

Weighted L^p Boundary Value Problems for Laplace’s Equation on (Semi-)Convex Domains

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Abstract. Let $n \geq 2$ and Ω be a bounded (semi-)convex domain in \mathbb{R}^n . Assume that $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, where $A_p(\partial\Omega)$ denotes the Muckenhoupt weight class on $\partial\Omega$, the boundary of Ω . In this article, the author proves that the Dirichlet and Neumann problems for Laplace’s equation on Ω with boundary data in the weighted space $L^p_\omega(\partial\Omega)$ are uniquely solvable. Moreover, the unique solvability of the Regularity problem for Laplace’s equation on Ω with boundary data in the weighted Sobolev space $\dot{W}^p_{1,\omega}(\partial\Omega)$ is also obtained. Furthermore, the weighted $L^p_\omega(\partial\Omega)$ -estimates for the Dirichlet, Regularity and Neumann problems are established.

1. Introduction

Let Ω be a bounded Lipschitz domain in \mathbb{R}^n with $n \geq 2$. Denote by $\nu := (\nu_1, \dots, \nu_n)$ the *outward unit normal* to $\partial\Omega$, the boundary of Ω . Let $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, where $A_p(\partial\Omega)$ denotes the Muckenhoupt weight class on $\partial\Omega$. Recall that the weighted space $L^p_\omega(\partial\Omega)$ is defined by

$$L^p_\omega(\partial\Omega) := \{f \text{ is measurable on } \partial\Omega : \|f\|_{L^p_\omega(\partial\Omega)} < \infty\},$$

where $\|f\|_{L^p_\omega(\partial\Omega)} := \{\int_{\partial\Omega} |f(x)|^p \omega(x) d\sigma(x)\}^{1/p}$ and $d\sigma$ is the surface measure on $\partial\Omega$. Moreover, for a measurable function F on Ω , the *non-tangential maximal function* of F is defined by, for any $P \in \partial\Omega$,

$$(1.1) \quad (F)^*(P) := \sup\{|F(x)| : x \in \Omega, |x - P| < 2 \text{ dist}(x, \partial\Omega)\}.$$

Then, the Dirichlet problem with the $L^p_\omega(\partial\Omega)$ -boundary data for Laplace’s equation is as follows:

$$(1.2) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f \in L^p_\omega(\partial\Omega) & \text{on } \partial\Omega, \\ (u)^* \in L^p_\omega(\partial\Omega), \end{cases}$$

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where $(u)^*$ is as in (1.1). Furthermore, the Neumann problem with the $L^p_\omega(\partial\Omega)$ -boundary data for Laplace’s equation is as follows:

$$(1.3) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = f \in L^p_\omega(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p_\omega(\partial\Omega), \end{cases}$$

where f satisfies $\int_{\partial\Omega} f(x) \, d\sigma(x) = 0$ and $(\nabla u)^*$ is as in (1.1). Moreover, for $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, let

$$\dot{W}^p_{1,\omega}(\partial\Omega) := \{f \text{ is measurable on } \partial\Omega : \nabla_t f \in L^p_\omega(\partial\Omega)\}$$

and for any $f \in \dot{W}^p_{1,\omega}(\partial\Omega)$, $\|f\|_{\dot{W}^p_{1,\omega}(\partial\Omega)} := \|\nabla_t f\|_{L^p_\omega(\partial\Omega)}$, where $\nabla_t f$ denotes the tangential gradient of f on $\partial\Omega$, namely, $\nabla_t f := \nabla f - (\nu \cdot \nabla f)\nu$. When $\omega \equiv 1$, denote the space $\dot{W}^p_{1,\omega}(\partial\Omega)$ simply by $\dot{W}^p_1(\partial\Omega)$. Then, the Regularity problem with the $\dot{W}^p_{1,\omega}(\partial\Omega)$ -boundary data for Laplace’s equation is as follows:

$$(1.4) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f \in \dot{W}^p_{1,\omega}(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla u)^* \in L^p_\omega(\partial\Omega). \end{cases}$$

We point out that in (1.2), (1.3) and (1.4), the boundary values are taken in the sense of the non-tangential convergence almost everywhere with respect to the surface measure on $\partial\Omega$. When $\omega \equiv 1$ in (1.2), (1.3) and (1.4), (1.2), (1.3) and (1.4) are just the classical $L^p(\partial\Omega)$ -Dirichlet problem, the $L^p(\partial\Omega)$ -Neumann problem and the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem for Laplace’s equation, respectively.

Assume that Ω is a bounded Lipschitz domain in \mathbb{R}^n . It is well known that there exists $\varepsilon \in (0, \infty)$, depending on n and the Lipschitz character of Ω , such that the $L^p(\partial\Omega)$ -Dirichlet problem and the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem are uniquely solvable for the ranges $p \in (2 - \varepsilon, \infty]$ and $p \in (1, 2 + \varepsilon)$, respectively (see, for example, [5, 23]). Moreover, the $L^p(\partial\Omega)$ -Neumann problem is uniquely solvable for $p \in (1, 2 + \varepsilon)$ (see, for example, [6, 13]). Moreover, it is also worth pointing out that the ranges $p \in (2 - \varepsilon, \infty]$ and $p \in (1, 2 + \varepsilon)$ are sharp for the $L^p(\partial\Omega)$ -Dirichlet problem, the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem and the $L^p(\partial\Omega)$ -Neumann problem on general Lipschitz domain Ω , respectively (see, for example, [5, 14, 23]). See also the monograph [14] for the unique solvability of the $L^p(\partial\Omega)$ -Dirichlet problem, the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem and the $L^p(\partial\Omega)$ -Neumann problem and more progress. However, if Ω is a bounded C^1 domain in \mathbb{R}^n , the $L^p(\partial\Omega)$ -Dirichlet problem, the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem and the $L^p(\partial\Omega)$ -Neumann problem are uniquely solvable for any $p \in (1, \infty)$ (see, for example, [8]). Furthermore, if Ω is a bounded (semi-)convex

domain in \mathbb{R}^n , it was proved in [11, 15, 18, 25] that the $L^p(\partial\Omega)$ -Dirichlet problem (see [18, Theorem 3.10]), the $\dot{W}_1^p(\partial\Omega)$ -Regularity problem (see [18, Theorem 3.11]) and the $L^p(\partial\Omega)$ -Neumann problem (see [11, 15, 25]) are uniquely solvable for any $p \in (1, \infty)$. Recall that C^1 domains and (semi-)convex domains in \mathbb{R}^n are special Lipschitz domains and convex domains in \mathbb{R}^n are semi-convex domains (see, for example, [17, 18]). Moreover, we refer the readers to [3, 12, 16, 27] for more recent progress about boundary value problems of Laplace's, second order elliptic and high order elliptic equations on non-smooth domains in \mathbb{R}^n .

Furthermore, the unique solvability of the $L_\omega^p(\partial\Omega)$ -Dirichlet problem, the $\dot{W}_{1,\omega}^p(\partial\Omega)$ -Regularity problem and the $L_\omega^p(\partial\Omega)$ -Neumann problem for Laplace's equation on bounded Lipschitz domains was studied in [7, 19, 21]. More precisely, let Ω be a bounded Lipschitz domain in \mathbb{R}^n . For the $L_\omega^p(\partial\Omega)$ -Dirichlet problem, it was proved in [19, 21] that there exists $\eta \in (0, 1]$, depending on n and the Lipschitz character of Ω , such that, for any $\omega \in A_{1+\eta}(\partial\Omega)$, the Dirichlet problem (1.2) with boundary datum $f \in L_\omega^2(\partial\Omega)$ has the unique solution u , which satisfies $\|(u)^*\|_{L_\omega^2(\partial\Omega)} \leq C\|f\|_{L_\omega^2(\partial\Omega)}$, where C is a positive constant independent of u and f . In particular, if Ω is a C^1 domain, one may take $\eta = 1$. For the $\dot{W}_{1,\omega}^p(\partial\Omega)$ -Regularity problem, it was proved in [19] that, there exists $\eta \in (0, 1]$, depending on n and the Lipschitz character of Ω , such that, for any $\omega \in A_{1+\eta}(\partial\Omega)$, the Regularity problem (1.4) with boundary datum $f \in \dot{W}_{1,1/\omega}^2(\partial\Omega)$ has the unique solution u satisfying $\|(\nabla u)^*\|_{L_{1/\omega}^2(\partial\Omega)} \leq C\|\nabla_t f\|_{L_{1/\omega}^2(\partial\Omega)}$, where C is a positive constant independent of u and f . Furthermore, if Ω is a C^1 domain, one may take $\eta = 1$. For the $L_\omega^p(\partial\Omega)$ -Neumann problem, it was proved in [21] that, there exists $\eta \in (0, 1]$, depending on n and the Lipschitz character of Ω , such that, for any $\omega \in A_{1+\eta}(\partial\Omega)$, the Neumann problem (1.3) with boundary datum $f \in L_{1/\omega}^2(\partial\Omega)$ satisfying $\int_{\partial\Omega} f(x) d\sigma(x) = 0$ has the unique solution u , which satisfies $\|(\nabla u)^*\|_{L_{1/\omega}^2(\partial\Omega)} \leq C\|f\|_{L_{1/\omega}^2(\partial\Omega)}$, where C is a positive constant independent of u and f . Furthermore, as pointed out in [21, Remark 8.4], the above condition $\omega \in A_{1+\eta}(\partial\Omega)$ for the Dirichlet and Neumann problems is sharp in the context of $A_p(\partial\Omega)$ weights.

In particular, if Ω is a bounded C^1 domain, by the unique solvability of the $L_\omega^2(\partial\Omega)$ -Dirichlet problem (1.2) and the $\dot{W}_{1,\omega}^p(\partial\Omega)$ -Regularity problem (1.4) with $\omega \in A_2(\partial\Omega)$ obtained in [19, 21], the extrapolation theorem (see, for example, [9]) and the unique solvability of the $L^p(\partial\Omega)$ -Dirichlet problem and the $\dot{W}_1^p(\partial\Omega)$ -Regularity problem for any $p \in (1, \infty)$ established in [8], we know that the $L_\omega^p(\partial\Omega)$ -Dirichlet problem (1.2) and the $\dot{W}_{1,\omega}^p(\partial\Omega)$ -Regularity problem (1.4) are uniquely solvable for any $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, which were remarked in [7, Remark 1.2]. Very recently, the unique solvability of the $L_\omega^p(\partial\Omega)$ -Neumann problem (1.3) for any $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$ was obtained in [7] via studying the boundedness, convergence and compactness of the boundary double

layer potential and its adjoint operator on the weighted space $L^p_\omega(\partial\Omega)$.

Then, there are the *natural questions* whether or not the $L^p_\omega(\partial\Omega)$ -Dirichlet problem (1.2), the $L^p_\omega(\partial\Omega)$ -Neumann problem (1.3) and the $\dot{W}^p_{1,\omega}(\partial\Omega)$ -Regularity problem (1.4) on the bounded (semi-)convex domain Ω are uniquely solvable for any $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$. The main purpose of this article is to give an affirmative answer to these questions. Furthermore, for any $f \in L^p_\omega(\partial\Omega)$ or $f \in \dot{W}^p_{1,\omega}(\partial\Omega)$ with $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, the weighted $L^p_\omega(\partial\Omega)$ -estimates

$$\|(u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|f\|_{L^p_\omega(\partial\Omega)}, \quad \|(\nabla u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|f\|_{L^p_\omega(\partial\Omega)}$$

and

$$\|(\nabla u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|\nabla_t f\|_{L^p_\omega(\partial\Omega)}$$

are obtained, respectively, for the Dirichlet problem (1.2), the Neumann problem (1.3) and the Regularity problem (1.4), where u denotes the unique solution for the Dirichlet problem, the Neumann problem or the Regularity problem and C is a positive constant independent of u and f . It is worth pointing out that the method dealing with the $L^p_\omega(\partial\Omega)$ -Neumann problem (1.3) in this paper is also valid for the case of bounded C^1 domains.

To state our main result, we first recall some necessary definitions and notation.

Definition 1.1. (i) Let O be an open set in \mathbb{R}^n . The collection of *semi-convex functions* on O consists of continuous functions $u: O \rightarrow \mathbb{R}$ with the property that there exists a positive constant C such that, for all $x, h \in \mathbb{R}^n$ with the ball $B(x, |h|) \subset O$,

$$2u(x) - u(x + h) - u(x - h) \leq C|h|^2.$$

The best constant C above is referred as the *semi-convexity constant* of u .

(ii) A nonempty, proper open subset Ω of \mathbb{R}^n is said to be *semi-convex* provided that there exist $b, c \in (0, \infty)$ with the property that, for every $x_0 \in \partial\Omega$, there exist an $(n - 1)$ -dimensional affine variety $H \subset \mathbb{R}^n$ passing through x_0 , a choice N of the unit normal to H , and an open set

$$\mathcal{C} := \{\tilde{x} + tN : \tilde{x} \in H, |\tilde{x} - x_0| < b, |t| < c\}$$

(called a *coordinate cylinder* near x_0 with axis along N) satisfying, for some semi-convex function $\varphi: H \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathcal{C} \cap \Omega &= \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t > \varphi(\tilde{x})\}, \\ \mathcal{C} \cap \partial\Omega &= \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t = \varphi(\tilde{x})\}, \\ \mathcal{C} \cap \overline{\Omega}^c &= \mathcal{C} \cap \{\tilde{x} + tN : \tilde{x} \in H, t < \varphi(\tilde{x})\}, \\ \varphi(x_0) &= 0 \quad \text{and} \quad |\varphi(\tilde{x})| < c/2 \quad \text{if} \quad |\tilde{x} - x_0| \leq b, \end{aligned}$$

where $\bar{\Omega}$ and $\bar{\Omega}^c$, respectively, denote the closure of Ω in \mathbb{R}^n and the complementary set of $\bar{\Omega}$ in \mathbb{R}^n .

Remark 1.2. (i) The Lipschitz domain in \mathbb{R}^n can be defined via replacing the semi-convex function φ by a Lipschitz function $\psi: H \rightarrow \mathbb{R}$ in Definition 1.1(ii).

If Ω is a bounded Lipschitz domain in \mathbb{R}^n , by the definition of Ω , we know that there exist finite sets $\{Q_j\}_{j=1}^{N_0} \subset \partial\Omega$ of points, $\{\delta_j\}_{j=1}^{N_0} \subset (0, \infty)$ of numbers and $\{\psi_j\}_{j=1}^{N_0}$ of Lipschitz functions such that

- (a) $\partial\Omega \subset \bigcup_{j=1}^{N_0} B(Q_j, \delta_j)$;
- (b) for any $j \in \{1, \dots, N_0\}$, $\psi_j(Q_j) = 0$;
- (c) for any $j \in \{1, \dots, N_0\}$,

$$\Omega \cap B(Q_j, \delta_j) = \{(x, t) : x \in \mathbb{R}^{n-1}, t \in \mathbb{R} \text{ and } t > \psi_j(x)\} \cap B(Q_j, \delta_j)$$

and

$$\partial\Omega \cap B(Q_j, \delta_j) = \{(x, t) : x \in \mathbb{R}^{n-1}, t \in \mathbb{R} \text{ and } t = \psi_j(x)\} \cap B(Q_j, \delta_j),$$

where N_0 is a positive integer depending on Ω and

$$B(Q_j, \delta_j) := \{x \in \mathbb{R}^n : |x - Q_j| < \delta_j\}$$

denotes the ball in \mathbb{R}^n with the center Q_j and the radius δ_j .

(ii) It is well known that bounded (semi-)convex domains in \mathbb{R}^n are bounded Lipschitz domains, and convex domains in \mathbb{R}^n are semi-convex domains (see, for example, [17, 18, 26]).

Now we recall the definition of $A_p(\partial\Omega)$ weights (see, for example, [22, p. 4] and [19, (7.1)]).

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ be a domain and $p \in (1, \infty)$. A non-negative and locally integrable function ω on $\partial\Omega$ is called an $A_p(\partial\Omega)$ weight, if there exists a positive constant C such that, for any $Q \in \partial\Omega$ and $r \in (0, \text{diam}(\partial\Omega))$,

$$(1.5) \quad \left\{ \frac{1}{r^{n-1}} \int_{I(Q,r)} w(x) d\sigma(x) \right\} \left\{ \frac{1}{r^{n-1}} \int_{I(Q,r)} [w(x)]^{-1/(p-1)} d\sigma(x) \right\}^{p-1} \leq C < \infty,$$

where $\text{diam}(\partial\Omega) := \sup\{|x - y| : x, y \in \partial\Omega\}$ and $I(Q, r) := B(Q, r) \cap \partial\Omega$. The smallest constant C such that (1.5) holds true is called the $A_p(\partial\Omega)$ weight constant of ω and denoted by $[\omega]_{A_p(\partial\Omega)}$.

Then the main results of this article read as follows.

Theorem 1.4. *Let Ω be a bounded (semi-)convex domain in \mathbb{R}^n with $n \geq 2$. Assume that $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$. Then the $L^p_\omega(\partial\Omega)$ -Dirichlet problem (1.2) on Ω is uniquely solvable. Namely, for any given $f \in L^p_\omega(\partial\Omega)$, there exists a harmonic function u in Ω such that $(u)^* \in L^p_\omega(\partial\Omega)$ and $u = f$ on $\partial\Omega$. Moreover, there exists a positive constant C , depending on n, p, ω and Ω , such that*

$$(1.6) \quad \|(u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|f\|_{L^p_\omega(\partial\Omega)}.$$

Theorem 1.5. *Let Ω be a bounded (semi-)convex domain in \mathbb{R}^n with $n \geq 2$. Assume that $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$. Then the $L^p_\omega(\partial\Omega)$ -Neumann problem (1.3) in Ω is uniquely solvable. Namely, for any given $f \in L^p_\omega(\partial\Omega)$ with $\int_{\partial\Omega} f(x) d\sigma(x) = 0$, there exists a harmonic function u in Ω , up to constants, such that $(\nabla u)^* \in L^p_\omega(\partial\Omega)$ and $\partial u/\partial\nu = f$ on $\partial\Omega$. Moreover, there exists a positive constant C , depending on n, p, ω and Ω , such that*

$$(1.7) \quad \|(\nabla u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|f\|_{L^p_\omega(\partial\Omega)}.$$

Theorem 1.6. *Let Ω be a bounded (semi-)convex domain in \mathbb{R}^n with $n \geq 2$. Assume that $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$. Then the $\dot{W}^p_{1,\omega}(\partial\Omega)$ -Regularity problem (1.4) on Ω is uniquely solvable. Namely, for any given $f \in \dot{W}^p_{1,\omega}(\partial\Omega)$, there exists a harmonic function u in Ω such that $(\nabla u)^* \in L^p_\omega(\partial\Omega)$ and $u = f$ on $\partial\Omega$. Moreover, there exists a positive constant C , depending on n, p, ω and Ω , such that*

$$(1.8) \quad \|(\nabla u)^*\|_{L^p_\omega(\partial\Omega)} \leq C\|\nabla_t f\|_{L^p_\omega(\partial\Omega)}.$$

We prove Theorem 1.4 by using the unique solvability of the $L^p(\partial\Omega)$ -Dirichlet problem on bounded (semi-)convex domains for any $p \in (1, \infty)$ obtained in [18, Theorem 3.10] and a key pointwise estimate for $(u)^*$ in [18, p. 2541] (see also (2.3) below).

The proof of Theorem 1.5 is based on a weighted real variable argument (see Lemma 2.4 below), which was obtained by Shen [21, Theorem 3.4] and inspired by [4] (see also [24]), a criterion for the solvability of the $L^p(\partial\Omega)$ -Neumann problem on bounded Lipschitz domains (see Lemma 2.7 below), which is essentially established in [15, Theorem 1.1] (see also [20]), and the unique solvability of the $L^p(\partial\Omega)$ -Neumann problem on bounded (semi-)convex domains for any $p \in (1, \infty)$ obtained in [11, 15, 25]. We also remark that a similar real variable argument with the different motivation was also used in [1, 2]. Moreover, the proof of Theorem 1.6 is similar to that of Theorem 1.5. More precisely, by the weighted real variable argument in Lemma 2.4, the criterion for the solvability of the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem on bounded Lipschitz domains obtained in Lemma 2.9 below and the unique solvability of the $\dot{W}^p_1(\partial\Omega)$ -Regularity problem on bounded (semi-)convex domains for any $p \in (1, \infty)$ established in [18, Theorem 3.11] (see also Lemma 2.10 below), we prove Theorem 1.6.

Remark 1.7. Let Ω be a bounded C^1 domain in \mathbb{R}^n . For any $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$, via replacing Lemma 2.8 below by the unique solvability of the $L^q(\partial\Omega)$ -Neumann problem for any $q \in (1, \infty)$ established in [8] and then repeating the proof of Theorem 1.5, we can obtain the unique solvability of the $L^p_\omega(\partial\Omega)$ -Neumann problem (1.3) on Ω and the weighted estimate (1.7). Thus, this gives another proof for [7, Theorem 1.3]. Recall that, [7, Theorem 1.3] was proved by the boundedness, convergence and compactness of the boundary double layer potential and its adjoint operator on the weighted space $L^p_\omega(\partial\Omega)$.

The layout of this article is as follows. In Section 2, we recall several key conclusions and then we give the proofs of Theorems 1.4, 1.5 and 1.6.

Finally we make some conventions on notation. Denote by C a *positive constant* which is independent of the solution u and the datum f , but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. We also let $\mathbb{N} := \{1, 2, \dots\}$. Finally, for the domain Ω in \mathbb{R}^n , $\omega \in A_p(\partial\Omega)$, with $p \in (1, \infty)$, and the measurable set $E \subset \partial\Omega$, let $\sigma(E) := \int_E d\sigma(x)$ and $\omega(E) := \int_E \omega(x) d\sigma(x)$, where $d\sigma$ denotes the surface measure on $\partial\Omega$.

2. Proofs of Theorems 1.4, 1.5 and 1.6

In this section, we give the proofs of Theorems 1.4, 1.5 and 1.6. To show Theorem 1.4, we first recall a property of $A_p(\partial\Omega)$ weight and the solvability result for the $L^p(\partial\Omega)$ -Dirichlet problem on bounded (semi-)convex domains.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$. Then there exists $q \in (1, p)$ such that $\omega \in A_q(\partial\Omega)$.*

The conclusion of Lemma 2.1 is well known (see, for example, [22, p. 5, Lemma 8]). The following Lemma 2.2 is just [18, Theorem 3.10].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded (semi-)convex domain and $p \in (1, \infty)$. Then, for any $f \in L^p(\partial\Omega)$, the $L^p(\partial\Omega)$ -Dirichlet problem with datum f (taking $\omega \equiv 1$ in (1.2)) is uniquely solvable. Moreover, the solution u satisfies*

$$\|(u)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)},$$

where C is a positive constant independent of u and f .

Now we prove Theorem 1.4 by using Lemmas 2.1 and 2.2.

Proof of Theorem 1.4. Let $p \in (1, \infty)$, $\omega \in A_p(\partial\Omega)$ and $f \in L^p_\omega(\partial\Omega)$. Then, it follows, from Lemma 2.1, that there exists $q \in (1, p)$ such that $\omega \in A_{p/q}(\partial\Omega)$. By the Hölder

inequality, we find that

$$(2.1) \quad \int_{\partial\Omega} |f(x)|^q d\sigma(x) \leq \left\{ \int_{\partial\Omega} |f(x)|^p \omega(x) d\sigma(x) \right\}^{q/p} \left\{ \int_{\partial\Omega} [\omega(x)]^{-\frac{1}{p/q-1}} d\sigma(x) \right\}^{\frac{q}{p} \left(\frac{p}{q} - 1 \right)}.$$

Moreover, from the fact that Ω is bounded, we deduce that there exist finitely many balls $\{B(Q_j, \delta_j)\}_{j=1}^{N_0}$, with $\{Q_j\}_{j=1}^{N_0} \subset \partial\Omega$ and $\{\delta_j\}_{j=1}^{N_0} \subset (0, \text{diam}(\partial\Omega))$, such that $\partial\Omega \subset \bigcup_{j=1}^{N_0} B(Q_j, \delta_j)$, where N_0 is a positive integer depending on Ω . By this and the definition of $A_{p/q}(\partial\Omega)$, we further conclude that

$$\begin{aligned} \left\{ \int_{\partial\Omega} [\omega(x)]^{-\frac{1}{p/q-1}} d\sigma(x) \right\}^{p/q-1} &\lesssim \sum_{j=1}^{N_0} \left\{ \int_{B(Q_j, \delta_j) \cap \partial\Omega} [\omega(x)]^{-\frac{1}{p/q-1}} d\sigma(x) \right\}^{p/q-1} \\ &\lesssim \sum_{j=1}^{N_0} \delta_j^{p(n-1)/q} \left\{ \int_{B(Q_j, \delta_j) \cap \partial\Omega} \omega(x) d\sigma(x) \right\}^{-1}, \end{aligned}$$

which, together with the fact that, for any $j \in \{1, \dots, N_0\}$, $\int_{B(Q_j, \delta_j) \cap \partial\Omega} \omega(x) d\sigma(x) > 0$ and $\delta_j < \text{diam}(\partial\Omega)$, implies that

$$\left\{ \int_{\partial\Omega} [\omega(x)]^{-\frac{1}{p/q-1}} d\sigma(x) \right\}^{p/q-1} \lesssim 1.$$

From this and (2.1), we deduce that

$$(2.2) \quad f \in L^p_\omega(\partial\Omega) \subset L^q(\partial\Omega) \quad \text{and} \quad \|f\|_{L^q(\partial\Omega)} \lesssim \|f\|_{L^p_\omega(\partial\Omega)},$$

which, combined with Lemma 2.2, further implies that the $L^p_\omega(\partial\Omega)$ -Dirichlet problem (1.2) with datum f is uniquely solvable.

Let u be the solution of the Dirichlet problem (1.2) with datum f . Now we prove (1.6). Indeed, for $f \in L^q(\partial\Omega)$, it was proved in [18, p. 2541] that, for any $x \in \partial\Omega$,

$$(2.3) \quad (u)^*(x) \lesssim M(f)(x),$$

where $M(f)$ denotes the *Hardy-Littlewood maximal function* of f on $\partial\Omega$, which is defined by, for any $x \in \partial\Omega$,

$$M(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{\sigma(B(x, r) \cap \partial\Omega)} \int_{B(x, r) \cap \partial\Omega} |f(y)| d\sigma(y).$$

Then, by (2.3) and the boundedness of M on $L^p_\omega(\partial\Omega)$ with $p \in (1, \infty)$ and $\omega \in A_p(\partial\Omega)$ (see, for example, [22, p. 5, Theorem 9]), we conclude that (1.6) holds true. This finishes the proof of Theorem 1.4. □

To state the weighted real variable argument established in [21, Theorem 3.4], we first recall the definition of the $A_p(\mathbb{R}^n)$ weight.

Definition 2.3. Let $n \in \mathbb{N}$ and $p \in (1, \infty)$. A non-negative and locally integrable function ω on \mathbb{R}^n is called an $A_p(\mathbb{R}^n)$ weight, if

$$(2.4) \quad [\omega]_{A_p(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B w(x) dx \right\} \left\{ \frac{1}{|B|} \int_B [w(x)]^{-1/(p-1)} dx \right\}^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Moreover, for any given measurable set $E \subset \mathbb{R}^n$, the $A_p(E)$ weight can be defined via replacing the ball B in (2.4) by $B \cap E$.

Then the weighted real variable argument established in [21, Theorem 3.4] is as follows.

Lemma 2.4. Let Q_0 be a cube in \mathbb{R}^n , $F \in L^1(2Q_0)$, $p_1 \in (1, \infty)$, $p_2 \in (1, p_1)$ and $f \in L^{p_2}(2Q_0)$. Let $0 < \beta < 1 < \alpha < \infty$. Assume further that, for any dyadic subcube Q of Q_0 with $|Q| \leq \beta|Q_0|$, there exist two functions F_Q and R_Q on $2Q$ such that $|F| \leq |F_Q| + |R_Q|$ on $2Q$,

$$\left\{ \frac{1}{|2Q|} \int_{2Q} |R_Q(x)|^{p_1} dx \right\}^{1/p_1} \leq C_1 \left[\frac{1}{|\alpha Q|} \int_{\alpha Q} |F(x)| dx + \sup_{\tilde{Q} \supset Q} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)| dx \right]$$

and

$$\frac{1}{|2Q|} \int_{2Q} |F_Q(x)| dx \leq C_2 \sup_{\tilde{Q} \supset Q} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)| dx,$$

where C_1 and C_2 are positive constants independent of F , f , F_Q , R_Q and Q , and the suprema are taken over all dyadic cubes $\tilde{Q} \subset Q_0$. Then, for any $\omega \in A_{p_2}(2Q_0)$ satisfying that there exist positive constants C and $\eta \in (p_2/p_1, \infty)$ such that, for any cube $Q \subset Q_0$ and any measurable $E \subset Q$, $\omega(E)/\omega(Q) \leq C[|E|/|Q|]^\eta$, where $\omega(E) := \int_E \omega(x) dx$, it holds true that

$$\begin{aligned} & \left\{ \frac{1}{\omega(Q_0)} \int_{Q_0} |F(x)|^{p_2} \omega(x) dx \right\}^{1/p_2} \\ & \leq \frac{C}{|2Q_0|} \int_{2Q_0} |F(x)| dx + C \left\{ \frac{1}{\omega(2Q_0)} \int_{2Q_0} |f(x)|^{p_2} \omega(x) dx \right\}^{1/p_2}, \end{aligned}$$

where C is a positive constant independent of F and f .

Let

$$(2.5) \quad D := \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ and } x_n > \psi(x')\},$$

where $\psi: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a Lipschitz function. The projection map $\Phi: \partial D \rightarrow \mathbb{R}^{n-1}$ is defined by $\Phi(x', \psi(x')) = x'$. Then it is said that $S \subset \partial D$ is a (surface) cube of ∂D , if $\Phi(S)$ is a cube of \mathbb{R}^{n-1} . Similarly, it is said that \tilde{S} is a dyadic subcube of S , if $\Phi(\tilde{S})$ is a dyadic subcube of $\Phi(S)$ in \mathbb{R}^{n-1} . Moreover, for any $a \in (0, \infty)$, the dilation aS of the cube S on ∂D may be defined by $aS := \Phi^{-1}(a\Phi(S))$.

Via the facts that

$$(2.6) \quad \|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim 1 \quad \text{and} \quad d\sigma(x) \sim dx \quad \text{on } \partial D$$

and similar to the proof of [7, Lemma 2.5], we obtain the following relation for $A_p(\partial D)$ and $A_p(\mathbb{R}^{n-1})$ weights, the details being omitted here.

Lemma 2.5. *Let D and ψ be as in (2.5), $p \in (1, \infty)$ and $\omega \in A_p(\partial D)$. Then $\omega(\cdot, \psi(\cdot)) \in A_p(\mathbb{R}^{n-1})$.*

Then it is easy to obtain the following weighted real variable argument on ∂D by Lemmas 2.4 and 2.5.

Lemma 2.6. *Let D be as in (2.5) and S_0 a cube of ∂D . Assume that $F \in L^1(2S_0)$, $p_1 \in (1, \infty)$, $p_2 \in (1, p_1)$ and $\tilde{f} \in L^{p_2}(2S_0)$. Let $0 < \beta < 1 < \alpha < \infty$. Assume further that, for any dyadic subcube S of S_0 with $\sigma(S) \leq \beta\sigma(S_0)$, there exist two functions F_S and R_S on $2S$ such that $|F| \leq |F_S| + |R_S|$ on $2S$,*

$$(2.7) \quad \left\{ \frac{1}{\sigma(2S)} \int_{2S} |R_S(x)|^{p_1} d\sigma(x) \right\}^{1/p_1} \leq C_3 \left\{ \frac{1}{\sigma(\alpha S)} \int_{\alpha S} |F(x)| d\sigma(x) + \sup_{\tilde{S} \supset S} \frac{1}{\sigma(\tilde{S})} \int_{\tilde{S}} |\tilde{f}(x)| d\sigma(x) \right\}$$

and

$$(2.8) \quad \frac{1}{\sigma(2S)} \int_{2S} |F_S(x)| d\sigma(x) \leq C_4 \sup_{\tilde{S} \supset S} \frac{1}{\sigma(\tilde{S})} \int_{\tilde{S}} |\tilde{f}(x)| d\sigma(x),$$

where C_3 and C_4 are positive constants independent of F , \tilde{f} , F_S , R_S and S , and the suprema are taken over all dyadic cubes $\tilde{S} \subset S_0$. Then, for any $\omega \in A_{p_2}(2S_0)$ satisfying that there exist positive constants C and $\eta \in (p_2/p_1, \infty)$ such that, for any cube $S \subset S_0$ and any measurable $E \subset S$,

$$(2.9) \quad \frac{\omega(E)}{\omega(S)} \leq C \left[\frac{\sigma(E)}{\sigma(S)} \right]^\eta,$$

it holds true that

$$\left\{ \frac{1}{\omega(S_0)} \int_{S_0} |F(x)|^{p_2} \omega(x) d\sigma(x) \right\}^{1/p_2} \leq \frac{C}{\sigma(2S_0)} \int_{2S_0} |F(x)| d\sigma(x) + C \left\{ \frac{1}{\omega(2S_0)} \int_{2S_0} |\tilde{f}(x)|^{p_2} \omega(x) d\sigma(x) \right\}^{1/p_2},$$

where C is a positive constant independent of F and \tilde{f} .

To show Theorem 1.5, we need the following equivalent characterization for the unique solvability of the $L^p(\partial\Omega)$ -Neumann problem on Ω .

Lemma 2.7. *Let $p_0 \in (1, \infty)$ and Ω be a bounded Lipschitz domain in \mathbb{R}^n with $n \geq 2$. Assume that the $L^{p_0}(\partial\Omega)$ -Neumann problem on Ω (taking $\omega \equiv 1$ in (1.3)) is uniquely solvable. Let $p \in (p_0, \infty)$. Then the following two statements are equivalent.*

- (i) *The $L^p(\partial\Omega)$ -Neumann problem on Ω is uniquely solvable. Moreover, the solution u satisfies*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)},$$

where C is a positive constant depending only on n, p and the Lipschitz character of Ω .

- (ii) *There exist positive constants $\tilde{C} \in (0, \infty)$ and $r_0 \in (0, \text{diam}(\Omega))$ such that, for any $r \in (0, r_0)$ and $x \in \partial\Omega$, the weak reverse Hölder inequality*

$$\begin{aligned} & \left\{ \frac{1}{r^{n-1}} \int_{B(x,r) \cap \partial\Omega} [(\nabla w)^*(x)]^p d\sigma(x) \right\}^{1/p} \\ & \leq \tilde{C} \left\{ \frac{1}{r^{n-1}} \int_{B(x,2r) \cap \partial\Omega} [(\nabla w)^*(x)]^{p_0} d\sigma(x) \right\}^{1/p_0} \end{aligned}$$

holds for any harmonic function w in Ω satisfying $(\nabla w)^* \in L^{p_0}(\partial\Omega)$ and $\partial w / \partial \nu = 0$ on $B(x, 3r) \cap \partial\Omega$.

We point out that Lemma 2.7 was essentially established in [15, Theorem 1.1]. Indeed, when $p_0 := 2$, Lemma 2.7 is just [15, Theorem 1.1]. For general $p_0 \in (1, \infty)$, by replacing [15, Lemma 2.2] with Lemma 2.6 (in this case, taking $\omega \equiv 1$) and then repeating the proof of [15, Theorem 2.1], we obtain the proof of (ii) implying (i) in Lemma 2.7. Notice that, for any bounded Lipschitz domain Ω , the $L^2(\partial\Omega)$ -Neumann problem is always uniquely solvable. Via this fact, replacing the exponent 2 with p_0 and repeating the proof of [15, Theorem 3.1], we conclude that (i) implies (ii) in Lemma 2.7. The details are omitted here.

The following Lemma 2.8 was established in [15, Theorem 1.2], [11, Theorem 1.1] and [25, Theorem 1.4].

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded (semi-)convex domain and $p \in (1, \infty)$. Then, for any $f \in L^p(\partial\Omega)$, the $L^p(\partial\Omega)$ -Neumann problem with datum f (taking $\omega \equiv 1$ in (1.3)) is uniquely solvable. Moreover, the solution u satisfies*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)},$$

where C is a positive constant independent of u and f .

Now we give the proof of Theorem 1.5 by using Lemmas 2.6, 2.7 and 2.8.

Proof of Theorem 1.5. Let $p \in (1, \infty)$, $\omega \in A_p(\partial\Omega)$ and $f \in L^p_\omega(\partial\Omega)$. Then, by (2.2) and Lemma 2.8, we conclude that the $L^p_\omega(\partial\Omega)$ -Neumann problem (1.3) with datum f is uniquely solvable.

Let u be the solution of the Neumann problem (1.3) with datum f . To end the proof of Theorem 1.5, we only need to show (1.7).

Let $Q \in \partial\Omega$ and $r \in (0, r_0)$, where r_0 is as in Lemma 2.7. Since Ω is a bounded semi-convex domain in \mathbb{R}^n and hence a Lipschitz domain, by rotation and translation, we may assume that $Q = 0$ and

$$(2.10) \quad B(Q, 100r_0\sqrt{n}) \cap \Omega = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > \psi(x')\} \cap B(Q, 100r_0\sqrt{n}),$$

where $\psi: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a Lipschitz function satisfying that $\psi(0) = 0$. Let

$$(2.11) \quad S_0 := S_0(r) := \{(x', \psi(x')) : |x_1| < r, \dots, |x_{n-1}| < r\}$$

be a surface cube on the Lipschitz graph ∂D , where D is as in (2.5).

Let $q \in (1, p)$ such that $\omega \in A_{p/q}(\partial\Omega)$. To apply Lemma 2.4, let $F := |(\nabla u)^*|^q$ and $\tilde{f} := |f|^q$. Furthermore, for any dyadic subcube S of S_0 with $\sigma(S) \leq \beta\sigma(S_0)$, define

$$g := f\chi_{8S} - \left\{ \frac{1}{\sigma(\partial\Omega \setminus 8S)} \int_{8S} f(x) d\sigma(x) \right\} \chi_{\partial\Omega \setminus 8S}.$$

Then $\int_{\partial\Omega} g(x) d\sigma(x) = 0$ and by (2.2), we find that $g \in L^q(\partial\Omega)$. Let

$$F_S := 2^{q-1}|(\nabla w)^*|^q \quad \text{and} \quad R_S := 2^{q-1}|(\nabla v)^*|^q,$$

where $v = u - w$ and w is the solution of the Neumann problem

$$(2.12) \quad \begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = g \in L^q(\partial\Omega) & \text{on } \partial\Omega, \\ (\nabla w)^* \in L^q(\partial\Omega). \end{cases}$$

It follows, from Lemma 2.8, that the Neumann problem (2.12) is uniquely solvable. It is easy to see that $\partial v / \partial \nu = 0$ on $8S$ and $|F| \leq |F_S| + |R_S|$ on $\partial\Omega$. By Lemma 2.8 again, we conclude that

$$\begin{aligned} \frac{1}{\sigma(2S)} \int_{2S} F_S(x) d\sigma(x) &= \frac{2^{q-1}}{\sigma(2S)} \int_{2S} [(\nabla w)^*(x)]^q d\sigma(x) \\ &\leq \frac{2^{q-1}}{\sigma(2S)} \int_{\partial\Omega} [(\nabla w)^*(x)]^q d\sigma(x) \lesssim \frac{1}{\sigma(2S)} \int_{\partial\Omega} |g(x)|^q d\sigma(x) \\ &\lesssim \frac{1}{\sigma(2S)} \int_{8S} |f(x)|^q d\sigma(x) + \frac{1}{\sigma(\partial\Omega \setminus 8S)} \int_{8S} |f(x)|^q d\sigma(x) \\ &\lesssim \frac{1}{\sigma(8S)} \int_{8S} |f(x)|^q d\sigma(x) \sim \frac{1}{\sigma(8S)} \int_{8S} |\tilde{f}(x)| d\sigma(x), \end{aligned}$$

which implies that (2.8) holds true.

Moreover, from the fact that Ω is a bounded semi-convex domain and Lemma 2.8, we deduce that, for any $s \in (1, \infty)$, the $L^{sq}(\partial\Omega)$ -Neumann problem is uniquely solvable, which, together with the fact that $\partial v/\partial\nu = 0$ on $8S$, Lemma 2.7 and (2.6), further implies that, for any $s \in (1, \infty)$,

$$\left\{ \frac{1}{\sigma(2S)} \int_{2S} [(\nabla v)^*(x)]^{sq} d\sigma(x) \right\}^{1/(sq)} \lesssim \left\{ \frac{1}{\sigma(4S)} \int_{4S} [(\nabla v)^*(x)]^q d\sigma(x) \right\}^{1/q}.$$

By this estimate, we conclude that, for any $s \in (1, \infty)$,

$$\begin{aligned} & \left\{ \frac{1}{\sigma(2S)} \int_{2S} [R_S(x)]^s d\sigma(x) \right\}^{1/s} \\ &= \left\{ \frac{2^{(q-1)s}}{\sigma(2S)} \int_{2S} [(\nabla v)^*(x)]^{sq} d\sigma(x) \right\}^{1/s} \\ (2.13) \quad & \lesssim \frac{1}{\sigma(4S)} \int_{4S} [(\nabla v)^*(x)]^q d\sigma(x) \\ & \lesssim \frac{1}{\sigma(4S)} \int_{4S} [(\nabla u)^*(x)]^q d\sigma(x) + \frac{1}{\sigma(4S)} \int_{4S} [(\nabla w)^*(x)]^q d\sigma(x) \\ & \lesssim \frac{1}{\sigma(4S)} \int_{4S} F(x) d\sigma(x) + \frac{1}{\sigma(8S)} \int_{8S} |\tilde{f}(x)| d\sigma(x), \end{aligned}$$

which implies that (2.7) holds true for $p_1 := s$.

Now we prove that ω satisfies (2.9). Recall that, for some $s \in (1, \infty]$, a non-negative and locally integrable function V on $\partial\Omega$ is said to belong to the *reverse Hölder class* $RH_s(\partial\Omega)$, if there exists a positive constant C such that, for any $Q \in \partial\Omega$ and $r \in (0, \text{diam}(\partial\Omega))$,

$$\left\{ \frac{1}{r^{n-1}} \int_{I(Q,r)} [V(x)]^s d\sigma(x) \right\}^{1/s} \leq \frac{C}{r^{n-1}} \int_{I(Q,r)} V(x) d\sigma(x),$$

here and hereafter, $I(Q, r) := B(Q, r) \cap \Omega$. Then it is well known that, for any $q \in (1, \infty)$,

$$(2.14) \quad A_q(\partial\Omega) \subset \bigcup_{s \in (1, \infty]} RH_s(\partial\Omega)$$

(see, for example, [22, p. 9, Theorem 15]). Moreover, if $V \in RH_s(\partial\Omega)$ with $s \in (1, \infty]$, then for any $Q \in \partial\Omega$, $r \in (0, \text{diam}(\partial\Omega))$ and measurable set $E \subset I(Q, r)$, $V(E)/V(I(Q, r)) \lesssim [\sigma(E)/\sigma(I(Q, r))]^{(s-1)/s}$. Indeed, from the definition of $RH_s(\partial\Omega)$, the Hölder inequality and the fact that $\sigma(I(Q, r)) \sim r^{n-1}$, it follows that

$$\begin{aligned} V(E) &\leq \left\{ \int_E [V(x)]^s d\sigma(x) \right\}^{1/s} [\sigma(E)]^{1/s'} \leq \left\{ \int_{I(Q,r)} [V(x)]^s d\sigma(x) \right\}^{1/s} [\sigma(E)]^{1/s'} \\ &\lesssim [\sigma(I(Q, r))]^{-1/s'} V(I(Q, r)) [\sigma(E)]^{1/s'}, \end{aligned}$$

which implies that

$$\frac{V(E)}{V(I(Q, r))} \lesssim \left[\frac{\sigma(E)}{\sigma(I(Q, r))} \right]^{(s-1)/s}.$$

By this and (2.14), we conclude that there exists $s_0 \in (1, \infty]$ such that $\omega \in RH_{s_0}(\partial\Omega)$ and, for any $Q \in \partial\Omega$, $r \in (0, \text{diam}(\partial\Omega))$ and measurable set $E \subset I(Q, r)$,

$$(2.15) \quad \frac{\omega(E)}{\omega(I(Q, r))} \lesssim \left[\frac{\sigma(E)}{\sigma(I(Q, r))} \right]^{(s_0-1)/s_0}.$$

Let $p_2 := p/q$. From the fact that (2.13) holds true for any $s \in (1, \infty)$, we deduce that there exists $p_1 := s$ such that $(s_0 - 1)/s_0 > p_2/p_1$ and (2.13) holds true for such p_1 , which, combined with (2.15), further implies that (2.9) holds true for such p_1 and p_2 . Thus, applying Lemma 2.6 to F and \tilde{f} , we find that

$$\begin{aligned} & \left\{ \frac{1}{\omega(S_0)} \int_{S_0} [(\nabla u)^*(x)]^p \omega(x) \, d\sigma(x) \right\}^{q/p} \\ &= \left\{ \frac{1}{\omega(S_0)} \int_{S_0} [F(x)]^{p_2} \omega(x) \, d\sigma(x) \right\}^{1/p_2} \\ &\lesssim \frac{1}{\sigma(2S_0)} \int_{2S_0} F(x) \, d\sigma(x) + \left\{ \frac{1}{\omega(2S_0)} \int_{2S_0} |\tilde{f}(x)|^{p_2} \omega(x) \, d\sigma(x) \right\}^{1/p_2} \\ &\sim \frac{1}{\sigma(2S_0)} \int_{2S_0} [(\nabla u)^*(x)]^q \, d\sigma(x) + \left\{ \frac{1}{\omega(2S_0)} \int_{2S_0} |f(x)|^p \omega(x) \, d\sigma(x) \right\}^{q/p}, \end{aligned}$$

which further implies that

$$(2.16) \quad \begin{aligned} & \left\{ \frac{1}{\omega(S_0)} \int_{S_0} [(\nabla u)^*(x)]^p \omega(x) \, d\sigma(x) \right\}^{1/p} \\ &\lesssim \left\{ \frac{1}{\sigma(2S_0)} \int_{2S_0} [(\nabla u)^*(x)]^q \, d\sigma(x) \right\}^{1/q} + \left\{ \frac{1}{\omega(2S_0)} \int_{2S_0} |f(x)|^p \omega(x) \, d\sigma(x) \right\}^{1/p}. \end{aligned}$$

Then, from (2.16) and a simple covering argument, it follows that

$$(2.17) \quad \begin{aligned} & \left\{ \frac{1}{\omega(I(Q, r))} \int_{I(Q, r)} [(\nabla u)^*(x)]^p \omega(x) \, d\sigma(x) \right\}^{1/p} \\ &\lesssim \left\{ \frac{1}{\sigma(I(Q, 2r))} \int_{I(Q, 2r)} [(\nabla u)^*(x)]^q \, d\sigma(x) \right\}^{1/q} \\ &\quad + \left\{ \frac{1}{\omega(I(Q, 2r))} \int_{I(Q, 2r)} |f(x)|^p \omega(x) \, d\sigma(x) \right\}^{1/p}. \end{aligned}$$

By the fact that Ω is bounded and Remark 1.2(i), we conclude that there exist a positive constant c_0 , finite sets $\{Q_j\}_{j=1}^{N_0} \subset \partial\Omega$ of points and $\{\psi_j\}_{j=1}^{N_0}$ of Lipschitz functions

such that $\partial\Omega \subset \bigcup_{j=1}^{N_0} I(Q_j, c_0r_0)$, $\{Q_j\}_{j=1}^{N_0}$ and $\{\psi_j\}_{j=1}^{N_0}$ satisfy (2.10), and the estimate (2.17) holds true for $I(Q_j, c_0r_0)$ with $j \in \{1, \dots, N_0\}$, where $N_0 \in \mathbb{N}$ is a positive integer depending on Ω and r_0 is as in Lemma 2.7. Then, via covering $\partial\Omega$ with the finite collection of surface balls $\{I(Q_j, c_0r_0)\}_{j=1}^{N_0}$, it follows, from (2.17), the Hölder inequality and (2.2) that

$$\begin{aligned} \|(\nabla u)^*\|_{L^p_\omega(\partial\Omega)} &\leq \sum_{j=1}^{N_0} \|(\nabla u)^*\|_{L^p_\omega(I(Q_j, c_0r_0))} \\ &\lesssim \sum_{j=1}^{N_0} \frac{[\omega(I(Q_j, c_0r_0))]^{1/p}}{[\sigma(I(Q_j, 2c_0r_0))]^{1/q}} \|f\|_{L^q(\partial\Omega)} + \sum_{j=1}^{N_0} \|f\|_{L^p_\omega(I(Q_j, 2c_0r_0))} \\ &\lesssim \|f\|_{L^q(\partial\Omega)} + \|f\|_{L^p_\omega(\partial\Omega)} \lesssim \|f\|_{L^p_\omega(\partial\Omega)}. \end{aligned}$$

Thus, (1.7) holds true, which completes the proof of Theorem 1.5. □

To show Theorem 1.6, we need the following two lemmas.

Lemma 2.9. *Let $p_0 \in (1, \infty)$ and Ω be a bounded Lipschitz domain in \mathbb{R}^n with $n \geq 2$. Assume that the $\dot{W}_1^{p_0}(\partial\Omega)$ -Regularity problem on Ω (taking $\omega \equiv 1$ in (1.4)) is uniquely solvable. Let $p \in (p_0, \infty)$. Then the following two statements are equivalent.*

- (i) *The $\dot{W}_1^p(\partial\Omega)$ -Regularity problem on Ω is uniquely solvable. Moreover, the solution u satisfies*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \|\nabla_t f\|_{L^p(\partial\Omega)},$$

where C is a positive constant depending only on n, p and the Lipschitz character of Ω .

- (ii) *There exist positive constants $\tilde{C} \in (0, \infty)$ and $r_0 \in (0, \text{diam}(\Omega))$ such that, for any $r \in (0, r_0)$ and $x \in \partial\Omega$, the weak reverse Hölder inequality*

$$\begin{aligned} (2.18) \quad &\left\{ \frac{1}{r^{n-1}} \int_{B(x,r) \cap \partial\Omega} [(\nabla w)^*(x)]^p d\sigma(x) \right\}^{1/p} \\ &\leq \tilde{C} \left\{ \frac{1}{r^{n-1}} \int_{B(x,2r) \cap \partial\Omega} [(\nabla w)^*(x)]^{p_0} d\sigma(x) \right\}^{1/p_0} \end{aligned}$$

holds for any harmonic function w in Ω satisfying $(\nabla w)^* \in L^{p_0}(\partial\Omega)$ and $w = 0$ on $B(x, 3r) \cap \partial\Omega$.

Proof. The proof of Lemma 2.9 is similar to that of [15, Theorem 1.1] or [10, Theorem 1.1]. For the sake of completeness, we give some details.

We first prove that (ii) implies (i). Let $f \in \dot{W}_1^p(\partial\Omega)$, $Q \in \partial\Omega$ and $r \in (0, r_0)$, where r_0 is as in Lemma 2.9(ii). By rotation and translation, we may assume that $Q = 0$ and

(2.10) holds true in this case. Let S_0 be as in (2.11) and S be a dyadic subcube of S_0 with $\sigma(S) \leq \beta\sigma(S_0)$. Assume that $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $8S$, $\varphi \equiv 0$ on $\mathbb{R}^n \setminus 16S$ and $|\nabla\varphi| \lesssim r^{-1}$. Let $f_1 := \varphi(f - c_f)$ and $f_2 := (1 - \varphi)(f - c_f)$, where

$$c_f := \frac{1}{\sigma(16S)} \int_{16S} f(x) d\sigma(x).$$

Assume that $u = u_1 + u_2 + c_f$, where u_1 and u_2 are the solutions of the $\dot{W}_1^{p_0}(\partial\Omega)$ -Regularity problem with data f_1 and f_2 , respectively. Let

$$F := |(\nabla u)^*|^{p_0}, \quad \tilde{f} := |\nabla_t f|^{p_0}, \quad F_S := 2^{p_0-1}|(\nabla u_1)^*|^{p_0}$$

and $R_S := 2^{p_0-1}|(\nabla u_2)^*|^{p_0}$. Then $u_2 = 0$ on $8S$ and $|F| \leq |F_S| + |R_S|$ on $\partial\Omega$. By the fact that u_1 is the solution of the $\dot{W}_1^{p_0}(\partial\Omega)$ -Regularity problem and the Poincaré inequality, we find that

$$\begin{aligned} & \frac{1}{\sigma(2S)} \int_{2S} F_S(x) d\sigma(x) \\ &= \frac{2^{p_0-1}}{\sigma(2S)} \int_{2S} [(\nabla u_1)^*(x)]^{p_0} d\sigma(x) \lesssim \frac{1}{\sigma(2S)} \int_{\partial\Omega} |\nabla_t f_1(x)|^{p_0} d\sigma(x) \\ &\lesssim \frac{1}{\sigma(2S)} \int_{16S} |\nabla_t f(x)|^{p_0} d\sigma(x) + \frac{1}{\sigma(2S)} \frac{1}{r^{p_0}} \int_{16S \setminus 8S} |f(x) - c_f|^{p_0} d\sigma(x) \\ &\lesssim \frac{1}{\sigma(16S)} \int_{16S} |\nabla_t f(x)|^{p_0} d\sigma(x) \sim \frac{1}{\sigma(16S)} \int_{16S} |\tilde{f}(x)| d\sigma(x), \end{aligned}$$

which implies that (2.8) holds true.

Furthermore, it is well known that the weak reverse Hölder inequality has the property of self-improving, which implies that if (2.18) holds true for some $p \in (p_0, \infty)$, then (2.18) also holds true for $p + \varepsilon$, where $\varepsilon \in (0, \infty)$ is a constant. From this and the fact that $u_2 = 0$ on $8S$, we deduce that

$$\left\{ \frac{1}{\sigma(2S)} \int_{2S} [(\nabla u_2)^*(x)]^{p+\varepsilon} d\sigma(x) \right\}^{1/(p+\varepsilon)} \lesssim \left\{ \frac{1}{\sigma(4S)} \int_{4S} [(\nabla u_2)^*(x)]^{p_0} d\sigma(x) \right\}^{1/p_0},$$

which further implies that

$$\begin{aligned} & \left\{ \frac{1}{\sigma(2S)} \int_{2S} [R_S(x)]^{(p+\varepsilon)/p_0} d\sigma(x) \right\}^{p_0/(p+\varepsilon)} \\ &\lesssim \frac{1}{\sigma(4S)} \int_{4S} [(\nabla u_2)^*(x)]^{p_0} d\sigma(x) \\ &\lesssim \frac{1}{\sigma(4S)} \int_{4S} [(\nabla u)^*(x)]^{p_0} d\sigma(x) + \frac{1}{\sigma(4S)} \int_{4S} [(\nabla u_1)^*(x)]^{p_0} d\sigma(x) \\ &\lesssim \frac{1}{\sigma(4S)} \int_{4S} F(x) d\sigma(x) + \frac{1}{\sigma(16S)} \int_{16S} |\tilde{f}(x)| d\sigma(x). \end{aligned}$$

Thus, (2.7) holds true for $p_1 := (p + \varepsilon)/p_0$ in this case. Then, by Lemma 2.6 with $\omega \equiv 1$, we conclude that, for any $q \in (p_0, p + \varepsilon)$,

$$(2.19) \quad \begin{aligned} & \left\{ \frac{1}{r^{n-1}} \int_{B(Q,r) \cap \partial\Omega} [(\nabla u)^*(x)]^q d\sigma(x) \right\}^{1/q} \\ & \lesssim \left\{ \frac{1}{r^{n-1}} \int_{B(Q,2r) \cap \partial\Omega} [(\nabla u)^*(x)]^{p_0} d\sigma(x) \right\}^{1/p_0} \\ & \quad + \left\{ \frac{1}{r^{n-1}} \int_{B(Q,2r) \cap \partial\Omega} |\nabla_t f(x)|^q d\sigma(x) \right\}^{1/q}. \end{aligned}$$

Thus, (2.19) holds true for $q = p$. Then, by a simple covering argument, (2.19) with $q = p$, the fact that the $\dot{W}_1^{p_0}(\partial\Omega)$ -Regularity problem is uniquely solvable and the Hölder inequality, we find that

$$\begin{aligned} \|(\nabla u)^*\|_{L^p(\partial\Omega)} & \lesssim |\partial\Omega|^{1/p-1/p_0} \|(\nabla u)^*\|_{L^{p_0}(\partial\Omega)} + \|\nabla_t f\|_{L^p(\partial\Omega)} \\ & \lesssim |\partial\Omega|^{1/p-1/p_0} \|\nabla_t f\|_{L^{p_0}(\partial\Omega)} + \|\nabla_t f\|_{L^p(\partial\Omega)} \lesssim \|\nabla_t f\|_{L^p(\partial\Omega)}. \end{aligned}$$

This finishes the proof of (ii) implying (i).

Now we show that (i) implies (ii). Let $r \in (0, r_0)$ and w be a harmonic function in Ω with the properties that $(\nabla w)^* \in L^{p_0}(\partial\Omega)$ and $w = 0$ on $S(3r)$, where $S(3r)$ is as in (2.11).

Let

$$Z(r) := \{(x', x_n) \in \mathbb{R}^n : |x_1| < r, \dots, |x_{n-1}| < r, \psi(x') < x_n < C_5 r\},$$

where the Lipschitz function ψ is as in (2.5) and $C_5 := 1 + 10\sqrt{n}\|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})}$. Observe that $w = 0$ on $S(3r)$ implies that the Poincaré inequality holds true for the function w on $Z(2r)$. Based on this observation and the fact that the $\dot{W}_1^p(\partial\Omega)$ -Regularity problem is uniquely solvable, repeating the proof of [15, Theorem 3.1], we know that (2.18) holds true. This finishes the proof of (i) implying (ii) and hence the proof of Lemma 2.9. \square

The following Lemma 2.10 is just [18, Theorem 3.11].

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded (semi-)convex domain and $p \in (1, \infty)$. Then, for any $f \in \dot{W}_1^p(\partial\Omega)$, the $\dot{W}_1^p(\partial\Omega)$ -Regularity problem with datum f (taking $\omega \equiv 1$ in (1.4)) is uniquely solvable. Moreover, the solution u satisfies*

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leq C \|\nabla_t f\|_{L^p(\partial\Omega)},$$

where C is a positive constant independent of u and f .

Now we give the proof of Theorem 1.6 by using Lemmas 2.6, 2.9 and 2.10.

Proof of Theorem 1.6. Let $p \in (1, \infty)$, $\omega \in A_p(\partial\Omega)$ and $f \in \dot{W}_{1,\omega}^p(\partial\Omega)$. Then, by (2.2) and Lemma 2.10, we find that the $\dot{W}_{1,\omega}^p(\partial\Omega)$ -Regularity problem (1.4) with datum f is uniquely solvable.

Let u be the solution of the Regularity problem (1.4) with datum f . To finish the proof of Theorem 1.6, we only need to show (1.8).

Let $Q \in \partial\Omega$ and $r \in (0, r_0)$, where r_0 is as in Lemma 2.9. Assume that $Q = 0$ and $S_0 := S_0(r)$ is as in (2.11). Let $q \in (1, p)$ such that $\omega \in A_{p/q}(\partial\Omega)$. Assume that S is a dyadic subcube of S_0 with $\sigma(S) \leq \beta\sigma(S_0)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $8S$, $\varphi \equiv 0$ on $\mathbb{R}^n \setminus 16S$ and $|\nabla\varphi| \lesssim r^{-1}$. Assume that $u = u_1 + u_2 + c_f$, $f_1 := \varphi(f - c_f)$ and $f_2 := (1 - \varphi)(f - c_f)$, where u_1 and u_2 are, respectively, the solutions of the $\dot{W}_1^{p_0}(\partial\Omega)$ -Regularity problem with data f_1 and f_2 , and

$$c_f := \frac{1}{\sigma(16S)} \int_{16S} f(x) d\sigma(x).$$

Let

$$F := |(\nabla u)^*|^q, \quad \tilde{f} := |\nabla_t f|^q, \quad F_S := 2^{q-1}|(\nabla u_1)^*|^q$$

and $R_S := 2^{q-1}|(\nabla u_2)^*|^q$. It is easy to see that $u_2 = 0$ on $8S$ and $|F| \leq |F_S| + |R_S|$ on $\partial\Omega$. Then, via replacing Lemmas 2.7 and 2.8, respectively, by Lemmas 2.9 and 2.10, and repeating the proof of (1.7), we conclude that (1.8) holds true, which completes the proof of Theorem 1.6. □

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