integral exists since $\int_0^1 \varphi^p(r)(1-r^2)^{q+s} r\,dr < \infty$ and $(1-|a|)/(1-|a|r) \le 1$. It is clear that

$$\lim_{|a|\to 1} \frac{(1-|a|)}{(1-|a|r)} = 0, \tag{2.11}$$

which implies

$$\lim_{|a|\to 1} \int_0^1 \varphi^p(r)(1-r^2)^{q+s} \frac{(1-|a|)^s}{(1-|a|r)^s} r\,dr = 0. \tag{2.12}$$

By Lebesgue's dominated convergence theorem, we get

$$\lim_{|a|\to 1} I_1(a) = 0. \tag{2.13}$$

Now, we consider the case $z \in \mathbb{D}(a, 1/2)$. Note that

$$\left|\frac{1-\overline{a}z}{a-z}\right| > 2 \tag{2.14}$$

in the case. By a well-known inequality (see, e.g., [12, page 3]), we have that

$$\frac{\big||z|-|a|\big|}{1-|a||z|} \le \frac{1}{2} \tag{2.15}$$

and consequently, for $z \in \mathbb{D}(a, 1/2)$, we have

$$|z| \le \frac{1+2|a|}{2+|a|}. \tag{2.16}$$

By the monotonicity of φ, we have

$$\begin{aligned}
I_2(a) &= \int_{\mathbb{D}(a,1/2)} \left|f'(z)\right|^p (1-|z|^2)^q \left(\log\frac{1}{\left|\varphi_a(z)\right|}\right)^s dA(z) \\
&\le \varphi^p\left(\frac{1+2|a|}{2+|a|}\right)\int_{\mathbb{D}(a,1/2)} (1-|z|^2)^q \left(\log\frac{1}{\left|\varphi_a(z)\right|}\right)^s dA(z) \\
&= \varphi^p\left(\frac{1+2|a|}{2+|a|}\right)\int_{|w|<1/2} \left(1-\left|\varphi_a(w)\right|^2\right)^q \left(\log\frac{1}{|w|}\right)^s \frac{(1-|a|^2)^2}{\left|1-\overline{a}w\right|^4} dA(w) \\
&= \varphi^p\left(\frac{1+2|a|}{2+|a|}\right)\int_{|w|<1/2} (1-|a|^2)^{q+2}(1-|w|^2)^q \left(\log\frac{1}{|w|}\right)^s \frac{1}{\left|1-\overline{a}w\right|^{2q+4}} dA(w) \\
&\le \frac{1}{\pi}\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)(1-|a|^2)^{q+2}\int_0^{1/2} (1-t^2)^q \left(\log\frac{1}{t}\right)^s \left[\int_0^{2\pi}\frac{d\theta}{(1-|\overline{a}|t)^{2q+4}}\right] t\,dt \\
&\le \frac{1}{\pi}\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)2^{q+2}(1-|a|)^{q+2}2\pi\int_0^{1/2} (1-t^2)^q \left(\log\frac{1}{t}\right)^s \frac{1}{(1-|\overline{a}|t)^{2q+4}} t\,dt \\
&\le \varphi^p\left(\frac{1+2|a|}{2+|a|}\right)2^{q+3}2^{2q+4}(1-|a|)^{q+2}\int_0^{1/2} (1-t^2)^q \left(\log\frac{1}{t}\right)^s t\,dt.
\end{aligned}$$

This implies

$$I_2(a) \le C\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)(1-|a|)^{q+2}, \tag{2.17}$$

since the following integral exists

$$\int_0^{1/2} (1-t^2)^q \left(\log\frac{1}{t}\right)^s t\,dt. \tag{2.18}$$

Choosing $s=1$, for every $r \in (0,1)$, it follows that

$$\int_r^1 \varphi^p(t)(1-t^2)^{q+1} t\,dt \ge \varphi^p(r)\int_r^1 (1-t^2)^{q+1} t\,dt = \frac{1}{2}\varphi^p(r)\frac{1}{q+2}(1-r^2)^{q+2}. \tag{2.19}$$

This and

$$\int_0^1 \varphi^p(r)(1-r^2)^{q+1} r\,dr < \infty \tag{2.20}$$

imply that

$$\lim_{r\to 1}\varphi^p(r)(1-r)^{q+2} = 0 \tag{2.21}$$

or

$$\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)\left(1-\frac{1+2|a|}{2+|a|}\right)^{q+2}=\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)\frac{(1-|a|)^{q+2}}{(2+|a|)^{q+2}}\longrightarrow 0, \tag{2.22}$$

as $|a|\to 1$. Consequently,

$$\lim_{|a|\to 1} I_2(a)\le C\lim_{|a|\to 1}\varphi^p\left(\frac{1+2|a|}{2+|a|}\right)(1-|a|)^{q+2}=0. \tag{2.23}$$

Combining (2.8), (2.13) with (2.23) we see that

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\left|f'(z)\right|^p\left(1-|z|^2\right)^q(g(z,a))^s dA(z)=0, \tag{2.24}$$

which means $f\in F_0(p,q,s)$. The proof is complete. □

Proof of Theorem 1.2. Consider the monotone increasing function

$$\varphi(r)=\sum_{k=1}^{\infty}n_k\left|a_k\right|r^{n_k-1},\quad 0<r<1. \tag{2.25}$$

For every $\theta\in[0,2\pi)$, we have

$$\left|f'\left(re^{i\theta}\right)\right|=\left|\sum_{k=1}^{\infty}n_k a_k r^{n_k-1}\right|\le\varphi(r), \tag{2.26}$$

By Theorem 1.1, we only need to prove

$$I=\int_0^1\left(\sum_{k=1}^{\infty}n_k\left|a_k\right|r^{n_k-1}\right)^p\left(1-r^2\right)^q K\left(1-r^2\right)r\,dr<\infty. \tag{2.27}$$

By the inequality $1-r^2\le 2\log(1/r)$, $r\in(0,1)$, and Lemma 2.2, there exists a constant C such that

$$K\left(1-r^2\right)\le K\left(2\log\frac{1}{r}\right)\approx CK\left(\log\frac{1}{r}\right). \tag{2.28}$$

Then for $0\le q<\infty$, we have

$$I\le C\int_0^1\left(\sum_{k=1}^{\infty}n_k\left|a_k\right|r^{n_k}\right)^p r^{1-p}\left(\log\frac{1}{r}\right)^q K\left(\log\frac{1}{r}\right)dr. \tag{2.29}$$

For $p\ge 1$, assume $I_n=\{k\colon 2^n\le k<2^{n+1},\ k\in\mathbb{N}\}$. Since

$$\sum_{n=0}^{\infty}2^{n/2}r^{2^n}\le 2^{1/2}\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}t^{-1/2}r^{t/2}dt\le 2^{1/2}\int_0^{\infty}t^{-1/2}r^{t/2}dt=2\Gamma\left(\frac{1}{2}\right)\left(\log\frac{1}{r}\right)^{-1/2}, \tag{2.30}$$

which together with Hölder's inequality gives

$$\begin{aligned}\left[\sum_{k=1}^{\infty} n_k|a_k|r^{n_k}\right]^p &= \left[\sum_{n=0}^{\infty}\sum_{n_k\in I_n} n_k|a_k|r^{n_k}\right]^p \le \left[\sum_{n=0}^{\infty}\sum_{n_k\in I_n} n_k|a_k|r^{2^n}\right]^p\\ &= \left[\sum_{n=0}^{\infty}\left(2^{n/2}r^{2^n}\right)^{1-1/p}\left(r^{2^n}2^{(1-p)n/2}\right)^{1/p}\sum_{n_k\in I_n} n_k|a_k|\right]^p\\ &\le \left[\sum_{n=0}^{\infty} r^{2^n}2^{((1-p)/2)n}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p\right]\left[\sum_{n=0}^{\infty}2^{n/2}r^{2^n}\right]^{p-1}\\ &\le C\left(\log\frac{1}{r}\right)^{-(p-1)/2}\sum_{n=0}^{\infty} r^{2^n}2^{((1-p)/2)n}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p.\end{aligned} \tag{2.31}$$

Hence,

$$I \le C\sum_{n=0}^{\infty}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p 2^{((1-p)/2)n}\int_0^1 r^{2^n+1-p}\left(\log\frac{1}{r}\right)^{q-((p-1)/2)} K\left(\log\frac{1}{r}\right)dr. \tag{2.32}$$

For $1 \le p \le 2$, $1 \le p-2q < 3$, by Lemma 2.4, choosing $\alpha = 2^n+2-p$, $\beta = (p-2q-1)/2$, we obtain

$$\begin{aligned} I &\le C\sum_{n=0}^{\infty}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p\left(\frac{(1/2)(2q-p+3)}{2^n+2-p}\right)^{(2q-p+3)/2} 2^{((1-p)/2)n}K\left(\frac{(1/2)(2q-p+3)}{2^n+2-p}\right)\\ &\le C\sum_{n=0}^{\infty}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p\left(\frac{1}{2^n}\right)^{(2q-p+3)/2}\left(\frac{1}{2^n}\right)^{(p-1)/2}K\left(\frac{1}{2^n}\right)\\ &= C\sum_{n=0}^{\infty}\left(\sum_{n_k\in I_n} n_k|a_k|\right)^p\left(\frac{1}{2^n}\right)^{1+q}K\left(\frac{1}{2^n}\right).\end{aligned} \tag{2.33}$$

If $n_k \in I_n$, then $n_k < 2^{n+1}$. The assumption that K is nondecreasing and Lemma 2.2 give

$$\left(\frac{1}{2^n}\right)^{1+q}K\left(\frac{1}{2^n}\right) \le \left(\frac{1}{2^n}\right)^{1+q}\frac{1}{2^{1+q}}K\left(\frac{1}{2^{n+1}}\right) < n_k^{-(1+q)}K\left(\frac{1}{n_k}\right). \tag{2.34}$$

Since $f(z)$ is a lacunary series, the Taylor series of f has most $[\log_\lambda 2]+1$ terms $a_k z^{n_k}$ such that $n_k \in I_n$. Combining this with the last inequality and Hölder's inequality, we obtain

$$\begin{aligned} I &\le C\sum_{n=0}^{\infty}\left(\sum_{n_k\in I_n} n_k^{1-((1+q)/p)}|a_k|K^{1/p}\left(\frac{1}{n_k}\right)\right)^p\\ &\le C\sum_{n=0}^{\infty}\left([\log_\lambda 2]+1\right)^{p-1}\sum_{n_k\in I_n} n_k^{p-q-1}|a_k|^p K\left(\frac{1}{n_k}\right)\\ &= C\left([\log_\lambda 2]+1\right)^{p-1}\sum_{k=1}^{\infty} n_k^{p-q-1}|a_k|^p K\left(\frac{1}{n_k}\right) < \infty.\end{aligned} \tag{2.35}$$

This shows that $f \in Q_{K,0}(p,q)$. □

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