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Research Article

Stability of Functional Inequalities with Cauchy-Jensen Additive Mappings

Young-Sun Cho and Hark-Mahn Kim

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We investigate the generalized Hyers-Ulam stability of the functional inequalities associated with Cauchy-Jensen additive mappings. As a result, we obtain that if a mapping satisfies the functional inequalities with perturbation which satisfies certain conditions, then there exists a Cauchy-Jensen additive mapping near the mapping.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x),h(x)) < \epsilon$ for all $x \in G$?

In 1941, Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers' inequality*

$$||f(x+y) - f(x) - f(y)|| \le \epsilon \tag{1.1}$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$||f(x) - L(x)|| \le \epsilon. \tag{1.2}$$

In 1978, Rassias [3] provided a generalization of Hyers' theorem which allows the *Cauchy difference to be unbounded*.

Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$.

Then, the limit $L(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.4}$$

for all $x \in E$. If $p < 0$, then inequality (1.3) holds for $x, y \neq 0$ and (1.4) for $x \neq 0$.

In 1991, Gajda [4], following the same approach as in Rassias [3], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [4] as well as by Rassias and Šemrl [5] that one cannot prove a Rassias-type theorem when $p = 1$. Inequality (1.3) that was introduced for the first time by Rassias [3] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept of stability is known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of Czerwik [6], Hyers et al. [7]).

Găvruţa [8] provided a further generalization of Rassias' theorem. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [9–14]).

Gilányi [15] and Rätz [16] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.5}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}). \tag{1.6}$$

Gilányi [17] and Fechner [18] proved the generalized Hyers-Ulam stability of the functional inequality (1.3).

Now, we consider the following functional inequalities:

$$\left\|f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z)\right\| \leq \left\|f\left(\frac{x+y}{2} + z\right)\right\| + \phi(x,y,z), \tag{1.7}$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + \phi(x,y,z), \tag{1.8}$$

which are associated with Jordan-von Neumann-type Cauchy-Jensen additive functional equations.

The purpose of this paper is to prove that if f satisfies one of the inequalities (1.7) and (1.8) which satisfies certain conditions, then we can find a Cauchy-Jensen additive mapping near f, and thus we prove the generalized Hyers-Ulam stability of the functional inequalities (1.7) and (1.8).

2. Stability of functional inequality (1.7)

We prove the generalized Hyers-Ulam stability of a functional inequality (1.7) associated with a Jordan-von Neumann-type 3-variable Cauchy-Jensen additive functional equation. Throughout this paper, let G be a normed vector space and Y a Banach space.

LEMMA 2.1. *Let $f : G \to Y$ be a mapping such that*

$$\left\|f\left(\frac{x-y}{2}-z\right)+f(y)+2f(z)\right\| \le \left\|f\left(\frac{x+y}{2}+z\right)\right\| \tag{2.1}$$

for all $x,y,z \in G$. Then, f is Cauchy-Jensen additive.

Proof. Letting $x,y,z := 0$ in (2.1), we get $\|4f(0)\| \le \|f(0)\|$. So, $f(0) = 0$.

And by setting $y := -x$ and $z := 0$ in (2.1), we get $\|f(x)+f(-x)\| \le \|f(0)\| = 0$ for all $x \in G$. Hence, $f(-x) = -f(x)$ for all $x \in G$.

Also by letting $x := 0$, $y := 2x$, and $z := -x$ in (2.1), we get $\|f(2x)+2f(-x)\| \le \|2f(0)\| = 0$ for all $x \in G$. Thus, $f(2x) = 2f(x)$ for all $x \in G$.

Letting $z = (-x-y)/2$ in (2.1), we get

$$\left\|f\left(\frac{x-y}{2}+\frac{x+y}{2}\right)+f(y)+2f\left(\frac{-x-y}{2}\right)\right\| \le ||f(0)|| = 0 \tag{2.2}$$

for all $x,y \in G$. Thus, $f(x+y) = f(x)+f(y)$ for all $x,y \in G$, as desired. □

THEOREM 2.2. *Assume that a mapping $f : G \to Y$ satisfies the inequality*

$$\left\|f\left(\frac{x-y}{2}-z\right)+f(y)+2f(z)\right\| \le \left\|f\left(\frac{x+y}{2}+z\right)\right\| + \phi(x,y,z) \tag{2.3}$$

and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the condition

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} 3^j \phi\left(\frac{x}{3^j},\frac{y}{3^j},\frac{z}{3^j}\right) < \infty \tag{2.4}$$

for all $x,y,z \in G$. Then, there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||A(x)-f(x)|| \le \Phi\left(-\frac{x}{3},x,-\frac{x}{3}\right)+\frac{3}{2}\Phi\left(\frac{x}{3},\frac{x}{3},-\frac{x}{3}\right) \tag{2.5}$$

for all $x \in G$.

Proof. Letting $y := x$ and $z := -x$ in (2.3), we get

$$||2f(x)+2f(-x)|| \le \phi(x,x,-x)+||f(0)|| \tag{2.6}$$

for all $x \in G$. And by letting $x := -x$, $y := 3x$, and $z := -x$ in (2.3), we get

$$||3f(-x)+f(3x)|| \le \phi(-x,3x,-x)+||f(0)|| \tag{2.7}$$

for all $x \in G$. It follows from (2.6) and (2.7) that

$$\|f(3x) - 3f(x)\| \leq \phi(-x,3x,-x) + \frac{3}{2}\phi(x,x,-x) + \frac{5}{2}\|f(0)\|. \tag{2.8}$$

Also letting $x,y,z := 0$ in (2.3), we get $3\|f(0)\| \leq \phi(0,0,0) = 0$. Hence, we have $f(0) = 0$.

Now, it follows from (2.8) that for all nonnegative integers m and l with $m > l$

$$\begin{aligned}\left\|3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|3^j f\left(\frac{x}{3^j}\right) - 3^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 3^j\left[\phi\left(-\frac{x}{3^{j+1}},\frac{x}{3^j},-\frac{x}{3^{j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{j+1}},\frac{x}{3^{j+1}},-\frac{x}{3^{j+1}}\right)\right]\end{aligned} \tag{2.9}$$

for all $x \in G$. It means that a sequence $\{3^n f(x/3^n)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{3^n f(x/3^n)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 3^n f(x/3^n)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get the approximation (2.5) of f by A.

Next, we claim that the mapping $A : G \to Y$ is Cauchy-Jensen additive. In fact, it follows easily from (2.3) and condition of ϕ that

$$\begin{aligned}\left\|A\left(\frac{x-y}{2} - z\right) + A(y) + 2A(z)\right\| &= \lim_{n\to\infty} 3^n\left\|f\left(\frac{1}{3^n}\left(\frac{x-y}{2} - z\right)\right) + f\left(\frac{y}{3^n}\right) + 2f\left(\frac{z}{3^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 3^n\left[\left\|f\left(\frac{1}{3^n}\left(\frac{x+y}{2} + z\right)\right)\right\| + \phi\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right)\right] \\ &= A\left(\frac{x+y}{2} + z\right).\end{aligned} \tag{2.10}$$

Thus, the mapping $A : G \to Y$ is Cauchy-Jensen additive by Lemma 2.1.

Now, let $T : G \to Y$ be another Cauchy-Jensen additive mapping satisfying (2.5). Then we obtain

$$\begin{aligned}&\|A(x) - T(x)\| \\ &\quad = 3^n\left\|A\left(\frac{x}{3^n}\right) - T\left(\frac{x}{3^n}\right)\right\| \\ &\quad \leq 3^n\left(\left\|A\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\| + \left\|T\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\|\right) \\ &\quad \leq 2\sum_{j=0}^{\infty} 3^j\left[\phi\left(-\frac{x}{3^{n+j+1}},\frac{x}{3^{n+j}},-\frac{x}{3^{n+j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{n+j+1}},\frac{x}{3^{n+j+1}},-\frac{x}{3^{n+j+1}}\right)\right] \\ &\quad \leq 2\sum_{j=n}^{\infty} 3^j\left[\phi\left(-\frac{x}{3^{j+1}},\frac{x}{3^j},-\frac{x}{3^{j+1}}\right) + \frac{3}{2}\phi\left(\frac{x}{3^{j+1}},\frac{x}{3^{j+1}},-\frac{x}{3^{j+1}}\right)\right],\end{aligned} \tag{2.11}$$

which tends to zero as $n \to \infty$. So, we can conclude that $A(x) = T(x)$ for all $x \in G$. This proves the uniqueness of A. Hence, the mapping $A : G \to Y$ is a unique Cauchy-Jensen additive mapping satisfying (2.5). □

THEOREM 2.3. *Assume that a mapping $f : G \to Y$ satisfies inequality (2.3) and that the map $\phi : G \times G \times G \to [0,\infty)$ satisfies the condition*

$$\Phi(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{3^j}\phi(3^j x, 3^j y, 3^j z) < \infty \tag{2.12}$$

for all $x,y,z \in G$.

Then, there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||A(x) - f(x)|| \le \frac{1}{3}\Phi(-x,x,-x) + \frac{1}{2}\Phi(x,x,-x) + \frac{5}{4}||f(0)|| \tag{2.13}$$

for all $x \in G$.

Proof. We get by (2.8)

$$\begin{aligned}
&\left\|\frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x)\right\| \\
&\quad \le \sum_{j=l}^{m-1}\left\|\frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x)\right\| \\
&\quad \le \sum_{j=l}^{m-1}\left[\left\|\frac{1}{3^j} f(3^j x) + \frac{1}{3^{j+1}} f(-3^{j+1} x)\right\| + \left\|\frac{1}{3^{j+1}} f(2^{j+1} x) + \frac{1}{3^{j+1}} f(-3^{j+1} x)\right\|\right] \\
&\quad \le \sum_{j=l}^{m-1}\frac{1}{3^{j+1}}\left[\phi(-3^j x, 3^{j+1} x, -3^j x) + \frac{3}{2}\phi(3^j x, 3^j x, -3^j x) + \frac{5}{2}||f(0)||\right]
\end{aligned} \tag{2.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{(1/3^n)f(3^n x)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{(1/3^n)f(3^n x)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty}(1/3^n)f(3^n x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.13).

The remaining proof goes through by the similar argument to Theorem 2.2. □

THEOREM 2.4. *Assume that a mapping $f : G \to Y$ satisfies inequality (2.3) and that the map $\phi : G \times G \times G \to [0,\infty)$ satisfies the condition*

$$\lim_{n\to\infty} 3^n \phi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0 \tag{2.15}$$

for all $x, y, z \in G$. If there exists a number L with $0 \le L < 1$ such that the mapping $x \mapsto \psi(x) := \phi(-x, 3x, -x) + (3/2)\phi(x, x, -x)$ satisfies

$$\psi(x) \le \frac{L}{3}\psi(3x), \tag{2.16}$$

then there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||f(x) - A(x)|| \le \frac{L \cdot \psi(x)}{3(1-L)} \tag{2.17}$$

for all $x \in G$.

Proof. We get by (2.8)

$$||f(3x) - 3f(x)|| \le \psi(x) = \phi(-x, 3x, -x) + \frac{3}{2}\phi(x, x, -x) \tag{2.18}$$

for all $x \in G$. Hence, we get

$$\begin{aligned} \left\|3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|3^j f\left(\frac{x}{3^j}\right) - 3^{j+1} f\left(\frac{x}{3^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 3^j \psi\left(\frac{x}{3^{j+1}}\right) \le \sum_{j=l}^{m-1} \frac{L^{j+1}}{3}\psi(x) \end{aligned} \tag{2.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{3^n f(x/3^n)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{3^n f(x/3^n)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 3^n f(x/3^n)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.19), we get (2.17).

The remaining proof goes through by the similar argument to Theorem 2.2. □

Corollary 2.5. *Assume that there exist nonnegative numbers θ and a real $p > 1$ such that a mapping $f : G \to Y$ satisfies the inequality*

$$\left\|f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z)\right\| \le \left\|f\left(\frac{x+y}{2} + z\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.20}$$

for all $x, y, z \in G$.

Then, there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||f(x) - A(x)|| \le \frac{\theta(13 + 2 \cdot 3^p)}{2(3^p - 3)}\|x\|^p \tag{2.21}$$

for all $x \in G$.

Theorem 2.6. *Assume that a mapping $f : G \to Y$ satisfies inequality (2.3) and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition*

$$\lim_{n\to\infty} \frac{1}{3^n}\phi(3^n x, 3^n y, 3^n z) = 0 \tag{2.22}$$

for all $x, y, z \in G$. If there exists a number L with $0 \le L < 1$ such that the mapping $x \mapsto \psi(x) := \phi(-x, 3x, -x) + (3/2)\phi(x, x, -x)$ satisfies

$$\psi(3x) \le 3L \cdot \psi(x), \tag{2.23}$$

then there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||f(x) - A(x)|| \le \frac{\psi(x)}{3(1-L)} + \frac{5}{4}||f(0)|| \tag{2.24}$$

for all $x \in G$.

Proof. We get by (2.8)

$$\begin{aligned}\left\|\frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x)\right\| &\le \sum_{j=l}^{m-1} \left\|\frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{3^{j+1}}\left[\psi(3^j x) + \frac{5}{2}||f(0)||\right] \\ &\le \sum_{j=l}^{m-1}\left[\frac{L^j \psi(x)}{3} + \frac{5}{2\cdot 3^{j+1}}||f(0)||\right]\end{aligned} \tag{2.25}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{(1/3^n) f(3^n x)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{(1/3^n) f(3^n x)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} (1/3^n) f(3^n x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.25), we get (2.24).

The remaining proof goes through by the similar argument to Theorem 2.3. □

Corollary 2.7. *Assume that there exist nonnegative numbers θ, δ, and a real $p < 1$ such that a mapping $f : G \to Y$ satisfies the inequality*

$$\left\|f\left(\frac{x-y}{2} - z\right) + f(y) + 2f(z)\right\| \le \left\|f\left(\frac{x+y}{2} + z\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) + \delta \tag{2.26}$$

for all $x, y, z \in G$.

Then, there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$||f(x) - A(x)|| \le \frac{\theta(13 + 2\cdot 3^p)\|x\|^p + 5\delta + 5||f(0)||}{2(3 - 3^p)} \tag{2.27}$$

for all $x \in G$.

3. Stability of functional inequality (1.8)

We prove the generalized Hyers-Ulam stability of a functional inequality (1.8) associated with a Jordan-von Neumann-type 3-variable Cauchy-Jensen additive functional equation.

Theorem 3.1. *Assume that a mapping $f : G \to Y$ satisfies the inequality*

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\| + \phi(x,y,z) \tag{3.1}$$

and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the conditions

(1) $\rho(x) := \sum_{j=0}^{\infty}(1/2^{j+1})[\phi(-2^{j+1}x,0,2^jx)+\phi_1(2^{j+1}x)] < \infty$,

(2) $\lim_{n\to\infty}(1/2^n)\phi(2^nx,2^ny,2^nz) = 0$ *for all $x,y,z\in G$,*

where

$$\phi_1(x) := \min\left\{\phi(x,-x,0)+4\|f(0)\|,\ \frac{1}{2}\phi(x,x,-x)+\|f(0)\|\right\}. \tag{3.2}$$

Then, there exists a unique Cauchy-Jensen additive mapping $A : G\to Y$ such that

$$\|A(x)-f(x)\| \le \rho(x) \tag{3.3}$$

for all $x\in G$.

Proof. Letting $x,y,z := 0$ in (3.1), we get $\|f(0)\| \le (1/2)\phi(0,0,0)$.

And by setting $x := 2x$, $y := 0$, and $z := -x$ in (3.1), we get

$$\|f(2x)+2f(-x)\| \le 3\|f(0)\| + \phi(2x,0,-x) \tag{3.4}$$

for all $x\in G$.

Also by letting $y := -x$ and $z := 0$ or by letting $y := x$ and $z := -x$ in (3.1), we get

$$\|f(x)+f(-x)\| \le \phi_1(x) = \min\left\{\phi(x,-x,0)+4\|f(0)\|,\ \frac{1}{2}\phi(x,x,-x)+\|f(0)\|\right\} \tag{3.5}$$

for all $x\in G$. Hence, we get by (3.4) and (3.5)

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^jx)-\frac{1}{2^{j+1}}f(2^{j+1}x)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left[\left\|\frac{1}{2^j}f(2^jx)+\frac{1}{2^{j+1}}f(-2^{j+1}x)\right\|+\left\|\frac{1}{2^{j+1}}f(2^{j+1}x)+\frac{1}{2^{j+1}}f(-2^{j+1}x)\right\|\right] \\
&\quad\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}[\phi(-2^{j+1}x,0,2^jx)+\phi_1(2^{j+1}x)]
\end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{(1/2^n)f(2^n x)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{(1/2^n)f(2^n x)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty}(1/2^n)f(2^n x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.3).

The remaining proof is similar to that of Theorem 2.3. □

Theorem 3.2. *Assume that a mapping $f : G \to Y$ satisfies inequality (3.1) and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the conditions*

(1) $\rho(x) := \sum_{j=0}^{\infty} 2^j\phi(x/2^j, 0, -x/2^{j+1}) + 2^{j+1}\phi_2(x/2^{j+1}) < \infty$,

(2) $\lim_{n\to\infty} 2^n\phi(x/2^n, y/2^n, z/2^n) = 0$ *for all $x, y, z \in G$,*

where

$$\phi_2(x) := \min\left\{\phi(x,-x,0), \frac{1}{2}\phi(x,x,-x)\right\}. \tag{3.7}$$

Then, there exists a unique Cauchy-Jensen additive mapping $A : G \to Y$ such that

$$\|A(x) - f(x)\| \le \rho(x) \tag{3.8}$$

for all $x \in G$.

Proof. Letting $x, y, z := 0$ in (3.1), we get $\|f(0)\| \le (1/2)\phi(0,0,0) = 0$. So $f(0) = 0$.

Now, it follows from (3.4) and (3.5) that for all nonnegative integers m and l with $m > l$,

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\quad\le \sum_{j=l}^{m-1}\left[\left\|2^j f\left(\frac{x}{2^j}\right) + 2^{j+1} f\left(-\frac{x}{2^{j+1}}\right)\right\| + \left\|2^{j+1} f\left(-\frac{x}{2^{j+1}}\right) + 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|\right] \\
&\quad\le \sum_{j=l}^{m-1}\left[2^j\phi\left(\frac{x}{2^j}, 0, -\frac{x}{2^{j+1}}\right) + 2^{j+1}\phi_2\left(\frac{x}{2^{j+1}}\right)\right]
\end{aligned} \tag{3.9}$$

for all $x \in G$. It means that a sequence $\{2^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{2^n f(x/2^n)\}$ converges. So, one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of proof is similar to that of Theorem 2.2. □

Remark 3.3. Assume that a mapping $f : G \to Y$ satisfies inequality (3.1) and that the map $\phi : G \times G \times G \to [0,\infty)$ satisfies the conditions

(1) $\rho(x) := \sum_{j=0}^{\infty}(1/2^{j+2})[\phi(-2^{j+1}x,0,2^{j}x)+\phi(2^{j+1}x,0,-2^{j}x)] < \infty$,

(2) $\lim_{n\to\infty}(1/2^n)\phi(2^n x,2^n y,2^n z) = 0$ for all $x,y,z \in G$.

Then, there exists a unique Cauchy-Jensen additive mapping $L : G \to Y$ such that

$$\left\|L(x) - \frac{f(x)-f(-x)}{2}\right\| \le \rho(x) + 3\|f(0)\| \tag{3.10}$$

for all $x \in G$.

Proof. Let $g(x) := (f(x)-f(-x))/2$. Then, we get by (3.4)

$$\begin{aligned}\|2g(x)-g(2x)\| &\le \left\|f(x)+\frac{1}{2}f(-2x)\right\| + \left\|f(-x)+\frac{1}{2}f(2x)\right\| \\ &\le \frac{1}{2}[\phi(-2x,0,x)+\phi(2x,0,-x)] + 3\|f(0)\|\end{aligned} \tag{3.11}$$

for all $x \in G$. Hence, we get by (3.11)

$$\begin{aligned}&\left\|\frac{1}{2^l}g(2^l x) - \frac{1}{2^m}g(2^m x)\right\| \\ &\quad\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}g(2^j x) - \frac{1}{2^{j+1}}g(2^{j+1}x)\right\| = \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\left[\left\|2g(2^j x) - g(2^{j+1}x)\right\|\right] \\ &\quad\le \sum_{j=l}^{m-1}\frac{1}{2^{j+2}}[\phi(-2^{j+1}x,0,2^j x)+\phi(2^{j+1}x,0,-2^j x)+6\|f(0)\|]\end{aligned} \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{(1/2^n)g(2^n x)\}$ is a Cauchy sequence for all $x \in G$. So, one can define a mapping $L : G \to Y$ by $L(x) := \lim_{n\to\infty}(1/2^n)g(2^n x) = \lim_{n\to\infty}(1/2^n)[(f(2^n x)-f(-2^n x))/2]$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.10). Next, we claim that the mapping $L : G \to Y$ is a Cauchy-Jensen additive mapping. Note that $L(-x) = -L(x)$ because $g(-x) = -g(x)$. Then

$$\|L(x)+L(y)-L(x+y)\| = \lim_{n\to\infty}\frac{1}{2^n}\|g(2^n x)+g(2^n y)-g(2^n x+y)\|, \tag{3.13}$$

and so we obtain by (3.1) and (3.4),

$$\begin{aligned}
&\frac{1}{2^n}\|g(2^n x)+g(2^n y)+g(2^n(-x-y))\| \\
&\quad\le \frac{1}{2^{n+1}}\|f(2^n x)+f(2^n y)+2f(2^{n-1}(-x-y))\| \\
&\qquad+\frac{1}{2^{n+1}}\|-f(-2^n x)-f(-2^n y)-2f(2^{n-1}(x+y))\| \\
&\qquad+\frac{1}{2^{n+1}}\|-2f(2^{n-1}(-x-y))-f(2^n(x+y))\| \\
&\qquad+\frac{1}{2^{n+1}}\|f(2^n(-x-y))+2f(2^{n-1}(x+y))\| \\
&\quad\le \frac{1}{2^{n+1}}[\phi(2^n x,2^n y,2^{n-1}(-x-y))+\phi(-2^n x,-2^n y,2^{n-1}(x+y))+4\|f(0)\|] \\
&\qquad+\frac{1}{2^{n+1}}[\|6f(0)\|+\phi(-2^n(x+y),0,2^{n-1}(x+y))+\phi(2^n(x+y),0,-2^{n-1}(x+y))],
\end{aligned} \tag{3.14}$$

which tends to zero as $n\to\infty$ for all $x\in G$. Hence, we see that L is additive.

The remaining proof is similar to the corresponding part of Theorem 2.3. □

Remark 3.4. Assume that a mapping $f: G\to X$ satisfies inequality (3.1) and that the map $\phi: G\times G\times G\to[0,\infty)$ satisfies the conditions

(1) $\rho(x):=\sum_{j=0}^{\infty}2^{j-1}[\phi(-x/2^j,0,x/2^{j+1})+\phi(x/2^j,0,-x/2^{j+1})]<\infty$,

(2) $\lim_{n\to\infty}2^n\phi(x/2^n,y/2^n,z/2^n)=0$ for all $x,y,z\in G$.

Then, there exists a unique Cauchy-Jensen additive mapping $L: G\to Y$ such that

$$\left\|L(x)-\frac{f(x)-f(-x)}{2}\right\|\le\rho(x) \tag{3.15}$$

for all $x\in G$.

Proof. Letting $x,y,z:=0$ in (3.1), we get $\|f(0)\|\le(1/2)\phi(0,0,0)=0$. So $f(0)=0$.

Let $g(x):=(f(x)-f(-x))/2$. Then, we get by (3.4)

$$\begin{aligned}
\|2g(x)-g(2x)\| &\le \left\|f(x)+\frac{1}{2}f(-2x)\right\|+\left\|f(-x)+\frac{1}{2}f(2x)\right\| \\
&\le \frac{1}{2}[\phi(-2x,0,x)+\phi(2x,0,-x)]
\end{aligned} \tag{3.16}$$

for all $x\in G$. Hence, we get by (3.16)

$$\begin{aligned}
\left\|2^l g\left(\frac{x}{2^l}\right)-2^m g\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|2^j g\left(\frac{x}{2^j}\right)-2^{j+1}g\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\le \sum_{j=l}^{m-1}2^{j-1}\left[\phi\left(-\frac{x}{2^j},0,\frac{x}{2^{j+1}}\right)+\phi\left(\frac{x}{2^j},0,-\frac{x}{2^{j+1}}\right)\right]
\end{aligned} \tag{3.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that the sequence $\{2^n g(x/2^n)\}$ is a Cauchy sequence for all $x \in G$. So, one can define a mapping $L : G \to Y$ by $L(x) := \lim_{n\to\infty} 2^n g(x/2^n) = \lim_{n\to\infty} 2^n[(f(x/2^n) - f(-x/2^n))/2]$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.17), we get (3.15).

Next, we claim that the mapping $L : G \to Y$ is a Cauchy-Jensen additive mapping. Note that $L(-x) = -L(x)$ because $g(-x) = -g(x)$. So, we obtain by (3.1) and (3.4)

$$\begin{aligned}
&\|L(x)+L(y)-L(x+y)\| \\
&\quad= \lim_{n\to\infty} 2^n\left\|g\left(\frac{x}{2^n}\right)+g\left(\frac{y}{2^n}\right)-g\left(\frac{x+y}{2^n}\right)\right\| \\
&\quad= \lim_{n\to\infty} 2^n\left\|g\left(\frac{x}{2^n}\right)+g\left(\frac{y}{2^n}\right)+g\left(\frac{-x-y}{2^n}\right)\right\| \\
&\quad\le \lim_{n\to\infty} \frac{2^n}{2}\left[\left\|f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+2f\left(\frac{-x-y}{2^{n+1}}\right)\right\|\right. \\
&\qquad\qquad\left.+\left\|-f\left(\frac{-x}{2^n}\right)-f\left(\frac{-y}{2^n}\right)-2f\left(\frac{x+y}{2^{n+1}}\right)\right\|\right] \\
&\qquad+\lim_{n\to\infty} \frac{2^n}{2}\left[\left\|-2f\left(\frac{-x-y}{2^{n+1}}\right)-f\left(\frac{x+y}{2^n}\right)\right\|+\left\|f\left(\frac{-x-y}{2^n}\right)+2f\left(\frac{x+y}{2^{n+1}}\right)\right\|\right] \\
&\quad\le \lim_{n\to\infty} 2^{n-1}\left[\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{-x-y}{2^{n+1}}\right)+\phi\left(\frac{-x}{2^n},\frac{-y}{2^n},\frac{x+y}{2^{n+1}}\right)\right] \\
&\qquad+\lim_{n\to\infty} 2^{n-1}\left[\phi\left(\frac{x+y}{2^n},0,\frac{-x-y}{2^{n+1}}\right)+\phi\left(\frac{-x-y}{2^n},0,\frac{x+y}{2^{n+1}}\right)\right]=0
\end{aligned} \tag{3.18}$$

from the condition of ϕ. So, we have $L(x+y) = L(x) + L(y)$.

The remaining proof is similar to that of Theorem 2.2. □

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Young-Sun Cho: Department of Mathematics, Chungnam National University,
Daejeon 305-764, South Korea
Email address: yscho@cnu.ac.kr

Hark-Mahn Kim: Department of Mathematics, Chungnam National University,
Daejeon 305-764, South Korea
Email address: hmkim@cnu.ac.kr