EXISTENCE AND REGULARITY OF WEAK SOLUTIONS TO THE PRESCRIBED MEAN CURVATURE EQUATION FOR A NONPARAMETRIC SURFACE

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Received 15 August 1998

1. Introduction

The prescribed mean curvature equation with Dirichlet condition for a nonparametric surface $X : \Omega \to \mathbb{R}^3$, $X(u,v) = (u,v,f(u,v))$ is the quasilinear partial differential equation

$$\begin{gathered}\left(1+f_v^2\right) f_{uu} + \left(1+f_u^2\right) f_{vv} - 2 f_u f_v f_{uv} = 2h(u,v,f)\left(1+|\nabla f|^2\right)^{3/2} \quad \text{in } \Omega, \\ f = g \quad \text{in } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g \in H^1(\Omega)$.

We call $f \in H^1(\Omega)$ a weak solution of (1.1) if $f \in g + H_0^1(\Omega)$ and for every $\varphi \in C_0^1(\Omega)$

$$\int_\Omega \left(\left(1+|\nabla f|^2\right)^{-1/2} \nabla f \nabla \varphi + 2h(u,v,f)\varphi\right) du\, dv = 0. \tag{1.2}$$

It is known that for the parametric Plateau's problem, weak solutions can be obtained as critical points of a functional (see [2, 6, 7, 8, 10, 11]).

The nonparametric case has been studied for $H = H(x,y)$ (and generally $H = H(x_1,\dots,x_n)$ for hypersurfaces in $\mathbb{R}^{n+1}$) by Gilbarg, Trudinger, Simon, and Serrin, among other authors. It has been proved [5] that there exists a solution for any smooth boundary data if the mean curvature H' of $\partial\Omega$ satisfies

$$H'\left(x_1,\dots,x_n\right) \ge \frac{n}{n-1}\left|H\left(x_1,\dots,x_n\right)\right| \tag{1.3}$$

for any $(x_1,\dots,x_n) \in \partial\Omega$, and $H \in C^1(\overline{\Omega},\mathbb{R})$ satisfying the inequality

$$\left|\int_\Omega H\varphi\right| \le \frac{1-\epsilon}{n} \int_\Omega |D\varphi| \tag{1.4}$$

Abstract and Applied Analysis 4:1 (1999) 61–69
1991 Mathematics Subject Classification: 35J20, 35J65
URL: http://aaa.hindawi.com/volume-4/S1085337599000019.html

for any $\varphi \in C_0^1(\Omega, \mathbb{R})$ and some $\epsilon > 0$. They also proved a non-existence result (see [5, Corollary 14.13]): if $H'(x_1, \dots, x_n) < (n/(n-1))|H(x_1, \dots, x_n)|$ for some $(x_1, \dots, x_n)$ and the sign of H is constant, then for any $\epsilon > 0$ there exists $g \in C^\infty(\overline{\Omega})$ such that $\|g\|_\infty \leq \epsilon$ and that Dirichlet's problem is not solvable.

We remark that the solutions obtained in [5] are classical. In this paper, we find weak solutions of the problem by variational methods.

We prove that for prescribed h there exists an associated functional to h, and under some conditions on h and g we find that this functional has a global minimum in a convex subset of $H^1(\Omega)$, which provides a weak solution of (1.1). We denote by $H^1(\Omega)$ the usual Sobolev space, [1].

2. The associated variational problem

Given a function $f \in C^2(\Omega)$, the generated nonparametric surface associated to this function is the graph of f in $\mathbb{R}^3$, parametrized as $X(u, v) = (u, v, f(u, v))$.

The mean curvature of this surface is

$$h(u, v, f) = \frac{1}{2} \frac{E f_{vv} - 2F f_{uv} + G f_{uu}}{\left(1 + f_u^2 + f_v^2\right)^{3/2}}, \tag{2.1}$$

where E, F, and G are the coefficients of the first fundamental form [4, 9].

For prescribed h, weak solutions of (1.1) can be obtained as critical points of a functional.

PROPOSITION 2.1. *Let $J_h : H^1(\Omega) \to \mathbb{R}$ be the functional defined by*

$$J_h(f) = \int_\Omega \left(\left(1 + |\nabla f|^2\right)^{1/2} + H(u, v, f)\right) du\, dv, \tag{2.2}$$

where $H(u, v, z) = \int_0^z 2h(u, v, t) dt$. Then (1.1) is the Euler Lagrange equation of (2.2).

Remark 2.2. If $f \in T = g + H_0^1(\Omega)$ is a critical point of J_h, then f is a weak solution of (1.1).

Proof. For $\varphi \in C_0^1(\Omega)$, integrating by parts we obtain

$$d J_h(f)(\varphi) = 2 \int_\Omega \left(\frac{1}{2} \frac{E f_{vv} - 2F f_{uv} + G f_{uu}}{\left(1 + f_u^2 + f_v^2\right)^{3/2}} - h(u, v, f) \right) \varphi\, du\, dv. \tag{2.3}$$

□

3. Behavior of the functional J_h

In this section, we study the behavior of the functional J_h restricted to T. For simplicity we write $J_h(f) = A(f) + B(f)$, with

$$A(f) = \int_\Omega \left(1 + |\nabla f|^2\right)^{1/2} du\, dv, \qquad B(f) = \int_\Omega H(u, v, f)\, du\, dv. \tag{3.1}$$

We will assume that h is bounded.

LEMMA 3.1. *The functional $A : T \to \mathbb{R}$ is continuous and convex.*

Proof. Continuity can be proved by a simple computation. Let $a, b \geq 0$ such that $a+b = 1$. By Cauchy inequality, it follows that

$$\sqrt{1+\left|\nabla\left(af+bf_0\right)\right|^2} \leq a\sqrt{1+|\nabla f|^2}+b\sqrt{1+|\nabla f_0|^2} \tag{3.2}$$

and convexity holds. □

Remark 3.2. As A is continuous and convex, then it is weakly lower semicontinuous in T.

LEMMA 3.3. *The functional B is weakly lower semicontinuous in T.*

Proof. Since h is bounded, we have

$$|H(u, v, z)| \leq c|z|+d. \tag{3.3}$$

From the compact immersion $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$ and the continuity of Nemytskii operator associated to H in $L^1(\Omega)$, we conclude that B is weakly lower semicontinuous in T (see [3, 12]). □

4. Weak solutions as critical points of J_h

Let us assume that $g \in W^{1,\infty}$, and consider for each $k > 0$, the following subset of T:

$$\overline{M}_k = \left\{f \in T : \ \|\nabla(f-g)\|_\infty \leq k\right\}. \tag{4.1}$$

$\overline{M}_k$ is nonempty, closed, convex, bounded, then it is weakly compact.

Remark 4.1. As $g \in W^{1,\infty}$, taking $p > 2$ we obtain, for any $f \in \overline{M}_k$:

$$\|f-g\|_p \leq c\|\nabla(f-g)\|_p. \tag{4.2}$$

Then, by Sobolev imbedding, $\|f-g\|_\infty \leq c_1\|f-g\|_{1,p} \leq \bar{c}k$ for some constant $\bar{c}$. We deduce that $f \in W^{1,\infty}$ and $f(\Omega) \subset K$ for some fixed compact $K \subset \mathbb{R}$. Thus, the assumption $\|h\|_\infty < \infty$ is not needed.

Let ρ be the slope of J_h in $\overline{M}_k$ defined by

$$\rho\left(f_0, \overline{M}_k\right) = \sup\left\{dJ_h\left(f_0\right)\left(f_0-f\right); \ f \in \overline{M}_k\right\} \tag{4.3}$$

(see [7, 11]), then the following result holds.

LEMMA 4.2. *If $f_0 \in \overline{M}_k$ verifies*

$$J_h\left(f_0\right) = \inf\left\{J_h(f) : f \in \overline{M}_k\right\}, \tag{4.4}$$

then $\rho(f_0, \overline{M}_k) = 0$.

Proof.

$$
\begin{aligned}
dJ_h\big(f_0\big)\big(f-f_0\big) &= \lim_{\varepsilon\to 0}\frac{J_h\big(f_0+\varepsilon\big(f-f_0\big)\big)-J_h\big(f_0\big)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{J_h\big((1-\varepsilon)f_0+\varepsilon f\big)-J_h\big(f_0\big)}{\varepsilon}.
\end{aligned}
\tag{4.5}
$$

When $0<\varepsilon<1$ we have that $(1-\varepsilon)f_0+\varepsilon f\in\overline{M}_k$, and then $dJ_h(f_0)(f_0-f)\le 0$ for all $f\in\overline{M}_k$. As $dJ_h(f_0)(f_0-f_0)=0$, we conclude that $\rho(f_0,\overline{M}_k)=0$. □

Remark 4.3. Let J_h be weakly semicontinuous and let $\overline{M}_k$ be a weakly compact subset of T, then J_h achieves a minimum f_0 in $\overline{M}_k$. By Lemma 4.2, $\rho(f_0,\overline{M}_k)=0$.

As in [7], if f_0 has zero slope, we call it a ρ-critical point. The following result gives sufficient conditions to assure that if f_0 is a ρ-critical point, then it is a critical point of J_h.

THEOREM 4.4. *Let $f_0\in\overline{M}_k$ such that $\rho(f_0,\overline{M}_k)=0$, and assume that one of the following conditions holds:*

(i) $dJ_h(f_0)(f_0-g)\ge 0$
(ii) $\|\nabla(f_0-g)\|_\infty<k$.

Then $dJ_h(f_0)=0$.

Proof. As $\rho(f_0,\overline{M}_k)=0$, we have that $dJ_h(f_0)(f_0-f)\le 0$, and then $dJ_h(f_0)(f_0-g)$ $\le dJ_h(f_0)(f-g)$ for any $f\in\overline{M}_k$.

We will prove that $dJ_h(f_0)(\varphi)=0$ for any $\varphi\in C_0^1$. Let $\widetilde{\varphi}=k\varphi/2\|\nabla\varphi\|_\infty$, then $\pm\widetilde{\varphi}+g\in\overline{M}_k$, and then $dJ_h(f_0)(f_0-g)\le\pm dJ_h(f_0)(\widetilde{\varphi})$.

Suppose that $dJ_h(f_0)(\widetilde{\varphi})\ne 0$, then $dJ_h(f_0)(f_0-g)<0$.

If (i) holds, we immediately get a contradiction. On the other hand, if (ii) holds, there exists $r>1$ such that $g+r(f_0-g)\in\overline{M}_k$. Then $dJ_h(f_0)(f_0-g)\le r\,dJ_h(f_0)(f_0-g)$, a contradiction. □

Examples

Let us assume that $\int_\Omega((\nabla(f-g)\nabla g)/\sqrt{1+|\nabla f|^2})du\,dv\ge 0$ for any $f\in\overline{M}_k$. Then condition (i) of Theorem 4.4 is fulfilled for example if

(a) $|h(u,v,z)|\le c(z-g(u,v))_+$ for every $(u,v)\in\Omega,\ z\in\mathbb{R}^3$, for some constant c small enough.

(b) $\int_\Omega h(u,v,f)(f-g)du\,dv\ge 0$ for every $f\in\overline{M}_k$. As a particular case, we may take $h(u,v,z)=c(z-g(u,v))$ for any $c\ge 0$.

(c) $h(u,v,z)=-c(z-g(u,v))$ for some $c>0$ small enough.

Indeed, in all the examples the inequality $dJ_h(f)(f-g) \geq 0$ holds for any $f \in \overline{M}_k$, since

$$\begin{aligned} dJ_h(f)(f-g) &= \int_\Omega \left(\frac{\nabla f \nabla (f-g)}{\sqrt{1+|\nabla f|^2}} + 2h(u,v,f)(f-g) \right) du\,dv \\ &= \int_\Omega \left(\frac{|\nabla (f-g)|^2}{\sqrt{1+|\nabla f|^2}} + 2h(f-g) \right) du\,dv + \int_\Omega \frac{\nabla (f-g) \nabla g}{\sqrt{1+|\nabla f|^2}} du\,dv \\ &\geq \int_\Omega \left(\frac{|\nabla (f-g)|^2}{\sqrt{1+|\nabla f|^2}} + 2h(f-g) \right) du\,dv. \end{aligned} \tag{4.6}$$

Then the result follows immediately in example (b). In examples (a) and (c), being $\|\nabla(f-g)\|_\infty \leq k$ we can choose $\tilde{k}$ such that $\sqrt{1+\|\nabla f\|_\infty^2} \leq \tilde{k}$. Then

$$\begin{aligned} \int_\Omega \left(\frac{|\nabla (f-g)|^2}{\sqrt{1+|\nabla f|^2}} + 2h(u,v,f)(f-g) \right) du\,dv &\geq \int_\Omega \left(\frac{|\nabla (f-g)|^2}{\tilde{k}} - 2c(f-g)^2 \right) du\,dv \\ &\geq \frac{1}{\tilde{k}} \|\nabla (f-g)\|_2^2 - 2cc_1^2 \|\nabla (f-g)\|_2^2 \\ &= \left(\frac{1}{\tilde{k}} - 2cc_1^2 \right) \|\nabla (f-g)\|_2^2, \end{aligned} \tag{4.7}$$

where c_1 is the Poincaré's constant associated to Ω.

Thus, the result holds for $c \leq 1/2\tilde{k}c_1^2$.

Remark 4.5. As in the preceding examples, it can be proved that if $dJ_h(f)(f-g) \geq 0$ for any $f \in \overline{M}_k$, then g is a weak solution of (1.1). Indeed, if $dJ_h(g) \neq 0$, from Theorem 4.4 it follows that $\rho(g, \overline{M}_k) > 0$. As J_h achieves a minimum in every $\overline{M}_k$, we may take $k \geq k_n \to 0$, and f_n such that $\rho(f_n, \overline{M}_{k_n}) = 0$. As $\overline{M}_{k_n} \subset \overline{M}_k$, condition (i) in Theorem 4.4 holds, and then $dJ_h(f_n) = 0$. It is immediate that $f_n \to g$ in $W^{1,\infty}$, and then it follows easily that $dJ_h(g) = 0$.

Furthermore, for constant g we can see that if $dJ_h(f)(f-g) \geq 0$ for any $f \in \overline{M}_k$, then g is a global minimum of J_h in $\overline{M}_k$: let us define $\varphi(t) = J_h(tf + (1-t)g)$, then $\varphi'(t) = dJ_h(tf + (1-t)g)(f-g)$. As $0 \leq dJ_h(tf + (1-t)g)(tf + (1-t)g - g) = t\,dJ_h(tf + (1-t)g)(f-g)$ it follows that $J_h(f) - J_h(g) = \varphi(1) - \varphi(0) = \varphi'(c) \geq 0$.

5. Multiple solutions

In this section, we study the multiplicity of weak solutions of (1.1). Consider

$$\overline{N}_k = \left\{ f \in \overline{M}_k \cap H^2 : \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_2 \leq k \right\}, \tag{5.1}$$

$\overline{N}_k$ is a nonempty, closed, bounded, and convex subset of T, therefore $\overline{N}_k$ is weakly compact.

Then we obtain the following theorem, which is a variant of the mountain pass lemma.

THEOREM 5.1. *Let $f_0 \in \overline{N}_k$ be a local minimum of J_h and assume that $J_h(f_1) < J_h(f_0)$ for some $f_1 \in \overline{N}_k$. Let*

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_h\big(\gamma(t)\big), \tag{5.2}$$

where $\Gamma = \{\gamma \in C([0,1], \overline{N}_k) : \gamma(0) = f_0,\ \gamma(1) = f_1\}$. Then there exists $f \in \overline{N}_k$ such that $J_h(f) = c$ and $\rho(f, \overline{N}_k) = 0$.

We remark that f is not a local minimum of J_h. This kind of f is called an unstable critical point.

The proof of Theorem 5.1 follows from Theorem 3 in [7] and Lemmas 5.2, 5.3, and 5.4 below.

LEMMA 5.2. *The functional J_h is $C^1(\overline{N}_k)$.*

Proof. Let $f, f_0 \in \overline{N}_k$. Then

$$\begin{aligned}
&\big|dJ_h(f)(\varphi) - dJ_h\big(f_0\big)(\varphi)\big| \\
&\quad \le \|\varphi\|_{H_0^1}\left(\left\|\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1+|\nabla f_0|^2}}\right\|_2 + \big\|N_h\big(f_0\big) - N_h(f)\big\|_2\right),
\end{aligned} \tag{5.3}$$

where N_h is the Nemytskii operator associated to h. Let

$$\begin{aligned}
\left\|\frac{\nabla f}{\sqrt{1+|\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1+|\nabla f_0|^2}}\right\|_2 &\le \left\|\sqrt{1+|\nabla f_0|^2}\nabla f - \sqrt{1+|\nabla f|^2}\nabla f_0\right\|_2 \\
&\le \kappa \left\|f_0 - f\right\|_{H_0^1}
\end{aligned} \tag{5.4}$$

and $N_h : L^2 \to L^2$ continuous, the result holds. □

LEMMA 5.3. *The slope ρ is H^1-continuous.*

Proof. Let $f_n \in \overline{N}_k$ such that $f_n \to f_0$ in H_0^1. For $\epsilon > 0$ we take $g_n \in \overline{N}_k$ such that $\rho(f_n, \overline{N}_k) - \epsilon/2 < dJ_h(f_n)(f_n - g_n)$. Then

$$\begin{aligned}
\rho\big(f_n, \overline{N}_k\big) - \rho\big(f_0, \overline{N}_k\big) &\le dJ_h\big(f_n\big)\big(f_n - g_n\big) + \frac{\epsilon}{2} - dJ_h\big(f_0\big)\big(f_0 - g_n\big) \\
&\le \big\|dJ_h\big(f_n\big)\big\|_{(H_0^1)^*}\big\|\big(f_n - f_0\big)\big\|_{H_0^1} \\
&\quad + \big\|dJ_h\big(f_n\big) - dJ_h\big(f_0\big)\big\|_{(H_0^1)^*}\big\|\big(f_0 - g_n\big)\big\|_{H_0^1} + \frac{\epsilon}{2} < \epsilon
\end{aligned} \tag{5.5}$$

for $n \ge n_0$. Operating in the same way with $\rho(f_0, \overline{N}_k) - \rho(f_n, \overline{N}_k)$, we conclude that $\rho(f_n, \overline{N}_k) \to \rho(f_0, \overline{N}_k)$. □

LEMMA 5.4 (Palais Smale condition). *Let* $(f_n)_{n\in N} \subset \overline{N}_k$ *such that* $\lim_{n\to\infty} \rho(f_n, \overline{N}_k) = 0$. *Then* $(f_n)_{n\in N}$ *has a convergent subsequence in* $H_0^1(\Omega)$.

Proof. As $f_n \in \overline{N}_k$, we may suppose that $f_n \to f$ weakly. Let $\Psi_n = f_n - f$. We will see that $\Psi_n \to 0$. Indeed,

$$\begin{aligned} dJ_h\big(f_n\big)\big(\Psi_n\big) &= \int_\Omega \left(\frac{\nabla f_n}{\sqrt{1+|\nabla f_n|^2}} \nabla \Psi_n + 2h\big(u,v,f_n\big)\Psi_n \right) du\,dv \\ &= \int_\Omega \frac{1}{\sqrt{1+|\nabla f_n|^2}} |\nabla \Psi_n|^2\, du\,dv + \int_\Omega \frac{\nabla \Psi_n}{\sqrt{1+|\nabla f_n|^2}} \nabla f\, du\,dv \\ &\quad + \int_\Omega 2h\big(u,v,f_n\big)\Psi_n\, du\,dv. \end{aligned} \tag{5.6}$$

Then for some constant c

$$c\big\|\nabla\Psi_n\big\|_2^2 \le \rho\big(f_n, \overline{N}_k\big) - \int_\Omega \frac{\nabla\Psi_n}{\sqrt{1+|\nabla f_n|^2}} \nabla f\, du\,dv - \int_\Omega 2h\big(u,v,f_n\big)\Psi_n\, du\,dv. \tag{5.7}$$

By Rellich-Kondrachov theorem $\Psi_n \to 0$ in $L^2(\Omega)$, and then

$$\left| \int_\Omega 2h\big(u,v,f_n\big)\Psi_n\, du\,dv \right| \le 2\|h\|_\infty |\Omega|^{1/2} \|\Psi_n\|_2 \longrightarrow 0, \tag{5.8}$$

$$\begin{aligned} &\left| \int_\Omega \frac{\nabla\Psi_n}{\sqrt{1+|\nabla f_n|^2}} \nabla f\, du\,dv \right| \\ &\quad = \left| - \int_\Omega \frac{\Delta f}{\sqrt{1+|\nabla f_n|^2}} \Psi_n\, du\,dv - \int_\Omega \Psi_n \nabla\big(1+|\nabla f_n|^2\big)^{-1/2} \nabla f\, du\,dv \right| \\ &\quad \le \|\Delta f\|_2 \|\Psi_n\|_2 + \|\nabla f_n\|_\infty \|\nabla f\|_\infty \big\|D^2 f_n\big\|_2 \|\Psi_n\|_2 \longrightarrow 0. \end{aligned} \tag{5.9}$$

□

Example 5.5. Now we will show with an example that problem (1.1) may have at least three ρ-critical points in N_k.

Let $g = g_0$ be a constant, and $h(u,v,z) = -c(z-g_0)$ for some constant $c > 0$. Then, g_0 is a minimum of J_h in $\overline{M}_{k_1}$ for k_1 small enough, and a local minimum in M_k for any $k \ge k_1$.

Moreover, taking $\Omega = B_R$, $f(u,v) = g_0 + R^2 - (u^2+v^2)$, it follows that

$$J_h(f) - J_h\big(g_0\big) = 2\pi\left(o\big(R^3\big) - \frac{c}{6}R^6 \right), \tag{5.10}$$

and taking $k = 2\sqrt{\pi}R$ it holds that $f \in \overline{N}_k$. Hence, if R is big enough, it follows that g_0 is not a global minimum in $\overline{N}_k$. Furthermore, we see that the proof of Lemma 4.2 may be repeated in $\overline{N}_k$, and then the minimum of J_h in $\overline{N}_k$ is a ρ-critical point. From Theorem 5.1 there is a third ρ-critical point which is not a local minimum of J_h.

6. Regularity

As we proved, problem (1.1) admits (for an appropriate $k > 0$) a weak solution in a subset $\overline{M}(k) = \{f \in T \,/\, \|\nabla(f-g)\|_\infty \le k\}$.

Consider $p > 2$, and $f_0 \in W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ a weak solution of (1.1). Then $L_{f_0} f_0 = 2h(u,v,f_0)(1+\nabla f_0^2)^{3/2}$ in Ω where for any $f \in C^1(\overline{\Omega})$ $L_f : W^{2,p} \to L^p$ is the strictly elliptic operator given by

$$L_f \phi = \left(1+f_v^2\right)\phi_{uu} + \left(1+f_u^2\right)\phi_{vv} - 2 f_u f_v \phi_{uv}. \tag{6.1}$$

In order to prove the regularity of f_0, we study equation (6.2)

$$L_{f_0}\phi = 2h(u,v,f_0)(1+\nabla f_0^2)^{3/2} \quad \text{in } \Omega,\ \phi = g \text{ in } \partial\Omega. \tag{6.2}$$

Proposition 6.1. *Let us assume that $\partial\Omega \in C^{2,\alpha}$, $g \in C^{2,\alpha}$, and $h \in C^\alpha$ for some $0 < \alpha \le 1-2/p$. Then, if $\phi \in W^{2,p}$ is a strong solution of (6.2), $\phi \in C^{2,\alpha}(\overline{\Omega})$.*

Proof. By Sobolev imbedding $\phi \in C^{1,\alpha}(\overline{\Omega})$. Then $L_{f_0}\phi \in C^\alpha(\overline{\Omega})$ and the coefficients of the operator L_{f_0} belong to C^α. By Theorem 6.14 in [5], the equation $Lw = L_{f_0}\phi$ in Ω, $w = g$ in $\partial\Omega$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega})$, and the result follows from the uniqueness in Theorem 9.15 in [5]. □

Remark 6.2. As a simple consequence, we obtain that $f_0 \in C^{2,\alpha}(\overline{\Omega})$, by the uniqueness in $W^{2,p}$ given by [5, Theorem 9.15].

Corollary 6.3. *Let us assume that $\partial\Omega \in C^{k+2,\alpha}$, $g \in C^{k+2,\alpha}$, and $h \in C^{k,\alpha}$ for some $0 < \alpha \le 1-2/p$. Then $f_0 \in C^{k+2,\alpha}(\overline{\Omega})$.*

Proof. It is immediate from Proposition 2.1 and Theorem 6.19 in [5]. □

Acknowledgement

The authors thank specially Prof. J. P. Gossez for the careful reading of the manuscript and his suggestions and remarks.

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