ON LINEAR SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATIONS IN ONE FUNCTIONAL SPACE

ANDREI SHINDIAPIN

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We use a special space of integrable functions for studying the Cauchy problem for linear functional-differential equations with nonintegrable singularities. We use the ideas developed by Azbelev and his students (1995). We show that by choosing the function ψ generating the space, one can guarantee resolubility and certain behavior of the solution near the point of singularity.

1. Linear Volterra operators in Δ_ψ spaces

We consider the following n-dimensional functional-differential equation:

$$\mathcal{L}x \overset{\text{def}}{=} \dot{x} + (K+S)\dot{x} + Ax(0) = f, \tag{1.1}$$

where

$$(Ky)(t) = \int_0^t K(t,s)y(s)ds, \tag{1.2}$$

$$(Sy)(t) = \begin{cases} B(t)y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1]. \end{cases} \tag{1.3}$$

The case where K and S are continuous on $L_p[0,1]$ operators is well studied (see, e.g., [1] and the references therein). Here we suppose that the functions $K(t,s)$ and $B(t)$ may be nonintegrable at $t=0$. More precisely, we will formulate conditions on operators K and S in Sections 2 and 3. Under such conditions, those operators are not bounded on $L[0,1]$ and one has to choose other functional spaces for studying (1.1). We propose a space of integrable functions on $[0,1]$ and show that it may be useful in such a case.

We call Δ_ψ^p space the space of all measurable functions $y:[0,1]\to\mathbb{R}^n$, for which

$$\|y\|_{\Delta_\psi^p} = \sup_{0<h\le 1} \frac{1}{\psi(h)} \left(\int_0^h |y(s)|^p ds \right)^{1/p} < \infty. \tag{1.4}$$

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We assume everywhere below that ψ is a nondecreasing, absolutely continuous function, $\psi(0)=0$.

Theorem 1.1. *The space Δ_ψ^p is a Banach space.*

Let $X[a,b]$, $Y[a,b]$ be spaces of functions defined on $[a,b]$.

We will call $V: X[0,1] \to Y[0,1]$ the *Volterra* operator [3] if for every $\xi \in [0,1]$ and for any $x_1, x_2 \in X[0,1]$ such that $x_1(t)=x_2(t)$ on $[0,\xi]$, $(Vx_1)(t)=(Vx_2)(t)$ for $t\in[0,1]$.

It is possible to say that each Volterra operator $V: X[0,1]\to Y[0,1]$ generates a set of operators $V_\xi : X[0,\xi] \to Y[0,\xi]$, where $\xi \in (0,1]$. By y_ξ, we denote the restriction of function y defined on $[0,1]$ onto segment $[0,\xi]$.

Theorem 1.2. *Let $V: L\to L$ be a linear bounded operator. Then V is a linear bounded operator in Δ_ψ^p and $\|V\|_{\Delta_\psi^p} \le \|V\|_{L^p}$.*

Proof. Let $y\in\Delta_\psi^p$. Then

$$\begin{aligned} ||Vy||_{\Delta_\psi^p} &= \sup_{0<h\le 1} \frac{1}{\psi(h)} ||(V_\xi y_\xi)||_{L[0,\xi]^p} \\ &\le \sup_{0<h\le 1} \frac{1}{\psi(h)} ||V_\xi||_{L[0,\xi]} ||y_\xi||_{L[0,\xi]} \le \|V\|_L \|y\|_{L^p}. \end{aligned} \tag{1.5}$$

□

Theorem 1.3. *Let $V: \Delta_{\psi_1}^p \to \Delta_{\psi_1}^p$ be linear bounded operator and let*

$$\sup_{t\in[0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty. \tag{1.6}$$

Then V is linear and bounded in $\Delta_{\psi_2}^p$ and

$$\|V\|_{\Delta_{\psi_2}^p} \le \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \tag{1.7}$$

Proof. Let $y\in\Delta_{\psi 2}$. Then

$$\begin{aligned} \|Vy\|_{\Delta_{\psi_2}^p} &\le \sup_{\xi\in[0,1]} \frac{||Vy_\xi||_{L[0,\xi]}\psi_1(\xi)}{\psi_2(\xi)\psi_1(\xi)} \le \sup_{\xi\in[0,1]} \frac{||Vy_\xi||_{\Delta_{\psi_1}^p[0,\xi]}\psi_1(\xi)}{\psi_2(\xi)} \\ &\le \|V\|_M {\psi_1}^p \sup \frac{||y_\psi||_{\Delta_{\psi_1}^p}\psi_1(\xi)}{\psi_2(\xi)} \\ &\le \|V\|_M {\psi_1}^p \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\psi]} \frac{||y_\tau||_{L[0,\tau]}\psi_1(\xi)\psi_2(\tau)}{\psi_1(\tau)\psi_2(\xi)\psi_2(\tau)} \\ &\le \|y\|_{\Delta_{\psi_2}^p} \|V\|_{\Delta_{\psi 1}^p} \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \end{aligned} \tag{1.8}$$

□

Corollary 1.4. *If $V_1: \Delta_{\psi_1}^p \to \Delta_{\psi_1}^p$ and $V_2: \Delta_{\psi_2}^p \to \Delta_{\psi_2}^p$ are linear continuous Volterra operators, then $V=V_1+V_2$ is continuous on space Δ_ψ^p generated by $\psi(t)=\min(\psi_1(t),\psi_2(t))$ and $\|V\|_{\Delta_\psi^p} \le \|V_1\|_{\Delta_{\psi_1}^p} + \|V_2\|_{\Delta_{\psi_2}^p}$.*

2. Operator K

In this section, we consider the integral operator (1.2). We will show that under certain conditions on matrix $K(t,s)$, a function ψ may be indicated such that K is bounded on Δ_ψ and its norm is limited by a given number.

We say that matrix $K(t,s)$ satisfies the $\mathcal{N}$ condition if for some p and p_1 such that $1 \le p \le p_1 < \infty$ and for any $\varepsilon \in (0,1]$,

$$||K_\varepsilon(t,\cdot)||_{L[0,t]} \in L_{p'}[\varepsilon,1]. \tag{2.1}$$

Here $K_\varepsilon(t,s)$ is a restriction of $K(t,s)$ onto $[\varepsilon,1]\times[0,t]$, $1/p+1/p'=1$.

The $\mathcal{N}$ condition admits a nonintegrable singularity at point $t=0$.

Lemma 2.1. *Let nonnegative function $\omega:[0,1]\to\mathbb{R}$ be nonincreasing and having a nonintegrable singularity at $t=0$.*

Then $\psi(t)=\exp[\int_1^t\omega(s)ds]$ is absolutely continuous on $[0,1]$, does not decrease, and is a solution of the equation $\int_1^t\omega(s)x(s)ds=x(t)$.

Denote

$$\psi(t)=\exp\left[\frac{1}{C}\int_1^t \operatorname*{vrai\,sup}_{s\in[0,\tau]}||K(\tau,s)||d\tau\right]. \tag{2.2}$$

Theorem 2.2. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $p=1$ and let C be some positive constant. Then operator K is bounded in Δ_ψ with function ψ defined by the equality (2.2) and $\|K\|_{\Delta_\psi}\le C$.*

Proof. Let $x\in\Delta_\psi$ and $y=Kx$. From the $\mathcal{N}$ condition it follows that for almost all $t\in[0,1]$, $K(\cdot,s)\in L_\infty$. Let $\omega(t)=\operatorname{vrai\,sup}_{s\in[0,\tau]}\|K(\tau,s)\|d\tau$. Then

$$\begin{aligned}\left(\int_0^t||y(s)||ds\right) &\le \left[\int_0^t\left(\int_0^\tau||K(\tau,s)||||x(s)||ds\right)d\tau\right]\\ &\le \int_0^t\left(\operatorname*{vrai\,sup}_{s\in[0,\tau]}||K(\tau,s)||\right)\left(\int_0^\tau||x(s)||ds\right)d\tau\\ &\le \|x\|_{\Delta_\psi}\int_0^t\omega(\tau)\psi(\tau)d\tau.\end{aligned} \tag{2.3}$$

According to Lemma 2.1, $\psi(t)=\exp[(1/C)\int_1^t\omega(s)ds]$ is a solution of the equation $\int_1^t\omega(s)\psi(s)ds=C\psi(t)$, does not decrease, is absolutely continuous, and $\psi(0)=0$. That implies

$$\left(\int_0^t||y(s)||ds\right)\le C\|x\|_{\Delta_\psi}\psi(t). \tag{2.4}$$

□

Remark 2.3. If $K(\cdot,s)$ has bounded variation on s, it is possible to indicate a "wider" space Δ_ψ for which conditions of Theorem 2.2 are satisfied by defining function ψ as

$$\psi(t)=\exp\left[\frac{1}{C}\int_1^t\left(||K(\tau,\tau)||d\tau+\int_0^\tau d_s\operatorname*{var}_{s\in[0,\tau]}||K(\tau,s)||\right)d\tau\right]. \tag{2.5}$$

Theorem 2.4. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $1<p<\infty$ and let C be some positive constant. Then operator K is bounded in space Δ_ψ^p generated by*

$$\psi(t)=\exp\left[\frac{1}{pC}\int_1^t\left(\int_0^\tau||K(\tau,s)||^{p'}ds\right)^{p_1/p'}d\tau\right] \tag{2.6}$$

and $\|K\|_{\Delta_\psi^p}\le C$.

Theorem 2.4 can be proved in a way similar to proof of Theorem 2.2.

Lemma 2.5. *Let $K:\Delta_\psi^p\to\Delta_\psi^p$ $(1<p<\infty)$ be a bounded operator and let its matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition. Then $K:\Delta_\psi^p\to L_p$ is a compact operator.*

Proof. For every $t\in[0,1]$, $(Ky)(t)$ is a linear bounded functional on L_p. Let $\{y_i\}$ be a sequence weakly converging to y_0 in L_p. If $\{y_i\}\subset\Delta_\psi^p$ and $\|y_i\|_{\Delta_\psi^p}\le 1$, then $\|y_0\|_{\Delta_\psi^p}\le 1$. Indeed, if for some $t_1\in[0,1]$, $((1/\psi(t_1))\int_0^{t_1}\|y(s)\|^p ds)^{1/p}>1$, then the sequence $ly_i=\int_0^1 l(s)y_i(s)ds$ does not converge to ly_0, where

$$l(s)=\begin{cases}1, & \text{if } s\le t_1,\\ 0, & \text{if } s>t_1.\end{cases} \tag{2.7}$$

Hence, for almost all $t\in[0,1]$, $\{(Ky_i)(t)\}$ converges and the set Ky is compact in measure. Thus, for the operator $K:\Delta_\psi^p\to L_p$ to be compact, it is necessary and sufficient that the norms of Ky are equicontinuous for $\|y\|_{\Delta_\psi^p}\le M$. Let $\delta\in(0,1)$. As $K:\Delta_\psi^p\to\Delta_\psi^p$ is a bounded operator,

$$\left(\frac{1}{\psi(\delta)}\int_0^\delta||(Ky)(s)||^p ds\right)^{1/p}\le\Delta_0. \tag{2.8}$$

This implies that for any $\varepsilon>0$, there exists $\delta_1>0$ such that if $\delta<\delta_1$, then $(\int_0^\delta\|(Ky)(s)\|^p ds)^{1/p}\le\varepsilon/2$.

Then, from the $\mathcal{N}$ condition, there exists δ_2 such that if $\operatorname{mes} e\le\delta_2$ for some $e\subset[\delta,1]$, then $(\int_e\|(Ky)(s)\|^p ds)^{1/p}\le\varepsilon/2$.

Finally, for $e_1\subset[\delta,1]$ such that $\operatorname{mes} e_1\le\min\{\delta_1,\delta_2\}$,

$$\left(\int_{e_1}||(Ky)(s)||^p ds\right)^{1/p}\le\left(\int_0^\delta||(Ky)(s)||^p ds\right)^{1/p}+\left(\int_\delta^1||(Ky)(s)||^p ds\right)^{1/p}\le\varepsilon. \tag{2.9}$$

□

Lemma 2.6. *Let $\{y_i\}\to y_0$ in L_p $(1<p<\infty)$ and let the sequence $\{(1/u)y_i\}$ be bounded in Δ_ψ^p for some continuous increasing function u, $u(0)=0$. Then $\{y_i\}\to y_0$ in Δ_ψ^p.*

Proof. We have

$$\left(\int_0^t \|y_i(s)\|^p ds\right)^{1/p} \le u(t)\left(\int_0^t \left\|\frac{y_i(s)}{u(s)}\right\|^p ds\right)^{1/p} \le Mu(t)\psi(t). \tag{2.10}$$

Thus, $y_i \in \Delta_\psi^p$. Beginning with some N for any $t \in [0,1]$ and for any given $\varepsilon > 0$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} \le \varepsilon. \tag{2.11}$$

Hence,

$$\begin{aligned}\left(\int_0^t \|y_0(s)\|^p ds\right)^{1/p} &\le \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} + \left(\int_0^t \|y_i(s)\|^p ds\right)^{1/p} \\ &\le \varepsilon + Mu(t)\psi(t) \le Mu(t)\psi(t), \\ \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} &\le 2Mu(t)\psi(t),\end{aligned} \tag{2.12}$$

beginning with some N_δ for any $\delta > 0$, $\|y_0 - y_i\|_{\Delta_\psi^p} < \delta$. Indeed, Lemma 2.5 guarantees the existence of $\tau \in (0,1]$ such that for all $t \in [0,\tau]$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} \le \delta\psi(t). \tag{2.13}$$

Let $t \in [\tau,1]$. Then for $\varepsilon = \delta\psi(\tau)$, (2.11) yields (2.13) for all $t \in [0,1]$. □

Let $u : [0,1] \to \mathbb{R}$ be a continuous increasing function, $u(0) = 0$. Denote

$$\psi(t) = \exp\left[\int_1^t \frac{1}{u(\tau)}\left(\int_0^\tau \|K(\tau,s)\|^{p'} ds\right)^{p/p'} d\tau\right]. \tag{2.14}$$

Lemmas 2.5 and 2.6 imply the following theorem.

Theorem 2.7. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$. And let ψ be defined by (2.14). Then $K : \Delta_\psi^p \to \Delta_\psi^p$ is a compact operator and its spectral radius is equal to zero.*

3. Operator S

Denote

$$\begin{aligned}(S_g y)(t) &= \begin{cases} y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1], \end{cases} \\ (Sy)(t) &= B(t)(S_g)(t).\end{aligned} \tag{3.1}$$

In [2], it is shown that S_g is bounded in L_p if $r = (\sup(\operatorname{mes} g^{-1}(E)/\operatorname{mes} E))^{1/p} < \infty$ and $\|S_g\|_{L_p} = r$, where sup is taken on all measurable sets from $[0,1]$.

Let Ω_m be a set of points from $[0,1]$ for which $g(t) \geq mt$, $\beta(t)$ is a nonincreasing majorant of function $\|B(t)\|$, and

$$\varphi(t) = \lim_{\operatorname{mes} e \to 0} \frac{\operatorname{mes} g^{-1}(e)}{\operatorname{mes} e}, \tag{3.2}$$

where e is a closed interval containing t.

We say that operator S_g satisfies the $\mathcal{M}$ condition if $\operatorname{vrai\,sup}_{t\in[\varepsilon,1]} \varphi(t) < \infty$ for any

$$\varepsilon \in (0,1] \operatorname*{vrai\,sup}_{t\in[\varepsilon,1]} \|B(t)\| < \infty, \tag{3.3}$$

and there exists $m \in [0,1)$ such that

$$\mu_m = \operatorname*{vrai\,sup}_{t\in g(\Omega_m)} \left(\beta(t)^p \varphi(t)\right) < \infty. \tag{3.4}$$

Lemma 3.1. *There exists nonincreasing function $u : (0,1] \to \mathbb{R}$ such that $\beta(t)^p \varphi(t) \leq u(t)$ and the function*

$$\psi(t) = \begin{cases} t^{u(t)} & \text{if } t \in (0,1], \\ 0 & \text{if } t = 0, \end{cases} \tag{3.5}$$

is absolutely continuous on $[0,1]$.

Proof. Let $\{t_i\}$ be a decreasing sequence, $t_1 = 1$, $t_i \to 0$. Denote

$$n_i = \operatorname*{vrai\,sup}_{t\in(t_{i+1},t_i)} \left(\beta(t)^p \varphi(t)\right), \qquad u(t) = \frac{n_{i+1} - n_i}{t_{i+1} - t_i}(t - t_i) + n_i, \tag{3.6}$$

where $t \in (t_{i+1}, t_i)$. Then $\beta(t)^p \varphi(t) \leq u(t)$, u increases and is absolutely continuous on $[0,1]$. $\square$

Let

$$\nu_m = m^{u(1)}\left[u(1) - \frac{1}{\ln m}\right]. \tag{3.7}$$

Theorem 3.2. *Let operator S_g satisfy the $\mathcal{M}$ condition and let function u satisfy conditions of Lemma 3.1. Then S_g is bonded in Δ_ψ^p with $\psi(t) = t^{u(t)}$ and*

$$\|S_g\|_{\Delta_\psi^p} \leq \left(\nu_m + \mu_m\right)^{1/p}. \tag{3.8}$$

Proof. Let $y \in \Delta_\psi^p$, $\|y\|_{\Delta_\psi^p} = 1$, and $\delta \in (0,1)$. Denote measures λ and μ on $[\delta,1]$ by $\lambda(e) = \int_e \beta(s)^p ds$ and $\mu(e) = \int_{g^{-1}(e)} \beta(s)^p ds$. Then by the Radon-Nikodym [2] theorem, we have

$$\begin{aligned} \left\| \int_\delta^t |(S_g y)(t)|^p ds \right\| &\leq \int_{g^{-1}([0,t])\cap[\delta,1]} \|y[g(s)]\|^p d\lambda(s) \\ &= \int_{g^{-1}([0,t])\cap[\delta,1]} \|y(s)\|^p \frac{d\mu}{d\lambda}(s) d\lambda(s). \end{aligned} \tag{3.9}$$

Then as $g(t) \le t$,

$$\frac{d\mu}{d\lambda}(s) = \lim_{\operatorname{mes} e \to 0} \frac{\int_{g^{-1}(e)} \beta(s)^p ds}{\int_e \beta(s)^p ds} \le \lim_{\operatorname{mes} e \to 0} \frac{\operatorname{vrai\,sup}_{g^{-1}(e)} \beta(s)^p ds}{\operatorname{vrai\,sup}_e \beta(s)^p} \varphi(s) = \varphi(s) \tag{3.10}$$

or

$$\begin{aligned} \left\| \int_\delta^t |(S_g y)(t)|^p ds \right\| &\le \int_{g^{-1}([0,t]\setminus\Omega_m)\cap[\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\quad + \int_{g^{-1}(\Omega_m)\cap[\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\le \int_0^{mt} \beta(s)^p \|y(s)\|^p \varphi(s) ds + \int_0^t \|y(s)\|^p \mu_m ds \\ &\le \int_0^{mt} \|y(s)\|^p u(s) ds + \mu_m \psi(t)^p. \end{aligned} \tag{3.11}$$

We denote function $u_k : (0,1] \to \mathbb{R}$ by $u_k(t) = u(t_i)$, where $t_i = (2^k - i)/2^k$, $i = 0,1,2,\dots,$ $2^k - 1$. From $u_k \to u$, it follows that

$$\int_0^{mt} \|y(s)\|^p u(s) ds = \lim_{k\to 0} \int_0^{mt} \|y(s)\|^p u_k(s) ds. \tag{3.12}$$

We write function u_k in the form

$$u_k(t) = \begin{cases} u(t_0), & \text{if } t \in (t_1, t_0], \\ u(t_0) + [u(t_1) - u(t_0)], & \text{if } t \in (t_2, t_1], \\ \quad\vdots & \quad\vdots \\ u(t_{k-2}) + [u(t_{k-1}) - u(t_{k-2})], & \text{if } t \in (t_k, t_{k-1}]. \end{cases} \tag{3.13}$$

The condition $t < t_i$ implies that $\int_0^{mt} \|y(s)\|^p ds \le \psi^p(mt) = (mt)^{u(mt)} \le m^{pu(t_i)} \psi^p(t)$ and

$$\begin{aligned} \int_0^{mt} \|y(s)\|^p u(s) ds &\le \sum_{i=1}^{2^k} m^{pu(t_i)} [u(t_i) - u(t_{i-1})] \psi^p(t) + u(1) m^{pu(1)} \psi^p(t) \\ &\le \psi^p(t) \left[\int_{u(1)}^\infty m^s ds + m^{u(1)} u(1) \right] \\ &\le \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right], \end{aligned} \tag{3.14}$$

simultaneously for all k. Finally,

$$\begin{aligned} \left\| \int_0^t |(S_g y)(s)|^p ds \right\| &= \lim_{\delta\to 0} \left\| \int_\delta^t |(S_g y)(s)|^p ds \right\| \\ &\le \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right] + \psi^p(t) \mu_m \\ &\le \psi^p(t)(\nu_m + \mu_m) \end{aligned} \tag{3.15}$$

which proves the theorem. □

Remark 3.3. From (3.7) and (3.8), it follows that if $\lim_{m\to 1} < 1$, then there exists function ψ such that the norm of operator $S_g : \Delta_\psi^p \to \Delta_\psi^p$ is less than 1.

In some particular cases, it is possible to give less strict conditions on function ψ generating the space Δ_ψ^p. Direct calculations prove the following theorem.

THEOREM 3.4. *Let $B(t) \le C_1/t^\alpha$ and $g(t) = C_2 t^\beta$ with $\beta > 1$. Then $\|S_g\|_{\Delta_\psi^p} \le C_1/C_2$, where $\psi(t) = t^\gamma$, $\gamma \ge (\alpha p + \beta - 1)/p(\beta - 1)$. If $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$, then the spectral radius of S_g is equal to zero.*

4. The Cauchy problem

We consider the Cauchy problem for (1.1):

$$(\mathscr{L}x)(t) = f(t), \qquad x(0) = \alpha. \tag{4.1}$$

The theorems of this section are immediate corollaries of Theorems 2.2, 2.4, 2.7, 3.2, and 3.4.

THEOREM 4.1. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition and let operator S_g satisfy the $\mathcal{M}$ condition. Let also $\operatorname{vraisup}_{t\in[0,1]} u(t) = \infty$, $(\mu_m)^{1/p} \le q < 1$, and let the function ψ_1 be given by (2.14). Then if $C < 1 - q$, the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^{u(t)}\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

Let ω be a solution of the equation

$$m^\omega \left(\omega - \frac{1}{\ln m}\right) \le C_1^p - q, \quad \gamma = \sup_{t\in[0,1]} \{u(t), \omega\}, \tag{4.2}$$

where $0 \le q \le C_1^p < 1$, and u satisfies conditions of Lemma 3.1.

THEOREM 4.2. *Let matrix $K(t,s)$ and operator S_g satisfy the $\mathcal{N}$ and $\mathcal{M}$ conditions, respectively. Let $\operatorname{vraisup}_{t\in[0,1]} u(t) < \infty$ and $(\mu_m)^{1/p} \le q < 1$. Then if $q < C_1$, $(C_1 + C_2) < 1$, then the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^\gamma\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

THEOREM 4.3. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition, $B(t) \le C_1/t^\alpha$, $g(t) = C_2 t^\beta$ $(\beta > 1)$, and $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$. Let also $C < 1$ and $\psi(t) = \min\{\psi_1(t), t^\gamma\}$. Then the Cauchy problem (4.1) has a unique solution for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

Example 4.4. The Cauchy problem

$$\begin{gathered} \dot{x}(t) + p(t)\frac{x[h(t)]}{t^k} + q(t)\dot{x}(t^2) = f(t), \quad t \in [0,1], \\ x(\xi) = 0, \quad \text{if } h(\xi) \le 0, \end{gathered} \tag{4.3}$$

where $h(t) \le t$, $k > 1$, and p and q are bounded functions, has a solution if $\int_0^t |f(s)|ds \le M\exp(-t^{1-k})$. If $(t - h(t)) \ge \tau > 0$, then it has a solution if $\int_0^t |f(s)|ds \le Mt^\gamma$ for $\gamma > 1$.

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Andrei Shindiapin: Department of Mathematics and Informatics, Eduardo Mondlane University, 257 Maputo, Mozambique
E-mail address: andrei@sm.luth.se