NULL CONTROLLABILITY OF A NONLINEAR HEAT EQUATION

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We study the internal exact null controllability of a nonlinear heat equation with homogeneous Dirichlet boundary condition. The method used combines the Kakutani fixed-point theorem and the Carleman estimates for the backward adjoint linearized system. The result extends to the case of boundary control.

1. Introduction

This work is concerned with the internal controllability of the equation

$$\begin{aligned} & y_t(x,t) - \Delta y(x,t) + a(x,t)y(x,t) + f\left(t, Hy(\cdot,t)\right)y(x,t) \\ & \qquad = m(x)u(x,t), \quad (x,t) \in Q = \Omega \times (0,T), \\ & y(x,t) = 0, \quad (x,t) \in \Sigma = \partial\Omega \times (0,T), \\ & y(x,0) = y_0(x), \quad x \in \Omega. \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is an open, bounded set with a boundary $\partial\Omega$, m is the characteristic function of a nonempty open subset ω of Ω, and Δ is the Laplace operator with respect to the variable x.

Here $a : \Omega \times (0,T) \to \mathbb{R}$ and $f : (0,T) \to \mathbb{R}$ are given functions satisfying the following conditions:

(i) $a \in L^\infty(\Omega \times (0,T))$,
(ii) f is nonnegative and continuous with respect to all variables,
(iii) $f(\cdot,0) \in L^\infty(0,T)$ and f is locally Lipschitz according to the second variable.

Also we assume that

(iv) $H : L^2(\Omega) \to \mathbb{R}$ is a locally Lipschitz continuous operator and $y_0 \in L^2(\Omega)$.

Equation (1.1) describes the heat propagation with a viscosity term.

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System (1.1) is said to be null controllable if for every $T > 0$ there are $(y, u) \in C([0,T]; L^2(\Omega)) \cap W^{1,2}_{\text{loc}}((0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)) \cap L^2_{\text{loc}}((0,T]; H^2(\Omega)) \times L^2(Q)$, which satisfy (1.1) and such that $y(x,T) = 0$ a.e. $x \in \Omega$.

The main result of the paper amounts to saying that system (1.1) is null controllable for all $y_0 \in L^2(\Omega)$.

Null controllability of the linear heat equation, when the control acts on a subset of the domain Ω, was established by Lebeau and Robbiano [6] and was extended later by Fursikov and Imanuvilov [5] to the semilinear equation,

$$y_t(x,t) - \Delta y(x,t) + f\big(y(x,t)\big) = m(x)u(x,t), \quad (x,t) \in Q, \tag{1.2}$$

where f is a sublinear function.

Fernández-Cara [4] established null controllability of superlinear control system of the form

$$y_t(x,t) - \Delta y(x,t) + f\big(y(x,t)\big)y(x,t) = m(x)u(x,t), \quad (x,t) \in Q, \tag{1.3}$$

with f satisfying the condition $f(y)(\log|y|+1)^{-1} \to 0$ as $|y| \to \infty$ while Barbu [3] established the same result in the case $f(y)(\log|y|+1)^{-3/2} \to 0$ as $|y| \to \infty$ if $1 \le n < 6$.

A general discussion on dissipative semilinear heat equation has been done by Aniţa and Tataru [1]. It has been proved that if f is nonnegative and is growing at infinity faster than a polynomial, then the equation is not null controllable.

This is not the case of our problem. Here, we show that system (1.1) is null controllable for any f satisfying the hypotheses mentioned above. Anyway, in (1.1) the nonlinear term $f(t, Hy(\cdot,t))$ does not depend explicitly on the spatial variable.

The paper is organized as follows. The main result is stated in Section 2 and proved in Section 3 via the Kakutani fixed-point theorem. The proof is based on Carleman inequality for the backward adjoint linearized system associated with (1.1). We do not impose asymptotic conditions on f (as in [3, 4]).

In what follows we use standard notations for the Sobolev spaces $H^2(\Omega)$, $H^1_0(\Omega)$, and $L^2(\Omega)$ on Ω and Q. Denote by $|\cdot|$ the usual norm of $\mathbb{R}^n$, and by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$. Moreover, we set

$$\begin{aligned} W^{1,2}(0,T; L^2(\Omega)) &= \left\{ y \in L^2(0,T; L^2(\Omega));\ \frac{dy}{dt} \in L^2(0,T; L^2(\Omega)) \right\}, \\ W^{1,2}_{\text{loc}}(0,T; L^2(\Omega)) &= \cap_{\delta \in (0,T)} W^{1,2}(\delta, T; L^2(\Omega)), \end{aligned} \tag{1.4}$$

where dy/dt is taken in the sense of distributions.

2. The main result

THEOREM 2.1. *Assume that conditions (i), (ii), (iii), and (iv) hold. Then for all* $y_0 \in L^2(\Omega)$ *and* $T > 0$, *there are* $u \in L^2(Q)$ *and* $y \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{\text{loc}}((0,T];L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \cap L^2_{\text{loc}}((0,T];H^2(\Omega))$, *which satisfy (1.1), and*

$$y(x,T) = 0 \quad a.e.\ x \in \Omega. \tag{2.1}$$

The result of Theorem 2.1 extends in a classical manner (see [3]) to the case of boundary control. More exactly we have the following result.

THEOREM 2.2. *Under assumptions (i), (ii), (iii), and (iv) for each* $T > 0$ *and* $y_0 \in H^1(\Omega)$, *there are* $v \in L^2(\Sigma)$ *and* $y \in W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, *which satisfy*

$$\begin{gathered} y_t - \Delta y + a(x,t)y + f(t,Hy)y = 0 \quad in\ Q, \\ y = v \quad on\ \Sigma, \\ y(x,0) = y_0 \quad in\ \Omega, \\ y(x,T) = 0 \quad in\ \Omega. \end{gathered} \tag{2.2}$$

Proof of Theorem 2.2. Let $\tilde{\Omega}$ be an open bounded set such that $\tilde{\Omega} \supset \Omega$. We set $\omega = \tilde{\Omega} \setminus \bar{\Omega}$ and apply Theorem 2.1 with $y_0 \in H^1(\Omega)$ to (1.1) on $\tilde{\Omega}$ with Dirichlet boundary condition, and the initial value condition $y(x,0) = \tilde{y}_0(x)$ on $\tilde{\Omega}$ where $\tilde{y}_0$ is an H^1_0-extension of y_0 to $\tilde{\Omega}$.

Consequently, there is $\tilde{y}$ satisfying (1.1) on $\tilde{\Omega} \times (0,T)$ such that $\tilde{y}(T) = 0$. So, by the trace theorem $v = \tilde{y}$ on $\partial\Omega \times (0,T)$ belongs to $L^2(\Sigma)$ and y, the restriction of $\tilde{y}$ to $\Omega \times (0,T)$ satisfies the requirements of Theorem 2.2. □

3. Proof of Theorem 2.1

Firstly, we prove Theorem 2.1 in the case $y_0 \in H^1_0(\Omega)$.

We fix $y_0 \in H^1_0(\Omega)$ and define the set

$$K = \{w \in L^\infty(0,T;L^2(\Omega));\ \|w(t)\|_{L^2(\Omega)} \le M,\ \text{a.e. } t \in (0,T)\}, \tag{3.1}$$

where M is a positive constant to be defined later.

For $w \in K$ and $\mu \in L^2(Q)$ consider the linear system

$$\begin{gathered} y_t - \Delta y + a(x,t)y + f(t,Hw(t))y = \mu \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(x,0) = y_0 \quad \text{in } \Omega. \end{gathered} \tag{3.2}$$

We note first that for all $w \in K$, $u \in L^2(Q)$, and $y_0 \in H_0^1(\Omega)$, (3.2) has a unique solution

$$y = y^u \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)). \tag{3.3}$$

We give a sketch of the proof for this assertion. Since $H : L^2(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous, for $w \in L^\infty(0,T;L^2(\Omega))$ it follows that $Hw \in L^\infty(0,T)$.

Now, assumptions (ii) and (iii) imply that $f(\cdot, Hw(\cdot)) \in L^\infty(0,T)$ for all $w \in K$.

Along with (i), the last implies that $\tilde{a} \in L^\infty(Q)$ where $\tilde{a}(x,t) = -a(x,t) - f(t,Hw(t))$, for all $w \in K$.

Let $S(t)$ be the C_0-semigroup generated on $L^2(\Omega)$ by the Laplace operator with Dirichlet boundary value conditions. Then, the solution y to (3.2) (if it exists) can be represented by the variation of constant formula,

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\big(\tilde{a}(s)y(s) + \mu(s)\big)\,ds. \tag{3.4}$$

In a standard way (see [2]) we show that (3.4) has a unique solution, $y \in C([0,T];L^2(\Omega))$, provided that the operator $\mathcal{T} : C([0,T];L^2(\Omega)) \to C([0,T];L^2(\Omega))$,

$$(\mathcal{T}y)(t) = \int_0^t S(t-s)\big(\tilde{a}(s)y(s) + \mu(s)\big)\,ds \tag{3.5}$$

is a contraction.

Multiplying now (3.2) by y and integrating on $(0,t) \times \Omega$, we obtain

$$\|y(t)\|_{L^2(\Omega)}^2 \le A + B\int_0^t \|y(s)\|_{L^2(\Omega)}^2\,ds, \tag{3.6}$$

where A and B are positive constants. Then, Gronwall's inequality gives

$$\|y(t)\|_{L^2(\Omega)} \le \bar{C} \quad \forall t \in [0,T], \tag{3.7}$$

$\bar{C}$ being a positive constant (independent of $w \in K$).

As $\tilde{a} \in L^\infty(Q)$, $y \in L^2(Q)$, and $u \in L^2(Q)$, it follows that $\tilde{a}y + \mu \in L^2(Q)$ and by [2, Theorem 2.1, page 189] we conclude that the solution y of (3.2) satisfies (3.3).

Multiplying now (3.2) by $y_t - \Delta y$ and having in mind (3.7), the following inequality is obtained

$$\begin{aligned} \|y(t)\|_{H_0^1(\Omega)}^2 &+ \int_Q \Big(y_t^2(x,t) + |\Delta y(x,t)|^2\Big)\,dx\,dt \\ &\le \tilde{\mu}\Big(M, \|y_0\|_{H_0^1(\Omega)}\Big) + \int_Q \mu^2\,dx\,dt, \end{aligned} \tag{3.8}$$

where $\tilde{\mu}$ is a constant depending on M and y_0.

Now consider the optimal control problem ($\varepsilon > 0$),

$$\text{Minimize} \left\{ \frac{1}{\varepsilon} \int_\Omega y^2(x,T)\,dx + \int_Q u^2\,dx\,dt \right\} \tag{3.9}$$

subject to (3.2).

It is easy to observe that (in (3.2)) the map $u \to y^u$ is closed in $(L^2(Q))_w \times L^2(Q)$, where by $(L^2(Q))_w$ we have denoted the space $L^2(Q)$ endowed with the weak topology. This implies that there exists an optimal pair $(y_\varepsilon, u_\varepsilon)$ for the functional (3.9).

The Pontryagin maximum principle yields that

$$u_\varepsilon(x,t) = m(x)p_\varepsilon(x,t) \quad \text{a.e. } (x,t) \in Q, \tag{3.10}$$

where p_ε is the solution to the backward adjoint system

$$\begin{gathered} (p_\varepsilon)_t + \Delta p_\varepsilon - a p_\varepsilon - f(t, Hw(t))p_\varepsilon = 0 \quad \text{in } Q, \\ p_\varepsilon = 0 \quad \text{on } \Sigma, \\ p_\varepsilon(x,T) = -\frac{1}{\varepsilon} y_\varepsilon(x,T) \quad \text{in } \Omega. \end{gathered} \tag{3.11}$$

Now, we prove an observability result for the solution $p \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ to the equation

$$p_t + \Delta p - a(x,t)p - f(t, Hw(t))p = 0 \quad \text{in } Q. \tag{3.12}$$

Lemma 3.1. *There is a constant C independent of w, M, and p such that*

$$\int_\Omega p^2(x,0)\,dx \le C \int_0^T \int_\omega p^2(x,t)\,dx\,dt. \tag{3.13}$$

Proof. Consider the problem

$$\begin{gathered} p_t + \Delta p - a p = 0 \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(x,T) = z(x) \quad \text{in } \Omega, \end{gathered} \tag{3.14}$$

where $z \in L^2(\Omega)$.

It is well known (see [5]) that the solution of (3.14) satisfies the Carleman inequality,

$$\int_\Omega p^2(x,0)\,dx \le C \int_0^T \int_\omega p^2(x,t)\,dx\,dt, \tag{3.15}$$

for all $z \in L^2(\Omega)$.

It is easy to observe that the solution to (3.12) is given by

$$p_\varepsilon(t) = e^{-\int_t^T f(s,Hw(s))\,ds} p(t), \tag{3.16}$$

which implies that

$$\begin{gathered}
\int_\Omega p_\varepsilon^2(x,0)\,dx = e^{-2\int_0^T f(s,Hw(s))\,ds} \int_\Omega p^2(x,0)\,dx, \\
\int_0^T \int_\Omega p_\varepsilon^2(x,t)\,dx\,dt = \int_0^T \left(e^{-2\int_t^T f(s,Hw(s))\,ds} \int_\Omega p^2(x,t)\,dx \right) dt.
\end{gathered} \tag{3.17}$$

Now inequality (3.15) and $f \geq 0$ imply that

$$\begin{aligned}
e^{-2\int_0^T f(s,Hw(s))\,ds} \int_\Omega p^2(x,0)\,dx &\leq Ce^{-2\int_0^T f(s,Hw(s))\,ds} \int_0^T \int_\omega p^2(x,t)\,dx\,dt \\
&\leq C \int_0^T e^{-2\int_t^T f(s,Hw(s))\,ds} \int_\omega p^2(x,t)\,dx\,dt \\
&= C \int_0^T \int_\omega p_\varepsilon^2(x,t)\,dx\,dt,
\end{aligned} \tag{3.18}$$

and thus p_ε verifies (3.13) ending the proof of the lemma. □

Remark 3.2. The result given by the lemma can be viewed as a uniform observability result for the linear adjoint system (3.11) with respect to $w \in K$.

Proof of Theorem 2.1 (continued). Multiplying (3.2) by p_ε, (3.11) by y_ε, and having in mind (3.10), we obtain, after integration on Q that

$$\begin{aligned}
\frac{1}{\varepsilon} \int_\Omega & y_\varepsilon^2(x,T)\,dx + \int_0^T \int_\omega u_\varepsilon^2(x,t)\,dx\,dt \\
&= -\int_\omega y_\varepsilon(x,0) p_\varepsilon(x,0)\,dx \\
&= -\int_\Omega y_0(x) p_\varepsilon(x,0)\,dx \\
&\leq \gamma \int_\Omega p_\varepsilon^2(x,0) + \frac{1}{\gamma} \int_\Omega y_0^2(x)\,dx \quad \forall \gamma > 0.
\end{aligned} \tag{3.19}$$

As p_ε satisfies (3.13), the latter implies that

$$\begin{aligned}
\frac{1}{\varepsilon} \int_\Omega & y_\varepsilon^2(x,T)\,dx + \int_0^T \int_\omega u_\varepsilon^2(x,t)\,dx\,dt \\
&\leq C\gamma \int_0^T \int_\omega u_\varepsilon^2(x,t)\,dx\,dt + \frac{1}{\gamma} \int_\Omega y_0^2(x)\,dx \quad \forall \gamma > 0
\end{aligned} \tag{3.20}$$

which gives

$$\frac{1}{\varepsilon}\int_{\Omega} y_{\varepsilon}^{2}(x,T)\,dx \le C_1, \qquad \int_0^T\int_{\omega} u_{\varepsilon}^{2}(x,t)\,dx\,dt \le C_1, \tag{3.21}$$

C_1 being a positive constant.

By estimates (3.21) it follows that, selecting a subsequence, we have

$$\begin{aligned} u_{\varepsilon} &\longrightarrow u \quad \text{weakly in } L^2(Q),\\ y_{\varepsilon} &\longrightarrow y \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap W^{1,2}(0,T;L^2(\Omega)), \end{aligned} \tag{3.22}$$

where (y,u) satisfies (3.2) and $y(T)\equiv 0$.

For each $w\in K$, we denote by $\Phi(w)\subset L^2(Q)$ the set of all solutions $y^u \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap W^{1,2}(0,T;L^2(\Omega))$ to (3.2) such that

$$y^u(T)=0, \quad \|u\|_{L^2(Q)} \le C_1^{1/2}. \tag{3.23}$$

By (3.21) and (3.22) we deduce that $\Phi(w)\neq\emptyset$ for each $w\in K$. Moreover, it is readily seen that $\Phi(w)$ is a convex subset of $L^2(Q)$. Since, by (3.8)

$$u_n \longrightarrow u \quad \text{weakly in } L^2(Q) \tag{3.24}$$

implies that

$$y^{u_n} \longrightarrow y \quad \text{in } L^2(Q), \tag{3.25}$$

it follows also that $\Phi(w)$ is a closed subset of $L^2(Q)$. At the same time from estimate (3.8) we deduce, via the Arzelà-Ascoli theorem that $\Phi(K)$ is relatively compact.

Multiplying once again (3.2) by y and integrating on $Q_t=\Omega\times(0,t)$, we obtain

$$\|y(t)\|_{L^2(\Omega)} \le M \quad \forall t\in(0,T), \tag{3.26}$$

which is a constant that we choose in the definition of K. So, we have proved that $\Phi(K)\subset K$. Finally, we prove that Φ is upper semicontinuous in $L^2(Q)\times L^2(Q)$. For this, let $w_n\in K$, $y_n\in\Phi(w_n)$, $y_n=y^{u_n}$ such that

$$\begin{aligned} w_n &\longrightarrow w \quad \text{in } L^2(Q),\\ y_n &\longrightarrow y \quad \text{in } L^2(Q). \end{aligned} \tag{3.27}$$

By estimate (3.8) it follows that, eventually on a subsequence,

$$\begin{aligned} u_n &\longrightarrow u \quad \text{weakly in } L^2(Q),\\ y_n &\longrightarrow y \quad \text{strongly in } C([0,T];L^2(\Omega)) \text{ and}\\ &\qquad\quad \text{weakly in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap W^{1,2}(0,T;L^2(\Omega)). \end{aligned} \tag{3.28}$$

So, we have

$$\begin{gathered} f(t,Hw_n(t))\,y_n(x,t) \longrightarrow f(t,Hw(t))\,y(x,t) \quad \text{a.e. in } Q, \\ f(t,Hw_n)\,y_n \longrightarrow \eta \quad \text{weakly in } L^2(Q). \end{gathered} \tag{3.29}$$

By Egorov's theorem we conclude that

$$\eta = f(t,Hw(t))\,y(x,t) \quad \text{a.e. in } Q. \tag{3.30}$$

Since y_n is a solution of

$$\begin{gathered} (y_n)_t - \Delta y_n + a(x,t)y_n + f(t,Hw_n)\,y_n = \mu_n \quad \text{in } Q, \\ y_n = 0 \quad \text{on } \Sigma, \\ y_n(x,0) = y_0(x), \quad y_n(x,T) = 0 \quad \text{in } \Omega, \end{gathered} \tag{3.31}$$

we get (by passing to the limit) that (y,u) satisfies (3.2) and (3.23), that is, $y \in \Phi(w)$ as claimed. By the Kakutani fixed-point theorem in $L^2(Q)$ satisfied by Φ, we infer that there is at least one $w \in K$ such that $w \in \Phi(w)$. Then, by the definition of Φ, this implies that there exists at least one pair (y,u) satisfying the conditions of Theorem 2.1. □

In the general case $y_0 \in L^2(\Omega)$, we can use the smoothing effect of the parabolic equation on the initial data. More exactly, for each $\varepsilon > 0$ there exists $\tilde{\varepsilon} \in (0,\varepsilon]$ such that $\bar{y}(\tilde{\varepsilon}) \in H_0^1(\Omega)$, $\bar{y}$ being the solution of (1.1) with $u \equiv 0$ on $\omega \times (0,\varepsilon)$ (see [2]).

Theorem 2.1 applies, for example, to the semilinear heat equation with a viscosity term,

$$\begin{gathered} y_t - \Delta y + ay + f\left(t, \int_\Omega y(x,t)\,dx\right) y = \mu \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(x,0) = y_0(x) \quad \text{in } \Omega. \end{gathered} \tag{3.32}$$

Here a and f satisfy conditions (i), (ii), and (iii).

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