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Reducibility, Lyapunov Exponent, Pure Point Spectra Property for Quasi-periodic Wave Operator

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Abstract. In the present paper, it is shown that the linear wave equation subject to Dirichlet boundary condition

$$u_{tt} - u_{xx} + \varepsilon V(\omega t, x)u = 0, \quad u(t,-\pi) = u(t,\pi) = 0$$

can be changed by a symplectic transformation into

$$v_{tt} - v_{xx} + \varepsilon M_\xi v = 0, \quad v(t,-\pi) = v(t,\pi) = 0,$$

where V is finitely smooth and time-quasi-periodic potential with frequency $\omega \in \mathbb{R}^n$ in some Cantor set of positive Lebeague measure and where M_ξ is a Fourier multiplier. Moreover, it is proved that the corresponding wave operator $\partial_t^2 - \partial_x^2 + \varepsilon V(\omega t, x)$ possesses the property of pure point spectra and zero Lyapunov exponent.

1. Introduction

If a self-adjoint differential operator with time-quasi-periodic coefficients can be reduced to one with constant coefficients, the spectrum property and Lyapunov exponent of the operator can be easily obtained. To this end, there are many literatures dealing with Schrödinger operator with time-quasi-periodic potential of the form

$$\mathbf{i}\dot{u} = (H_0 + \varepsilon W(\omega t, x, -\mathbf{i}\nabla))u, \quad x \in \mathbb{R}^d \text{ or } x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d,$$

where $H_0 = -\Delta + V(x)$ or an abstract self-adjoint (unbounded) operator and the perturbation W is quasiperiodic in time t and it may or may not depend on x or/and ∇. When $x \in \mathbb{R}^d$, there are many interesting and important results. See [2, 4, 10–12, 30], and the references therein. When $x \in \mathbb{T}^d$ with any integer $d \geq 1$, it is in [13], proved that

$$\dot{u} = -\mathbf{i}(\Delta u - \varepsilon W(\phi_0 + \omega t, x; \omega)u), \quad x \in \mathbb{T}^d \tag{1.1}$$

is reduced to an autonomous equation for most values of the frequency vector ω, where W is analytic in (t, x) and quasiperiodic in time t with frequency vector ω. The reduction is

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made by means of Töplitz-Lipschitz property of operator and very hard KAM technique. The basic difficulty is that the frequencies of the unperturbed operator $-\Delta$, denoted by λ_k ($k \in \mathbb{Z}$), have multiplicity

$$\lambda_k^\sharp \simeq |k|^{d-1} \to \infty \quad \text{as } |k| \to \infty \text{ if } d > 1.$$

Fortunately, the frequencies have a good separation property and

$$|\lambda_k - \lambda_{k'}| \geq 1 \quad \text{when } \lambda_k \neq \lambda_{k'}.$$

When the reducibility for a linear wave equation

$$u_{tt} = (-\Delta + \varepsilon V(\phi_0 + \omega t, x; \omega))u, \quad x \in \mathbb{T}^d, \tag{1.2}$$

the unperturbed operator is $\sqrt{-\Delta}$ by writing (1.2) as a system of order 1. At this time, a serious difficulty is that the frequencies of $\sqrt{-\Delta}$, still denoted by λ_k ($k \in \mathbb{Z}$), have no good separation property and $|\lambda_k - \lambda_{k'}|$ is dense, at least, in some interval of $\mathbb{R}$, when $d > 1$. Thus, the reducibility for (1.2) with $d > 1$ is a challenging open problem. See [24] for recent progress. However, the reducibility for (1.2) with $d = 1$ can be derived from the earlier KAM theorem (see [20] and [25]) for nonlinear partial differential equations, assuming V is analytic in (t, x). Also see, [21].

According to our knowledge, the reducibility for (1.2) with perturbation of finite smoothness has not been treated explicitly in the literatures. Usually, the spectrum property of operators depends heavily on the smoothness of the perturbation. For example, the Anderson localization and positivity of the Lyapunov exponent for one frequency discrete quasi-period Schrödinger operator with analytic potential occur in non-perturbative sense (the largeness of the potential does not depend on the Diophantine condition. See [8], for the detail). However, one can only get perturbative results when the analytic property of the potential is weakened to Gevrey regularity (see [19]). Thus, the reducibility is worth studying when the perturbation V is of finite smoothness in (t, x). We also mention the papers by Baldi-Berti-Montalto [1] for KdV equation and Feola-Procesi [15] for nonlinear Schrödinger equation where the reducibility is obtained in the finite differentiable case (using "tame" estimates on Sobolev spaces) and in the case of unbounded nonlinearities.

Let us consider a linear wave equation with quasi-periodic coefficient:

$$u_{tt} - u_{xx} + \varepsilon V(\omega t, x)u = 0 \tag{1.3}$$

subject to the boundary condition

$$u(t, -\pi) = u(t, \pi) = 0. \tag{1.4}$$

For $p \geq 0$, let $\mathscr{H}^p[-\pi,\pi]$ be the usual Sobolev space. Define

$$\mathscr{H}_0^p[-\pi,\pi] = \left\{u \in \mathscr{H}^p[-\pi,\pi] : \int_{-\pi}^{\pi} u(x)\,dx = 0, u(-\pi) = u(\pi) = 0\right\}.$$

Assumption A. Assume V is a C^N-smooth and quasi-periodic in time t with frequency $\omega \in \mathbb{R}^n$: that is, there is a hull function $\mathscr{V}(\theta,x) \in C^N(\mathbb{T}^n \times [-\pi,\pi],\mathbb{R})$ such that

$$V(\omega t, x) = \mathscr{V}(\theta,x)\big|_{\theta=\omega t}, \quad \mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n,$$

where $N > 200n$.

Assumption B. We also assume that V is an even function of x, with zero-average:

$$\int_{-\pi}^{\pi} V(\omega t, x)\,dx \equiv 0.$$

Assumption C. Assume $\omega = \tau\omega_0$, where ω_0 is Diophantine:

$$|\langle k, \omega_0\rangle| \geq \frac{\gamma}{|k|^{n+1}}, \quad k \in \mathbb{Z}^n \setminus \{0\},$$

where γ is a constant and $0 < \gamma \ll 1$, $\tau \in [1,2]$ is a parameter.

Let $w = u_t$. Endow $L^2[-\pi,\pi] \times L^2[-\pi,\pi]$ with the symplectic form $dw \wedge du$. Take $(L^2[-\pi,\pi] \times L^2[-\pi,\pi], dw \wedge du)$ as phase space. Then (1.3) is a hamiltonian system with hamiltonian functional

$$H(u,w) = \int_{-\pi}^{\pi} \left(\frac{1}{2}(w^2 + u_x^2) + \frac{1}{2}\varepsilon V(\omega t, x)u^2\right)\,dx,$$

and the Hamiltonian equation

$$u_t = \frac{\delta H}{\delta w}, \quad w_t = -\frac{\delta H}{\delta u}.$$

With these three assumptions, we can describe the main results of this paper.

Theorem 1.1. *With Assumptions* A, B, C, *for given* $1 \gg \gamma > 0$, *there exists* ε^* *with* $0 < \varepsilon^* = \varepsilon^*(n,\gamma) \ll \gamma$, *and exists a subset* $\Pi \subset [1,2]$ *with*

$$\operatorname{mes}\Pi \geq 1 - O(\gamma^{1/3})$$

such that for any $0 < \varepsilon < \varepsilon^*$ *and for any* $\tau \in \Pi$, *there is a quasi-periodic symplectic change* $u = \Phi(\theta,x)v\big|_{\theta=\omega t}$ *with the map* $\theta \mapsto \Phi(\theta,\cdot)$ *being of class* $C^{N-\mu}(\mathbb{T}^n, L(\mathscr{H}_0^N[-\pi,\pi], \mathscr{H}_0^N[-\pi,\pi]))$ *for any* $\mu \in (0,1)$ *and satisfying*

$$\|\Phi(\theta,\cdot) - \operatorname{id}\|_{L(\mathscr{H}_0^N[-\pi,\pi],\mathscr{H}_0^N[-\pi,\pi])} \leq C_\mu\varepsilon,$$

where id *is the identity from* $\mathscr{H}_0^N[-\pi,\pi] \to \mathscr{H}_0^N[-\pi,\pi]$, C_μ *is a constant depending* μ *and* $L(\mathscr{H}_0^N[-\pi,\pi], \mathscr{H}_0^N[-\pi,\pi])$ *is the class of all bounded linear operators from* $\mathscr{H}_0^N[-\pi,\pi]$ *to itself which changes* (1.3) *subject to* (1.4) *into*

$$v_{tt} - v_{xx} + \varepsilon M_\xi v = 0, \quad v(t,-\pi) = v(t,\pi) = 0, \tag{1.5}$$

where M_ξ *is a real Fourier multiplier:*

$$M_\xi \sin kx = \xi_k \sin kx, \quad k \in \mathbb{N}$$

with constants $\xi_k \in \mathbb{R}$ *and* $|\xi_k| \le C/|k|$, *where* C *is an absolute constant,* $\operatorname{mes}\Pi$ *denotes Lebesgue measure for set* Π.

Corollary 1.2. *With Assumptions* A, B, C, *for any* $\tau \in \Pi$ *and* $0 < \varepsilon < \varepsilon^*$, *the wave operator*

$$\mathscr{L}u(t,x) = (\partial_t^2 - \partial_x^2 + \varepsilon V(\omega t, x))u(t,x), \quad u(t,-\pi) = u(t,\pi) = 0$$

is of pure point spectrum property and of zero Lyapunov exponent.

Corollary 1.3. *With Assumptions* A, B, C, *for any* $\tau \in \Pi$ *and* $0 < \varepsilon < \varepsilon^*$, *the initial problem of the linear wave equation*

$$(\partial_t^2 - \partial_x^2 + \varepsilon V(\omega t, x))u(t,x) = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = \widetilde{u}_0(x) \tag{1.6}$$

with $(u_0, \widetilde{u}_0) \in \mathscr{H}_0^N[-\pi,\pi] \times \mathscr{H}_0^{N-1}[-\pi,\pi]$ *has a unique solution* $u(t,x)$ *which is almost periodic in time and obeys the following estimate*

$$\begin{aligned}
&(1 - c\sqrt{\varepsilon})\big(\|u_0\|_{\mathscr{H}^N}^2 + \|\widetilde{u}_0\|_{\mathscr{H}^{N-1}}^2\big) \le \|u(t)\|_{\mathscr{H}^N}^2 \le (1 + c\sqrt{\varepsilon})\big(\|u_0\|_{\mathscr{H}^N}^2 + \|\widetilde{u}_0\|_{\mathscr{H}^{N-1}}^2\big),\\
&(1 - c\sqrt{\varepsilon})\big(\|u_0\|_{\mathscr{H}^N}^2 + \|\widetilde{u}_0\|_{\mathscr{H}^{N-1}}^2\big) \le \|u_t(t)\|_{\mathscr{H}^{N-1}}^2 \le (1 + c\sqrt{\varepsilon})\big(\|u_0\|_{\mathscr{H}^N}^2 + \|\widetilde{u}_0\|_{\mathscr{H}^{N-1}}^2\big),
\end{aligned}$$

where c *is a positive constant which might be different in different places.*

Remark 1.4. In references [7, 23], there are solutions $u(t,x)$ obey

$$\|u(t)\|_{\mathscr{H}^N} = C(\log|t|)^C \to \infty \quad \text{as } |t| \to \infty$$

for the nonlinear Schrödinger equation (1.1). In [21], it is proved that for the following linear wave equation

$$u_{tt} - u_{xx} + Mu - \varepsilon(\cos 2t)u = 0$$

subject to Dirichlet boundary condition on $[0,\pi]$, there is a solution $u(t,x)$ obeying

$$\|u(t_j)\|_{\mathscr{H}^1(\mathbb{T})} \to \infty, \quad |t_j| \to \infty.$$

The above results show that one can not avoid restricting the choice of parameters ω (or τ) to a Cantor type subset Π of the parameter set $[1,2]$ in Corollary 1.3.

Remark 1.5. In [5], it is proved that there is a quasi-periodic solution for d-dimensional nonlinear wave equation with a quasi-periodic in time nonlinearity like

$$u_{tt} - \Delta u - V(x)u = \varepsilon f(\omega t, x, u), \quad x \in \mathbb{T}^d,$$

where the multiplicative potential V is in $C^q(\mathbb{T}^d; \mathbb{R})$, $\omega \in \mathbb{R}^n$ is a non-resonant frequency vector and $f \in C^q(\mathbb{T}^n \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R})$. Because of the application of Nash-Moser iteration, it is not clear whether the obtained quasi-periodic solution is linearly stable or has zero Lyapunov exponent. As a corollary of Theorem 1.1, we can prove that the quasi-periodic solution by [5] is linearly stable and has zero Lyapunov exponent, when $d = 1$.

Remark 1.6. As mentioned above, the reducibility of the linear wave equation with time quasi-periodic and analytic coefficients can be implicitly derived from the KAM theory dealing with the existence of KAM tori for nonlinear wave equation. Here we should note the difference between the analytic coefficient and the finitely smooth one. By passing to Fourier coefficients, the wave equation (1.3) can be written as a linear Hamiltonian system with Hamiltonian

$$H = \langle \Lambda z, \overline{z} \rangle + \varepsilon\big[\langle R^{zz}(\theta)z, z\rangle + \langle R^{z\overline{z}}(\theta)z, \overline{z}\rangle + \langle R^{\overline{z}\overline{z}}(\theta)\overline{z}, \overline{z}\rangle\big],$$

where the symplectic form is $\mathbf{i}dz \wedge d\overline{z}$. The basic task is to search a series of symplectic coordinate changes to eliminate the perturbations $R^{zz}(\theta)$, $R^{z\overline{z}}(\theta)$ and $R^{\overline{z}\overline{z}}(\theta)$ except for the averages of the diagonal terms of $R^{z\overline{z}}(\theta)$. To this end, the symplectic coordinate changes are the time-1 map of the flow for the Hamiltonian εF, where F is of the form

$$F = \langle F^{zz}(\theta,\tau)z, z\rangle + \langle F^{z\overline{z}}(\theta,\tau)z, \overline{z}\rangle + \langle F^{\overline{z}\overline{z}}(\theta,\tau)\overline{z}, \overline{z}\rangle.$$

- When the potential $V(\theta)$ ($\theta = \omega t$) is analytic in some strip domain $|\operatorname{Im}\theta| \le s_\nu^*$, (where ν is the KAM iteration step), the perturbations $R^{zz}(\theta)$, $R^{z\overline{z}}(\theta)$ and $R^{\overline{z}\overline{z}}(\theta)$ are also analytic in $|\operatorname{Im}\theta| \le s_\nu^*$. An important fact in this analytic case is that s_ν^*'s have a uniform non-zero below bound:
$$s_\nu^* \ge \frac{s_0}{2}, \quad s_0 > 0 \quad \text{for all } \nu = 1, 2, \ldots.$$
- When the potential $V(\theta)$ is finitely smooth of order N, by using Jackson-Moser-Zehnder approximate lemma, we can still make sure that $R^{zz}(\theta)$, $R^{z\overline{z}}(\theta)$ and $R^{\overline{z}\overline{z}}(\theta)$ are analytic in $|\operatorname{Im}\theta| \le s_\nu$ at the ν-th KAM step. However, the strip width s_ν's have no non-zero below bound. Actually, s_ν goes to zero very rapidly:
$$s_\nu = \varepsilon_{\nu+1}^{1/N}, \quad \varepsilon_\nu = \varepsilon^{(4/3)^\nu}, \quad \nu = 1, 2, \ldots.$$
- For analytic case, we can prove the Hamiltonian $\varepsilon F = O(\varepsilon_\nu)$ at the ν-th KAM step, because of $s_\nu^* \ge s_0/2$. It follows immediately that the new perturbation is $\{\varepsilon F, \varepsilon R\} = O(\varepsilon_\nu^2) = O(\varepsilon_{\nu+1})$.

- For the finitely smooth case, the situation is much more complicated. At this case, we find $\varepsilon F = O(\varepsilon_\nu^{1-6(n+1)/N})$ at the ν-th KAM step. Thus, for the finitely smooth potential $V \in C^N$, the new perturbation is $\{\varepsilon F, \varepsilon R\} = O(\varepsilon_\nu^{2-6(n+1)/N})$. In order to guarantee the quadratic convergence of the KAM iterations, that is, $O(\varepsilon_\nu^{2-6(n+1)/N}) = O(\varepsilon_\nu^{4/3}) = O(\varepsilon_{\nu+1})$, it is necessary to assume the smoothness order $N \gg 1$. It is enough to assume $N > 200n$. Clearly, this is not sharp. In this paper, we do not pursue the lowest smoothness for the potential V.

Finally, we list some related results:

In [3], Bambusi and Graffi eliminated by KAM methods the time dependence in 1-dimensional Schrödinger equation

$$H(t)\psi(x,t) = \mathbf{i}\partial_t\psi(x,t), \quad x \in \mathbb{R}; \quad H(t) := -\frac{d^2}{dx^2} + Q(x) + \varepsilon V(x,\omega t), \quad \varepsilon \in \mathbb{R}, \tag{1.7}$$

where $Q(x) \in C^\infty(\mathbb{R};\mathbb{R})$, $Q(x) \sim |x|^\alpha$ for $\alpha > 2$ as $|x| \to \infty$ and $V(x,\phi)$ is a $C^\infty(\mathbb{R};\mathbb{R})$-valued holomorphic function of $\phi \in \mathbb{T}^n$, with $|V(x,\phi)||x|^{-\beta}$ bounded as $|x| \to \infty$ for some $\beta < (\alpha-2)/2$. The proof is based on Kuksin's estimate of solutions of homological equations with nonconstant coefficients. When $\alpha > 2$ and $\beta = (\alpha-2)/2$, the methods used in [3] will become invalid. Afterward, Liu and Yuan [22] solved this case by a new estimate for the solution of the homological equation. Wang [29] proved the pure-point nature of the spectrum of the Floquet operator K_F:

$$K_F = -\mathbf{i}\sum_{k=1}^{n} \omega_k \frac{\partial}{\partial\theta_k} - \frac{d^2}{dx^2} + x^2 + \varepsilon V(\theta,x).$$

The spectral properties of the Floquet operator K_F is closely related to the long-time behavior of the solution $\psi(t,x)$ of the equation (1.7) with $Q(x) = x^2$. The author considered $V(x,\theta) = e^{-x^2}\sum_{k=1}^n \cos\theta_k$, which has exponential decay. The case $\beta < (\alpha-2)/2 = 0$ was solved by Grébert and Thomann [17], where $V(x,\theta)$ has polynomial decay. In [30] the above results were improved, in which $V(x,\theta)$ has logarithmic decay.

Grébert and Paturel [16] proved that a linear d-dimensional Schrödinger equation on $\mathbb{R}^d$ with harmonic potential $|x|^2$ and small t-quasiperiodic potential

$$\mathbf{i}\partial_t u - \Delta u + |x|^2 u + \varepsilon V(t\omega, x)u = 0, \quad x \in \mathbb{R}^d$$

reduces to an autonomous system for most values of the frequency vector $\omega \in \mathbb{R}^n$.

In [14], Fang, Han and Wang proved Anderson localization for the Klein-Gordon operator under non-resonant perturbations. The authors showed that the Sobolev norms of solutions to the corresponding Klein-Gordon equations remain bounded for all time.

Remark 1.7. In [28], it is proved that the wave equation of time quasi-periodic coefficients

$$u_{tt} - u_{xx} + Mu + \varepsilon(V_0(\omega t)u_{xx} + V(\omega t, x)u) = 0$$

subject to periodic boundary condition $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ can be reduced by a time quasi-periodic symplectic change to a linear Hamiltonian system of constant coefficients

$$\dot{\widetilde{q}} = (\Lambda + \varepsilon\widetilde{Q})\widetilde{p}, \quad \dot{\widetilde{p}} = -(\Lambda + \varepsilon\widetilde{Q})\widetilde{q}, \tag{1.8}$$

where $\Lambda = \mathrm{diag}(\Lambda_j : j = 0, 1, 2, \ldots)$, $\Lambda_0 = \rho\sqrt{M}$, $\Lambda_j = \rho\sqrt{j^2 + M}E_{22}$, ρ is a constant close to 1, E_{22} is a 2×2 unit matrix, and $\widetilde{Q} = \mathrm{diag}(\widetilde{Q}_i : i = 0, 1, 2, \ldots)$ is independent of time with $\widetilde{Q}_0 \in \mathbb{R}$, $\widetilde{Q}_i$ being a real 2×2 matrix, and $|\widetilde{Q}_i| \le C/i$, $i = 1, 2, \ldots$.

Since the eigenvalues of the differential operator $-\partial_{xx}$ with the periodic boundary condition $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ possesses multiplicity 2, the reduced linear operator $\Lambda + \varepsilon\widetilde{Q}$ is not diagonal, although it is block diagonal. Thus (1.8) can not be written as a linear wave equation with Fourier multiplier M_ξ:

$$u_{tt} - u_{xx} + M_\xi u = 0.$$

In the present paper, the eigenvalues of the differential operator $-\partial_{xx}$ with Dirichlet boundary condition are simple. By this fact, we can reduce the wave equation

$$u_{tt} - u_{xx} + \varepsilon V(\omega t, x)u = 0$$

with boundary condition

$$u(t, -\pi) = u(t, \pi) = 0$$

to a new equation with Fourier multiplier M_ξ:

$$v_{tt} - v_{xx} + \varepsilon M_\xi v = 0.$$

2. Passing to Fourier coefficients

Consider the differential equation

$$\mathscr{L}u = u_{tt} - u_{xx} + \varepsilon V(\omega t, x)u = 0 \tag{2.1}$$

subject to the boundary condition

$$u(t, -\pi) = u(t, \pi) = 0.$$

It is well-known that the Sturm-Liouville problem

$$-y'' = \lambda y$$

with the boundary condition

$$y(-\pi) = y(\pi) = 0$$

has the eigenvalues and eigenfunctions, respectively,

$$\lambda_k = k^2, \qquad k = 1, 2, \ldots,$$
$$\phi_k(x) = \sin kx, \qquad k = 1, 2, \ldots.$$

Make the ansatz

$$u(t,x) = \mathscr{S}(u_k) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x). \tag{2.2}$$

Note that V is an even function of x such that $\int_{-\pi}^{\pi} V(\omega t, x)\,dx \equiv 0$. Write

$$V(\omega t, x) = \sum_{k=1}^{\infty} v_k(\omega t)\varphi_k(x),$$

where $\varphi_k(x) = \cos kx$, $k = 1, 2, \ldots$. Let

$$\frac{du_k}{dt} = w_k.$$

By the fact that

$$\varphi_j \phi_l = \sum_{k=1}^{\infty} \langle \varphi_j \phi_l, \phi_k \rangle \phi_k, \quad j, l = 1, 2, \ldots,$$

then (2.1) can be expressed as

$$\sum_{k=1}^{\infty} \left(\frac{dw_k}{dt} + \lambda_k u_k + \varepsilon \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} c_{jlk} v_j u_l \right) \phi_k = 0,$$

which implies that

$$\frac{dw_k}{dt} = -\lambda_k u_k - \varepsilon \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} c_{jlk} v_j u_l,$$

where

$$c_{jlk} = \langle \varphi_j \phi_l, \phi_k \rangle = \int_{-\pi}^{\pi} \cos jx \cdot \sin lx \cdot \sin kx\,dx = \begin{cases} 0 & \text{if } k \neq \pm l \pm j, \\ \pi/2 & \text{if } k = l \pm j \geq 1, \\ -\pi/2 & \text{if } k = -l \pm j \geq 1. \end{cases} \tag{2.3}$$

Rescale

$$\mathscr{T}: \quad w_k = \sqrt[4]{\lambda_k} p_k, \quad u_k = \frac{1}{\sqrt[4]{\lambda_k}} q_k.$$

Then

$$\dot{q}_k = \sqrt{\lambda_k} p_k, \quad \dot{p}_k = -\sqrt{\lambda_k} q_k - \varepsilon \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} c_{jlk} \frac{v_j(\theta)}{\sqrt[4]{\lambda_k \lambda_l}} q_l.$$

This is a linear Hamiltonian system

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}, \tag{2.4}$$

where the symplectic structure is $dp \wedge dq = \sum_{j=1}^{\infty} dp_j \wedge dq_j$ and the Hamiltonian function is

$$H(p,q) = \sum_{k=1}^{\infty} \frac{\sqrt{\lambda_k}(p_k^2 + q_k^2)}{2} + \varepsilon \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} c_{jlk} \frac{v_j(\theta)}{\sqrt[4]{\lambda_k \lambda_l}} q_l q_k, \quad \theta = \omega t.$$

Introduce complex variables:

$$z_j = \frac{1}{\sqrt{2}}(q_j - \mathbf{i} p_j), \quad \overline{z}_j = \frac{1}{\sqrt{2}}(q_j + \mathbf{i} p_j),$$

which is a symplectic transformation with $dp \wedge dq = \mathbf{i} dz \wedge d\overline{z}$. Thus (2.4) is changed into

$$\mathscr{G}: \quad \dot{z}_k = \mathbf{i}\frac{\partial H}{\partial \overline{z}_k}, \quad \dot{\overline{z}}_k = -\mathbf{i}\frac{\partial H}{\partial z_k},$$

where

$$H(z,\overline{z}) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} z_k \overline{z}_k + \varepsilon \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} c_{jlk} \frac{v_j(\theta)}{\sqrt[4]{\lambda_k \lambda_l}} \left(\frac{z_l + \overline{z}_l}{\sqrt{2}}\right)\left(\frac{z_k + \overline{z}_k}{\sqrt{2}}\right).$$

For two sequences $x = (x_j \in \mathbb{C}, j = 1,2,\ldots)$, $y = (y_j \in \mathbb{C}, j = 1,2,\ldots)$, define

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j.$$

Then we can write

$$H = \langle \Lambda z, \overline{z} \rangle + \varepsilon\big[\langle R^{zz}(\theta) z, z\rangle + \langle R^{z\overline{z}}(\theta) z, \overline{z}\rangle + \langle R^{\overline{z}\overline{z}}(\theta)\overline{z}, \overline{z}\rangle\big],$$

where

$$\Lambda = \operatorname{diag}\left(\sqrt{\lambda_j} : j = 1,2,\ldots\right), \quad \theta = \omega t,$$

$$R^{zz}(\theta) = \big(R^{zz}_{kl}(\theta) : k,l = 1,2,\ldots\big), \quad R^{zz}_{kl}(\theta) = \frac{1}{2}\sum_{j=1}^{\infty} \frac{c_{jlk} v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}}, \tag{2.5}$$

$$R^{z\overline{z}}(\theta) = \big(R^{z\overline{z}}_{kl}(\theta) : k,l = 1,2,\ldots\big), \quad R^{z\overline{z}}_{kl}(\theta) = \sum_{j=1}^{\infty} \frac{c_{jlk} v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}},$$

$$R^{\overline{z}\overline{z}}(\theta) = \big(R^{\overline{z}\overline{z}}_{kl}(\theta) : k,l = 1,2,\ldots\big), \quad R^{\overline{z}\overline{z}}_{kl}(\theta) = \frac{1}{2}\sum_{j=1}^{\infty} \frac{c_{jlk} v_j(\theta)}{\sqrt[4]{\lambda_k}\sqrt[4]{\lambda_l}}.$$

Define a Hilbert space h_N as follows:

$$h_N = \{z = (z_k \in \mathbb{C} : k = 1,2,\ldots)\}.$$

Let

$$\langle y, z\rangle_N := \sum_{k=1}^{\infty} k^{2N} y_k \overline{z}_k, \quad \forall\, y, z \in h_N, \quad \text{and} \quad \|z\|_N^2 = \langle z, z\rangle_N.$$

Recall that

$$\mathscr{V}(\theta, x) \in C^N(\mathbb{T}^n \times [-\pi, \pi], \mathbb{R}).$$

Note that the Fourier transformation (2.2) is isometric from $u \in \mathscr{H}^N[-\pi,\pi]$ to $(u_k : k = 1, 2, \ldots) \in h_N$, where $\mathscr{H}^N[-\pi,\pi]$ is the usual Sobolev space.

Now we need the following lemmas.

Lemma 2.1.

$$\begin{aligned} &\sup_{\theta\in\mathbb{T}^n} \Big\| \sum_{|\alpha|=N} \partial_\theta^\alpha J R^{zz}(\theta) J \Big\|_{h_N\to h_N} \le C,\\ &\sup_{\theta\in\mathbb{T}^n} \Big\| \sum_{|\alpha|=N} \partial_\theta^\alpha J R^{z\overline{z}}(\theta) J \Big\|_{h_N\to h_N} \le C, \qquad (2.6)\\ &\sup_{\theta\in\mathbb{T}^n} \Big\| \sum_{|\alpha|=N} \partial_\theta^\alpha J R^{\overline{zz}}(\theta) J \Big\|_{h_N\to h_N} \le C, \end{aligned}$$

where $\|\cdot\|_{h_N\to h_N}$ *is the operator norm from* h_N *to* h_N*, and* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$*,* $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$*,* α_j*'s are positive integers, and* $J = \operatorname{diag}(\sqrt[4]{\lambda_j} : j = 1, 2, \ldots) = \operatorname{diag}(\sqrt{j} : j = 1, 2, \ldots)$.

Proof. By (2.5),

$$\partial_\theta^\alpha J R^{zz}(\theta) J = \Big(\frac{1}{2} \sum_{j=1}^{\infty} C_{jlk} \partial_\theta^\alpha v_j(\theta) : l, k = 1, 2, \ldots \Big),$$

where C_{jlk} is defined as (2.3). For any $z = (z_k \in \mathbb{C} : k = 1, 2, \ldots) \in h_N$,

$$\Big(\sum_{|\alpha|=N} \partial_\theta^\alpha J R^{zz}(\theta) J \Big) z = \Big(\frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jlk} \Big(\sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big) z_k : l = 1, 2, \ldots \Big).$$

Let

$$\gamma_{lj} = \frac{(\pm l \pm j) j}{l}, \quad \text{where } (\pm l \pm j) j l \ne 0.$$

Thus,

$$\begin{aligned} &\Big\| \Big(\sum_{|\alpha|=N} \partial_\theta^\alpha J R^{zz}(\theta) J \Big) z \Big\|_N^2 \\ &= \sum_{l=1}^{\infty} l^{2N} \Big| \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{jlk} \Big(\sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big) z_k \Big|^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=1}^{\infty} l^{2N} \Big| \frac{1}{2} \sum_{j=1}^{\infty} C_{jl(\pm l \pm j)} \Big(\sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big) z_{\pm l \pm j} \Big|^2 \\
&= \frac{1}{4} \sum_{l=1}^{\infty} l^{2N} \Big| \sum_{j=1}^{\infty} \frac{1}{\gamma_{lj}^N} \cdot \gamma_{lj}^N C_{jl(\pm l \pm j)} \Big(\sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big) z_{\pm l \pm j} \Big|^2 \\
&\le C \sum_{l=1}^{\infty} \Big(\sum_{j=1}^{\infty} \frac{1}{\gamma_{lj}^{2N}} \Big) \Big(\sum_{j=1}^{\infty} |j|^{2N} \Big| \sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big|^2 | \pm l \pm j|^{2N} |z_{\pm l \pm j}|^2 \Big) \\
&\le C \Big(\sum_{j=1}^{\infty} |j|^{2N} \Big| \sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big|^2 \sum_{l=1}^{\infty} | \pm l \pm j|^{2N} |z_{\pm l \pm j}|^2 \Big) \\
&\le C \sum_{j=1}^{\infty} |j|^{2N} \Big| \sum_{|\alpha|=N} \partial_\theta^\alpha v_j(\theta) \Big|^2 \|z\|_N^2 \\
&\le C \sup_{(\theta,x) \in \mathbb{T}^n \times [-\pi,\pi]} \Big| \sum_{|\alpha|=N} \partial_\theta^\alpha \partial_x^N \mathscr{V}(\theta, x) \Big| \|z\|_N^2 \le C \|z\|_N^2,
\end{aligned}
$$

where C is a universal constant which might be different in different places. It follows

$$
\sup_{\theta \in \mathbb{T}^n} \Big\| \sum_{|\alpha|=N} \partial_\theta^\alpha J R^{zz}(\theta) J \Big\|_{h_N \to h_N} \le C. \tag{2.7}
$$

The proofs of the last two inequalities in (2.6) are similar to that of (2.7). □

3. Analytical approximation lemma

We need to find a series of operators which are analytic in some complex strip domains to approximate the operators $R^{zz}(\theta)$, $R^{z\bar{z}}(\theta)$ and $R^{\bar{z}\bar{z}}(\theta)$. To this end, we use an approximation lemma developed in [18, 26, 27]. This method is used in [31], too.

We start by recalling some definitions and setting some new notations. Assume X is a Banach space with the norm $\|\cdot\|_X$. First recall that $C^\mu(\mathbb{R}^n; X)$ for $0 < \mu < 1$ denotes the space of bounded Hölder continuous functions $f\colon \mathbb{R}^n \mapsto X$ with the form

$$
\|f\|_{C^\mu, X} = \sup_{0 < |x-y| < 1} \frac{\|f(x) - f(y)\|_X}{|x-y|^\mu} + \sup_{x \in \mathbb{R}^n} \|f(x)\|_X.
$$

If $\mu = 0$ then $\|f\|_{C^\mu, X}$ denotes the sup-norm. For $\ell = k + \mu$ with $k \in \mathbb{N}$ and $0 \le \mu < 1$, we denote by $C^\ell(\mathbb{R}^n; X)$ the space of functions $f\colon \mathbb{R}^n \mapsto X$ with Hölder continuous partial derivatives, i.e., $\partial^\alpha f \in C^\mu(\mathbb{R}^n; X_\alpha)$ for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with the assumption that $|\alpha| := |\alpha_1| + \cdots + |\alpha_n| \le k$ and X_α is the Banach space of bounded operators $T\colon \prod^{|\alpha|}(\mathbb{R}^n) \mapsto X$ with the norm

$$
\|T\|_{X_\alpha} = \sup\{\|T(u_1, u_2, \ldots, u_{|\alpha|})\|_X : \|u_i\| = 1, 1 \le i \le |\alpha|\}.
$$

We define the norm

$$\|f\|_{C^\ell} = \sup_{|\alpha|\le\ell} \|\partial^\alpha f\|_{C^\mu, X_\alpha}.$$

Lemma 3.1 (Jackson-Moser-Zehnder). *Let $f \in C^\ell(\mathbb{R}^n; X)$ for some $\ell > 0$ with finite C^ℓ norm over $\mathbb{R}^n$. Let ϕ be a radial-symmetric, C^∞ function, having as supporting the closure of the unit ball centered at the origin, where ϕ is completely flat and takes value 1, let $K = \widehat{\phi}$ be its Fourier transform. For all $\sigma > 0$ define*

$$f_\sigma(x) := K_\sigma * f = \frac{1}{\sigma^n}\int_{\mathbb{R}^n} K\left(\frac{x-y}{\sigma}\right) f(y)\,dy.$$

Then there exists a constant $C \ge 1$ depending only on ℓ and n such that the following holds: For any $\sigma > 0$, the function $f_\sigma(x)$ is a real-analytic function from $\mathbb{C}^n/(\pi\mathbb{Z})^n$ to X such that if Δ^n_σ denotes the n-dimensional complex strip of width σ,

$$\Delta^n_\sigma := \{x \in \mathbb{C}^n \mid |\operatorname{Im} x_j| \le \sigma, 1 \le j \le n\},$$

then $\forall\, \alpha \in \mathbb{N}^n$ such that $|\alpha| \le \ell$ one has

$$\sup_{x\in\Delta^n_\sigma} \left\| \partial^\alpha f_\sigma(x) - \sum_{|\beta|\le\ell-|\alpha|} \frac{\partial^{\beta+\alpha} f(\operatorname{Re} x)}{\beta!} (\mathbf{i}\operatorname{Im} x)^\beta \right\|_{X_\alpha} \le C\|f\|_{C^\ell}\sigma^{\ell-|\alpha|},$$

and for all $0 \le s \le \sigma$,

$$\sup_{x\in\Delta^n_s} \|\partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x)\|_{X_\alpha} \le C\|f\|_{C^\ell}\sigma^{\ell-|\alpha|}.$$

The function f_σ preserves periodicity (i.e., if f is T-periodic in any of its variable x_j, so is f_σ). Finally, if f depends on some parameter $\xi \in \Pi \subset \mathbb{R}^n$ and if

$$\|f(x,\xi)\|^{\mathscr{L}}_{C^\ell(X)} := \sup_{\xi\in\Pi} \|\partial_\xi f(x,\xi)\|_{C^\ell(X)}$$

are uniformly bounded by a constant C then all the above estimates hold with $\|\cdot\|$ replaced by $\|\cdot\|^{\mathscr{L}}$.

The proof of this lemma consists in a direct check which is based on standard tools from calculus and complex analysis. It is used to deal with KAM theory for finitely smooth systems by Zehnder [32]. Also see [9] and [31] and references therein, for example. For ease of notation, we shall replace $\|\cdot\|_X$ by $\|\cdot\|$. Now let us apply this lemma to the perturbation $P(\phi)$.

Fix a sequence of fast decreasing numbers $s_\nu \downarrow 0$, $\upsilon \ge 0$, and $s_0 \le 1/2$. For an X-valued function $P(\phi)$, construct a sequence of real analytic functions $P^{(\upsilon)}(\phi) = P_{s_\upsilon}(\phi)$ such that the following conclusions hold:

(1) $P^{(\upsilon)}(\phi)$ is real analytic on the complex strip $\mathbb{T}^n_{s_\upsilon}$ of the width s_υ around $\mathbb{T}^n$.

(2) The sequence of functions $P^{(\upsilon)}(\phi)$ satisfies the bounds:

$$\sup_{\phi\in\mathbb{T}^n} \|P^{(\upsilon)}(\phi) - P(\phi)\| \leq C\|P\|_{C^\ell} s_\upsilon^\ell, \tag{3.1}$$
$$\sup_{\phi\in\mathbb{T}^n_{s_{\upsilon+1}}} \|P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)\| \leq C\|P\|_{C^\ell} s_\upsilon^\ell,$$

where C denotes (different) constants depending only on n and ℓ.

(3) The first approximate $P^{(0)}$ is "small" with the perturbation P. Precisely speaking, for arbitrary $\phi \in \mathbb{T}^n_{s_0}$, we have

$$\begin{aligned}\|P^{(0)}(\phi)\| &\leq \left\| P^{(0)}(\phi) - \sum_{|\alpha|\leq\ell} \frac{\partial^\alpha P(\operatorname{Re}\phi)}{\alpha!}(\mathbf{i}\operatorname{Im}\phi)^\alpha \right\| + \left\| \sum_{|\alpha|\leq\ell} \frac{\partial^\alpha P(\operatorname{Re}\phi)}{\alpha!}(\mathbf{i}\operatorname{Im}\phi)^\alpha \right\| \\ &\leq C\left(\|P\|_{C^\ell} s_0^\ell + \sum_{0\leq m\leq\ell} \|P\|_{C^m} s_0^m \right) \leq C\|P\|_{C^\ell} \sum_{m=0}^{\ell} s_0^m \\ &\leq C\|P\|_{C^\ell} \sum_{m=0}^{\infty} s_0^m \leq C\|P\|_{C^\ell},\end{aligned}$$

where constant C is independent of s_0, and the last inequality holds true due to the hypothesis that $s_0 \leq 1/2$.

(4) From the first inequality (3.1), we have the equality below. For arbitrary $\phi \in \mathbb{T}^n$,

$$P(\phi) = P^{(0)}(\phi) + \sum_{\upsilon=0}^{+\infty} (P^{(\upsilon+1)}(\phi) - P^{(\upsilon)}(\phi)). \tag{3.2}$$

Now take a sequence of real numbers $\{s_\upsilon \geq 0\}_{\upsilon=0}^\infty$ with $s_\upsilon > s_{\upsilon+1}$ goes fast to zero. Let $R^{p,q}(\theta) = P(\theta)$ for $p, q \in \{z, \overline{z}\}$. Then by (3.2) we can write, for $p, q \in \{z, \overline{z}\}$,

$$R^{p,q}(\theta) = R_0^{p,q}(\theta) + \sum_{l=1}^{\infty} R_l^{p,q}(\theta), \tag{3.3}$$

where $R_0^{p,q}(\theta)$ is analytic in $\mathbb{T}^n_{s_0}$ with

$$\sup_{\theta\in\mathbb{T}^n_{s_0}} \|R_0^{p,q}(\theta)\|_{h_N\to h_N} \leq C, \tag{3.4}$$

and $R_l^{p,q}(\theta)$ $(l \geq 1)$ is analytic in $\mathbb{T}^n_{s_l}$ with

$$\sup_{\theta\in\mathbb{T}^n_{s_l}} \|JR_l^{p,q}(\theta)J\|_{h_N\to h_N} \leq Cs_{l-1}^N. \tag{3.5}$$

4. Iterative parameters of domains

Let

- $\varepsilon_0 = \varepsilon$, $\varepsilon_\nu = \varepsilon^{(4/3)^\nu}$, $\nu = 0, 1, 2, \ldots$, which measures the size of perturbation at ν-th step.
- $s_\nu = \varepsilon_{\nu+1}^{1/N}$, $\nu = 0, 1, 2, \ldots$, which measures the strip-width of the analytic domain $\mathbb{T}^n_{s_\nu}$, $\mathbb{T}^n_{s_\nu} = \{\theta \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |\operatorname{Im}\theta| \le s_\nu\}$.
- $C(\nu)$ is a constant which may be different in different places, and it is of the form
$$C(\nu) = C_1 2^{C_2\nu},$$
where C_1, C_2 are constants.
- $K_\nu = 100 s_\nu^{-1} 2^\nu |\log\varepsilon|$.
- $\gamma_\nu = \gamma/2^\nu$, $0 < \gamma \ll 1$.
- a family of subsets $\Pi_\nu \subset [1, 2]$ with $[1, 2] \supset \Pi_0 \supset \cdots \supset \Pi_\nu \supset \cdots$, and
$$\operatorname{mes}\Pi_\nu \ge \operatorname{mes}\Pi_{\nu-1} - C\gamma_{\nu-1}^{1/3}.$$
- For an operator-value (or a vector-value) function $B(\theta, \tau)$, whose domain is $(\theta, \tau) \in \mathbb{T}^n_{s_\nu} \times \Pi_\nu$, set
$$\|B\|_{\mathbb{T}^n_{s_\nu}\times\Pi_\nu} = \sup_{(\theta,\tau)\in\mathbb{T}^n_{s_\nu}\times\Pi_\nu} \|B(\theta, \tau)\|_{h_N\to h_N},$$
where $\|\cdot\|_{h_N\to h_N}$ is the operator norm, and set
$$\|B\|^{\mathscr{L}}_{\mathbb{T}^n_{s_\nu}\times\Pi_\nu} = \sup_{(\theta,\tau)\in\mathbb{T}^n_{s_\nu}\times\Pi_\nu} \|\partial_\tau B(\theta, \tau)\|_{h_N\to h_N}.$$

5. Iterative lemma

In the following, for a function $f(\omega)$, denote by ∂_ω the derivative of $f(\omega)$ with respect to ω in Whitney's sense.

Lemma 5.1. *For $p, q \in \{z, \overline{z}\}$, let $R^{p,q}_{0,0} = R^{p,q}_0$, $R^{p,q}_{l,0} = R^{p,q}_l$, where $R^{p,q}_0$, $R^{p,q}_l$ are defined by* (3.3), (3.4) *and* (3.5). *Assume that we have a family of Hamiltonian functions H_ν:*

$$H_\nu = \sum_{j=1}^\infty \lambda_j^{(\nu)} z_j\overline{z}_j + \sum_{l\ge\nu}^\infty \varepsilon_l\big(\langle R^{zz}_{l,\nu}z, z\rangle + \langle R^{z\overline{z}}_{l,\nu}z, \overline{z}\rangle + \langle R^{\overline{z}\overline{z}}_{l,\nu}\overline{z}, \overline{z}\rangle\big), \quad \nu = 0, 1, \ldots, m, \tag{5.1}$$

where $R^{zz}_{l,\nu}$, $R^{z\overline{z}}_{l,\nu}$, $R^{\overline{z}\overline{z}}_{l,\nu}$ are operator-valued functions defined on the domain $\mathbb{T}^n_{s_\nu} \times \Pi_\nu$, and $\theta = \omega t$.

$(\mathrm{A1})_\nu$

(5.2) $$\lambda_j^{(0)} = \sqrt{\lambda_j} = j, \quad \lambda_j^{(\nu)} = \sqrt{\lambda_j} + \sum_{i=0}^{\nu-1} \varepsilon_i \mu_j^{(i)}, \quad \nu \geq 1$$

and $\mu_j^{(i)} = \mu_j^{(i)}(\tau)\colon \Pi_i \to \mathbb{R}$ *with*

(5.3) $$|\mu_j^{(i)}|_{\Pi_i} := \sup_{\tau\in\Pi_i} |\mu_j^{(i)}(\tau)| \leq C(i)/j, \quad 0 \leq i \leq \nu - 1,$$

(5.4) $$|\mu_j^{(i)}|_{\Pi_i}^{\mathscr{L}} := \sup_{\tau\in\Pi_i} |\partial_\tau \mu_j^{(i)}(\tau)| \leq C(i)/j, \quad 0 \leq i \leq \nu - 1.$$

$(\mathrm{A2})_\nu$ *For* $p, q \in \{z, \overline{z}\}$, $R_{l,\nu}^{p,q} = R_{l,\nu}^{p,q}(\theta, \tau)$ *is defined in* $\mathbb{T}_{s_l}^n \times \Pi_\nu$ *with* $l \geq \nu$, *and is analytic in* θ *for fixed* $\tau \in \Pi_\nu$, *and*

(5.5) $$\|JR_{l,\nu}^{p,q}J\|_{\mathbb{T}_{s_l}^n\times\Pi_\nu} \leq C(\nu),$$

(5.6) $$\|JR_{l,\nu}^{p,q}J\|_{\mathbb{T}_{s_l}^n\times\Pi_\nu}^{\mathscr{L}} \leq C(\nu).$$

Then there exists a compact set $\Pi_{m+1} \subset \Pi_m$ *with*

$$\operatorname{mes}\Pi_{m+1} \geq \operatorname{mes}\Pi_m - C\gamma_m^{1/3}$$

and symplectic coordinate changes

(5.7) $$\Psi_m\colon \mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1} \to \mathbb{T}_{s_m}^n \times \Pi_m,$$

(5.8) $$\|\Psi_m - \mathrm{id}\,\|_{h_N\to h_N} \leq \varepsilon^{1/2}, \quad (\theta,\tau) \in \mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}$$

such that the Hamiltonian function H_m *is changed into*

(5.9) $$\begin{aligned} H_{m+1} &\triangleq H_m \circ \Psi_m \\ &= \sum_{j=1}^{\infty} \lambda_j^{(m+1)} z_j \overline{z}_j + \sum_{l\geq m+1}^{\infty} \varepsilon_l \big[\langle R_{l,m+1}^{zz} z, z\rangle + \langle R_{l,m+1}^{z\overline{z}} z, \overline{z}\rangle + \langle R_{l,m+1}^{\overline{z}\overline{z}} \overline{z}, \overline{z}\rangle\big], \end{aligned}$$

which is defined on the domain $\mathbb{T}_{s_{m+1}}^n \times \Pi_{m+1}$, *and* $\lambda_j^{(m+1)}$*'s satisfy Assumption* $(\mathrm{A1})_{m+1}$ *and* $R_{l,m+1}^{p,q}$ *(*$p, q \in \{z, \overline{z}\}$*) satisfy Assumption* $(\mathrm{A2})_{m+1}$.

6. Derivation of homological equations

Our end is to find a symplectic transformation Ψ_ν such that the terms $R_{l,v}^{zz}$, $R_{l,v}^{z\overline{z}}$, $R_{l,v}^{\overline{z}\overline{z}}$ (with $l = v$) disappear. To this end, let F be a linear Hamiltonian of the form

(6.1) $$F = \langle F^{zz}(\theta,\tau) z, z\rangle + \langle F^{z\overline{z}}(\theta,\tau) z, \overline{z}\rangle + \langle F^{\overline{z}\overline{z}}(\theta,\tau) \overline{z}, \overline{z}\rangle,$$

where $\theta = \omega t$, $(F^{zz}(\theta,\tau))^T = F^{zz}(\theta,\tau)$, $(F^{z\overline{z}}(\theta,\tau))^T = F^{z\overline{z}}(\theta,\tau)$, $(F^{\overline{z}\overline{z}}(\theta,\tau))^T = F^{\overline{z}\overline{z}}(\theta,\tau)$. Moreover, let

$$\Psi = \Psi_m = X^t_{\varepsilon_m F}\big|_{t=1}, \tag{6.2}$$

where $X^t_{\varepsilon_m F}$ is the flow of the Hamiltonian, $X_{\varepsilon_m F}$ is the vector field of the Hamiltonian $\varepsilon_m F$ with the symplectic structure $\mathbf{i}dz \wedge d\overline{z}$. Let

$$H_{m+1} = H_m \circ \Psi_m. \tag{6.3}$$

By (5.1), we write

$$H_m = N_m + R_m \tag{6.4}$$

with

$$N_m = \sum_{j=1}^{\infty} \lambda_j^{(m)} z_j \overline{z}_j, \tag{6.5}$$

$$R_m = \sum_{l=m}^{\infty} \varepsilon_l R_{lm}, \tag{6.6}$$

$$R_{lm} = \langle R^{zz}_{l,m}(\theta) z, z\rangle + \langle R^{z\overline{z}}_{l,m}(\theta) z, \overline{z}\rangle + \langle R^{\overline{z}\overline{z}}_{l,m}(\theta) \overline{z}, \overline{z}\rangle, \tag{6.7}$$

where $(R^{zz}_{l,m}(\theta))^T = R^{zz}_{l,m}(\theta)$, $(R^{z\overline{z}}_{l,m}(\theta))^T = R^{z\overline{z}}_{l,m}(\theta)$, $(R^{\overline{z}\overline{z}}_{l,m}(\theta))^T = R^{\overline{z}\overline{z}}_{l,m}(\theta)$. Since the Hamiltonian $H_m = H_m(\omega t, z, \overline{z})$ depends on time t, we introduce a fictitious action $I =$ constant, and let $\theta = \omega t$ be angle variable. Then the non-autonomous $H_m(\omega t, z, \overline{z})$ can be written as

$$\omega I + H_m(\theta, z, \overline{z})$$

with symplectic structure $dI \wedge d\theta + \mathbf{i}dz \wedge d\overline{z}$. By combination of (6.1)–(6.7) and Taylor formula, we have

$$\begin{aligned} H_{m+1} &= H_m \circ X^1_{\varepsilon_m F} \\ &= N_m + \varepsilon_m\{N_m, F\} + \varepsilon_m^2 \int_0^1 (1-\tau)\{\{N_m, F\}, F\} \circ X^{\tau}_{\varepsilon_m F}\, d\tau + \varepsilon_m \omega \cdot \partial_\theta F \\ &\quad + \varepsilon_m R_{mm} + \left(\sum_{l=m+1}^{\infty} \varepsilon_l R_{lm}\right) \circ X^1_{\varepsilon_m F} + \varepsilon_m^2 \int_0^1 \{R_{mm}, F\} \circ X^{\tau}_{\varepsilon_m F}\, d\tau, \end{aligned}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket with respect to $\mathbf{i}dz \wedge d\overline{z}$, that is

$$\{H(z,\overline{z}), F(z,\overline{z})\} = \mathbf{i}\left(\frac{\partial H}{\partial z} \cdot \frac{\partial F}{\partial \overline{z}} - \frac{\partial H}{\partial \overline{z}} \cdot \frac{\partial F}{\partial z}\right).$$

Let Γ_{K_m} be a truncation operator. For any

$$f(\theta) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k)e^{\mathbf{i}\langle k,\theta\rangle}, \quad \theta \in \mathbb{T}^n,$$

define, for given $K_m > 0$,

$$\Gamma_{K_m} f(\theta) = (\Gamma_{K_m} f)(\theta) \triangleq \sum_{|k|\le K_m} \widehat{f}(k)e^{\mathbf{i}\langle k,\theta\rangle},$$
$$(1-\Gamma_{K_m}) f(\theta) = ((1-\Gamma_{K_m}) f)(\theta) \triangleq \sum_{|k|> K_m} \widehat{f}(k)e^{\mathbf{i}\langle k,\theta\rangle}.$$

Then

$$f(\theta) = \Gamma_{K_m} f(\theta) + (1-\Gamma_{K_m}) f(\theta).$$

Let

$$\omega\cdot\partial_\theta F + \{N_m, F\} + \Gamma_{K_m} R_{mm} = \langle [R^{z\overline{z}}_{mm}]z, \overline{z}\rangle, \tag{6.8}$$

where

$$[R^{z\overline{z}}_{mm}] := \operatorname{diag}\left(\widehat{R}^{z\overline{z}}_{mmjj}(0) : j = 1,2,\dots\right),$$

and $R^{z\overline{z}}_{mmij}(\theta)$ is the matrix element of $R^{z\overline{z}}_{m,m}(\theta)$ and $\widehat{R}^{z\overline{z}}_{mmij}(k)$ is the k-Fourier coefficient of $R^{z\overline{z}}_{mmij}(\theta)$. Then

$$H_{m+1} = N_{m+1} + C_{m+1}R_{m+1},$$

where

$$N_{m+1} = N_m + \varepsilon_m\langle [R^{z\overline{z}}_{mm}]z, \overline{z}\rangle = \sum_{j=1}^{\infty} \lambda_j^{(m+1)} z_j\overline{z}_j,$$

$$\lambda_j^{(m+1)} = \lambda_j^{(m)} + \varepsilon_m \widehat{R}^{z\overline{z}}_{mmjj}(0) = \sqrt{\lambda_j} + \sum_{l=1}^{m} \varepsilon_l \mu_j^{(l)}, \quad \mu_j^{(m)} := \widehat{R}^{z\overline{z}}_{mmjj}(0). \tag{6.9}$$

$$C_{m+1}R_{m+1} = \varepsilon_m(1-\Gamma_{K_m})R_{mm} \tag{6.10}$$
$$+\, \varepsilon_m^2 \int_0^1 (1-\tau)\{\{N_m, F\}, F\}\circ X^\tau_{\varepsilon_m F}\, d\tau \tag{6.11}$$
$$+\, \varepsilon_m^2 \int_0^1 \{R_{mm}, F\}\circ X^\tau_{\varepsilon_m F}\, d\tau \tag{6.12}$$
$$+ \left(\sum_{l=m+1}^{\infty} \varepsilon_l R_{lm}\right)\circ X^1_{\varepsilon_m F}. \tag{6.13}$$

The equation (6.8) is called homological equation. Developing the Poisson bracket $\{N_m, F\}$ and comparing the coefficients of z_iz_j, $z_i\overline{z}_j$, $\overline{z}_i\overline{z}_j$ $(i,j=1,2,\ldots)$, we get

$$\omega\cdot\partial_\theta F^{zz}(\theta,\tau)+\mathbf{i}(\Lambda^{(m)}F^{zz}(\theta,\tau)+F^{zz}(\theta,\tau)\Lambda^{(m)})=\Gamma_{K_m}R^{zz}_{mm}(\theta), \tag{6.14}$$

$$\omega\cdot\partial_\theta F^{\overline{z}\overline{z}}(\theta,\tau)-\mathbf{i}(\Lambda^{(m)}F^{\overline{z}\overline{z}}(\theta,\tau)+F^{\overline{z}\overline{z}}(\theta,\tau)\Lambda^{(m)})=\Gamma_{K_m}R^{\overline{z}\overline{z}}_{mm}(\theta), \tag{6.15}$$

$$\omega\cdot\partial_\theta F^{z\overline{z}}(\theta,\tau)+\mathbf{i}(F^{z\overline{z}}(\theta,\tau)\Lambda^{(m)}-\Lambda^{(m)}F^{z\overline{z}}(\theta,\tau))=\Gamma_{K_m}R^{z\overline{z}}_{mm}(\theta)-[R_{mm}], \tag{6.16}$$

where

$$\Lambda^{(m)}=\operatorname{diag}(\lambda_j^{(m)}:j=1,2,\ldots),$$

and we assume

$$\Gamma_{K_m}F^{zz}(\theta,\tau)=F^{zz}(\theta,\tau),\quad \Gamma_{K_m}F^{z\overline{z}}(\theta,\tau)=F^{z\overline{z}}(\theta,\tau),\quad \Gamma_{K_m}F^{\overline{z}\overline{z}}(\theta,\tau)=F^{\overline{z}\overline{z}}(\theta,\tau).$$

Here $F^{zz}_{ij}(\theta)$, $F^{z\overline{z}}_{ij}(\theta)$, $F^{\overline{z}\overline{z}}_{ij}(\theta)$ are the matrix elements of $F^{zz}(\theta,\tau)$, $F^{z\overline{z}}(\theta,\tau)$, $F^{\overline{z}\overline{z}}(\theta,\tau)$, respectively. Then (6.14)–(6.16) can be rewritten as

$$\omega\cdot\partial_\theta F^{zz}_{ij}(\theta)+\mathbf{i}(\lambda_i^{(m)}+\lambda_j^{(m)})F^{zz}_{ij}(\theta)=\Gamma_{K_m}R^{zz}_{mmij}(\theta), \tag{6.17}$$

$$\omega\cdot\partial_\theta F^{\overline{z}\overline{z}}_{ij}(\theta)-\mathbf{i}(\lambda_i^{(m)}+\lambda_j^{(m)})F^{\overline{z}\overline{z}}_{ij}(\theta)=\Gamma_{K_m}R^{\overline{z}\overline{z}}_{mmij}(\theta), \tag{6.18}$$

$$\begin{aligned}\omega\cdot\partial_\theta F^{z\overline{z}}_{ij}(\theta)-\mathbf{i}(\lambda_i^{(m)}-\lambda_j^{(m)})F^{z\overline{z}}_{ij}(\theta)&=\Gamma_{K_m}R^{z\overline{z}}_{mmij}(\theta),\quad i\neq j,\\ \omega\cdot\partial_\theta F^{z\overline{z}}_{ii}(\theta)&=\Gamma_{K_m}R^{z\overline{z}}_{mmii}(\theta)-\widehat{R}_{mmii}(0),\end{aligned} \tag{6.19}$$

where $i,j=1,2,\ldots$.

7. Solutions of the homological equations

Lemma 7.1. *There exists a compact subset $\Pi^{+-}_{m+1}\subset\Pi_m$ with*

$$\operatorname{mes}(\Pi^{+-}_{m+1})\geq\operatorname{mes}\Pi_m-C\gamma_m^{1/3} \tag{7.1}$$

such that for any $\tau\in\Pi^{+-}_{m+1}$ (Recall $\omega=\tau\omega_0$), the equation (6.19) *has a unique solution $F^{z\overline{z}}(\theta,\tau)$, which is defined on the domain $\mathbb{T}^n_{s_{m+1}}\times\Pi^{+-}_{m+1}$, with*

$$\begin{aligned}\|JF^{z\overline{z}}(\theta,\tau)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{+-}_{m+1}}&\leq C(m+1)\varepsilon_m^{-6(n+1)/N},\\ \|JF^{z\overline{z}}(\theta,\tau)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{+-}_{m+1}}&\leq C(m+1)\varepsilon_m^{-12(n+1)/N}.\end{aligned}$$

Proof. By passing to Fourier coefficients, we can rewrite (6.19) as

$$(-\langle k,\omega\rangle+\lambda_i^{(m)}-\lambda_j^{(m)})\widehat{F}^{z\overline{z}}_{ij}(k)=\mathbf{i}\widehat{R}^{z\overline{z}}_{mmij}(k), \tag{7.2}$$

where $i,j=1,2,\ldots$, $k\in\mathbb{Z}^n$ with $|k|\leq K_m$. Recall $\omega=\tau\omega_0$.

Let

$$A_k = |k|^{2n+3} + 8,$$

and let

$$Q^{(m)}_{kij} \triangleq \left\{ \tau \in \Pi_m \;\middle|\; \left| -\langle k, \omega_0\rangle \tau + \lambda^{(m)}_i - \lambda^{(m)}_j \right| < \frac{(|i-j|+1)\gamma_m}{A_k} \right\}, \tag{7.3}$$

where $i, j = 1, 2, \ldots$, $k \in \mathbb{Z}^n$ with $|k| \leq K_m$, and $k \neq 0$ when $i = j$. Let

$$\Pi^{+-}_{m+1} = \Pi_m \setminus \bigcup_{|k|\leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q^{(m)}_{kij}.$$

Then for any $\tau \in \Pi^{+-}_{m+1}$, we have

$$\left| -\langle k, \omega\rangle + \lambda^{(m)}_i - \lambda^{(m)}_j \right| \geq \frac{(|i-j|+1)\gamma_m}{A_k}. \tag{7.4}$$

Recall that $R^{z\bar{z}}_{mm}(\theta)$ is analytic in the domain $\mathbb{T}^n_{s_m}$ for any $\tau \in \Pi_m$,

$$|\widehat{R}^{z\bar{z}}_{mmij}(k)| \leq \frac{C(m)}{\sqrt{ij}} e^{-s_m|k|}.$$

It follows

$$\begin{aligned} |\widehat{F}^{z\bar{z}}_{ij}(k)| &= \left| \frac{\widehat{R}^{z\bar{z}}_{mmij}(k)}{-\langle k, \omega\rangle + \lambda^{(m)}_i - \lambda^{(m)}_j} \right| \leq \frac{A_k}{\gamma_m(|i-j|+1)} \cdot |\widehat{R}^{z\bar{z}}_{mmij}(k)| \\ &\leq \frac{(|k|^{2n+3}+8)}{\gamma_m(|i-j|+1)} \cdot \frac{C(m)}{\sqrt{ij}} e^{-s_m|k|}. \end{aligned} \tag{7.5}$$

Now we need the following lemmas:

Lemma 7.2. [6] *For $0 < \delta < 1$, $\nu > 1$, one has*

$$\sum_{k\in\mathbb{Z}^n} e^{-2|k|\delta} |k|^{\nu} < \left(\frac{\nu}{e}\right)^{\nu} \frac{(1+e)^n}{\delta^{\nu+n}}.$$

Lemma 7.3. [25] *If $A = (A_{ij})$ is a bounded linear operator on ℓ^2, then also $B = (B_{ij})$ with*

$$B_{ij} = \frac{|A_{ij}|}{|i-j|}, \quad i \neq j,$$

and $B_{ii} = 0$ is a bounded linear operator on ℓ^2, and $\|B\| \leq \left(\frac{\pi}{\sqrt{3}}\right)\|A\|$, where $\|\cdot\|$ is $\ell^2 \to \ell^2$ operator norm.

Remark 7.4. Lemma 7.3 holds true for the weight norm $\|\cdot\|_N$.

Therefore, by (7.5), we have

$$\begin{aligned}
&\sup_{\theta\in\mathbb{T}^n_{s'_m}\times\Pi_{m+1}}\big(\sqrt{i}|F^{z\overline{z}}_{ij}(\theta,\tau)|\sqrt{j}\big)\\
&\le\Bigg(\sum_{|k|\le K_m}(|k|^{2n+3}+8)e^{-(s_m-s'_m)|k|}\Bigg)\cdot\frac{C(m)}{\gamma_m(|i-j|+1)}\\
&\le C\left(\frac{2n+3}{e}\right)^{2n+3}(1+e)^n\left(\frac{2}{s_m-s'_m}\right)^{3n+3}\cdot\frac{C(m)}{\gamma_m(|i-j|+1)}\quad\text{(by Lemma 7.2)}\\
&\le C\cdot\frac{C(m)}{(s_m-s'_m)^{3(n+1)}}\cdot\frac{1}{\gamma_m(|i-j|+1)}\\
&\le C\cdot\varepsilon_m^{-6(n+1)/N}\cdot\frac{C(m)}{\gamma_m(|i-j|+1)},
\end{aligned}$$

where C is a constant depending on n, $s'_m=s_m-(s_m-s_{m+1})/4$.

By Lemma 7.3 and Remark 7.4, we have

$$\|JF^{z\overline{z}}(\theta,\tau)J\|_{\mathbb{T}^n_{s'_m}\times\Pi^{+-}_{m+1}}\le C\cdot C(m)\gamma_m^{-1}\varepsilon_m^{-6(n+1)/N}\le C(m+1)\varepsilon_m^{-6(n+1)/N}. \tag{7.6}$$

It follows from $s'_m>s_{m+1}$ that

$$\|JF^{z\overline{z}}(\theta,\tau)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{+-}_{m+1}}\le\|JF^{z\overline{z}}(\theta,\tau)J\|_{\mathbb{T}^n_{s'_m}\times\Pi^{+-}_{m+1}}\le C(m+1)\varepsilon_m^{-6(n+1)/N}.$$

Applying ∂_τ to both sides of (7.2), we have

$$\big(-\langle k,\omega\rangle+\lambda_i^{(m)}-\lambda_j^{(m)}\big)\partial_\tau\widehat{F}^{z\overline{z}}_{ij}(k)=\mathbf{i}\partial_\tau\widehat{R}^{z\overline{z}}_{mmij}(k)+(*), \tag{7.7}$$

where

$$(*)=-\big(-\langle k,\omega_0\rangle+\partial_\tau(\lambda_i^{(m)}-\lambda_j^{(m)})\big)\widehat{F}^{z\overline{z}}_{ij}(k).$$

Recalling $|k|\le K_m=100s_m^{-1}2^m|\log\varepsilon|$, and using (5.2) and (5.3) with $\nu=m$, and using (7.6), we have, on $\tau\in\Pi_{m+1}$,

$$\sqrt{i}|(*)|\sqrt{j}\le C(m)K_m|\widehat{F}^{z\overline{z}}_{ij}(k)|. \tag{7.8}$$

According to (5.6),

$$\big|\sqrt{i}\partial_\tau\widehat{R}^{z\overline{z}}_{mmij}(k)\sqrt{j}\big|\le C(m+1)e^{-s^*_m|k|}. \tag{7.9}$$

By (7.4), (7.7), (7.8) and (7.9), we have

$$\big|\sqrt{i}\partial_\tau\widehat{F}^{z\overline{z}}_{ij}(k)\sqrt{j}\big|\le\frac{A_k}{\gamma_m(|i-j|+1)}\cdot C\cdot C(m+1)K_m\gamma_m^{-1}\varepsilon_m^{-6(n+1)/N}e^{-s'_m|k|}\quad\text{for } i\ne j.$$

Note that $s_m > s'_m > s_{m+1}$. Again using Lemmas 7.2 and 7.3, we have

$$\begin{aligned}\|JF^{z\overline{z}}(\theta,\tau)J\|^{\mathscr{L}}_{\mathbb{T}_{s_{m+1}}\times\Pi^{+-}_{m+1}} &= \|J\partial_\tau F^{z\overline{z}}(\theta,\tau)J\|_{\mathbb{T}_{s_{m+1}}\times\Pi^{+-}_{m+1}}\\ &\leq C^2\cdot C(m+1)K_m\gamma_m^{-1}\varepsilon_m^{-12(n+1)/N} \leq C(m+1)\varepsilon_m^{-12(n+1)/N}.\end{aligned}$$

The proof of the measure estimate (7.1) will be postponed to Section 10. This completes the proof of Lemma 7.1. □

Lemma 7.5. *There exists a compact subset $\Pi^{++}_{m+1}\subset\Pi_m$ with*

$$\operatorname{mes}(\Pi^{++}_{m+1}) \geq \operatorname{mes}\Pi_m - C\gamma_m^{1/3} \tag{7.10}$$

such that for any $\tau\in\Pi^{++}_{m+1}$ (Recall $\omega=\tau\omega_0$), the equation (6.17) *has a unique solution $F^{zz}(\theta)$, which is defined on the domain $\mathbb{T}^n_{s_{m+1}}\times\Pi^{++}_{m+1}$, with*

$$\begin{aligned}\|JF^{zz}(\theta,\tau)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{++}_{m+1}} &\leq C(m+1)\varepsilon_m^{-6(n+1)/N},\\ \|JF^{zz}(\theta,\tau)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{++}_{m+1}} &\leq C(m+1)\varepsilon_m^{-12(n+1)/N}.\end{aligned}$$

Lemma 7.6. *There exists a compact subset $\Pi^{--}_{m+1}\subset\Pi_m$ with*

$$\operatorname{mes}(\Pi^{--}_{m+1}) \geq \operatorname{mes}\Pi_m - C\gamma_m^{1/3} \tag{7.11}$$

such that for any $\tau\in\Pi^{--}_{m+1}$ (Recall $\omega=\tau\omega_0$), the equation (6.18) *has a unique solution $F^{\overline{zz}}(\theta)$, which is defined on the domain $\mathbb{T}^n_{s_{m+1}}\times\Pi^{--}_{m+1}$, with*

$$\begin{aligned}\|JF^{\overline{zz}}(\theta,\tau)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{--}_{m+1}} &\leq C(m+1)\varepsilon_m^{-6(n+1)/N},\\ \|JF^{\overline{zz}}(\theta,\tau)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi^{--}_{m+1}} &\leq C(m+1)\varepsilon_m^{-12(n+1)/N}.\end{aligned}$$

The proofs of Lemmas 7.5 and 7.6 are a little bit simpler than that of Lemma 7.1. So we omit them.

Let

$$\Pi_{m+1} = \Pi^{+-}_{m+1}\cap\Pi^{++}_{m+1}\cap\Pi^{--}_{m+1}.$$

By (7.1), (7.10) and (7.11), we have

$$\operatorname{mes}\Pi_{m+1} \geq \operatorname{mes}\Pi_m - C\gamma_m^{1/3}.$$

8. Coordinate change Ψ by $\varepsilon_m F$

Recall $\Psi=\Psi_m = X^t_{\varepsilon_m F}\big|_{t=1}$, where $X^t_{\varepsilon_m F}$ is the flow of the Hamiltonian $\varepsilon_m F$, vector field $X_{\varepsilon_m F}$ with symplectic $\mathbf{i}dz\wedge d\overline{z}$. So

$$\mathbf{i}\dot{z} = \varepsilon_m\frac{\partial F}{\partial\overline{z}},\quad -\mathbf{i}\dot{\overline{z}} = \varepsilon_m\frac{\partial F}{\partial z},\quad \dot{\theta}=\omega.$$

More exactly,

$$\begin{aligned}\mathbf{i}\dot{z} &= \varepsilon_m(F^{z\overline{z}}(\theta,\tau)z + 2F^{\overline{z}\overline{z}}(\theta,\tau)\overline{z}), \quad \theta=\omega t,\\ -\mathbf{i}\dot{\overline{z}} &= \varepsilon_m(2F^{zz}(\theta,\tau)z + F^{z\overline{z}}(\theta,\tau)\overline{z}), \quad \theta=\omega t,\\ \dot{\theta} &= \omega.\end{aligned}$$

Let $u = \left(\begin{smallmatrix} z \\ \overline{z}\end{smallmatrix}\right)$,

$$B_m = \begin{pmatrix} -\mathbf{i}F^{z\overline{z}}(\theta,\tau) & -2\mathbf{i}F^{\overline{z}\overline{z}}(\theta,\tau) \\ 2\mathbf{i}F^{zz}(\theta,\tau) & \mathbf{i}F^{z\overline{z}}(\theta,\tau)\end{pmatrix}, \quad \theta=\omega t.$$

Then

$$\frac{du(t)}{dt} = \varepsilon_m B_m(\theta)u, \quad \dot{\theta}=\omega. \tag{8.1}$$

Let $u(0)=u_0\in h_N\times h_N$, $\theta(0)=\theta_0\in\mathbb{T}^n_{s_{m+1}}$ be initial value. Then

$$u(t) = u_0 + \int_0^t \varepsilon_m B_m(\theta_0+\omega s)u(s)\,ds, \quad \theta(t)=\theta_0+\omega t. \tag{8.2}$$

By Lemmas 7.1, 7.5 and 7.6,

$$\begin{aligned}\|JB_m(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le C(m+1)\varepsilon_m^{-6(n+1)/N},\\ \|JB_m(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le C(m+1)\varepsilon_m^{-12(n+1)/N}.\end{aligned}$$

It follows from (8.2) that

$$u(t)-u_0 = \int_0^t \varepsilon_m B_m(\theta_0+\omega s)u_0\,ds + \int_0^t \varepsilon_m B_m(\theta_0+\omega s)(u(s)-u_0)\,ds.$$

Moreover, for $t\in[0,1]$, $\|u_0\|_N\le 1$,

$$\|u(t)-u_0\|_N \le \varepsilon_m C(m+1)\varepsilon_m^{-6(n+1)/N} + \int_0^t \varepsilon_m\|B_m(\theta_0+\omega s)\|\|u(s)-u_0\|_N\,ds,$$

where $\|\cdot\|$ is the operator norm from $h_N\times h_N\to h_N\times h_N$.

By Gronwall's inequality,

$$\|u(t)-u_0\|_N \le C(m+1)\varepsilon_m^{1-6(n+1)/N}\cdot\exp\left(\int_0^t \varepsilon_m\|B_m(\theta_0+\omega s)\|\,ds\right) \le \varepsilon_m^{1/2}.$$

Thus,

$$\Psi_m\colon \mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}\to\mathbb{T}^n_{s_m}\times\Pi_m,$$

and

$$\|\Psi_m - \mathrm{id}\,\|_{h_N\to h_N} \le \varepsilon_m^{1/2}. \tag{8.3}$$

Since (8.1) is linear, so Ψ_m is linear coordinate change. According to (8.2), construct Picard sequence

$$u_0(t) = u_0, \quad u_{j+1}(t) = u_0 + \int_0^t \varepsilon_m B(\theta_0 + \omega s) u_j(s)\, ds, \quad j = 0, 1, 2, \dots.$$

By (8.3), this sequence with $t = 1$ goes to

$$\Psi_m(u_0) = u(1) = (\mathrm{id} + P_m(\theta_0)) u_0,$$

where id is the identity from $h_N \times h_N \to h_N \times h_N$, and $P_m(\theta_0)$ is an operator form $h_N \times h_N \to h_N \times h_N$ for any fixed $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$, $\tau \in \Pi_{m+1}$, and is analytic in $\theta_0 \in \mathbb{T}^n_{s_{m+1}}$, with

$$\|P_m(\theta_0)\|_{\mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \le \varepsilon_m^{1/2}.$$

Note that (8.1) is a Hamiltonian system. So $P_m(\theta_0)$ is a symplectic linear operator from $h_N \times h_N$ to $h_N \times h_N$.

9. Estimates of remainders

The section is aimed to estimate the remainders:

$$C_{m+1} R_{m+1} = (6.10) + \cdots + (6.13).$$

Case 1*: Estimate of* (6.10). By (6.7), let

$$\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta) = \begin{pmatrix} R^{zz}_{m,m}(\theta) & \frac{1}{2} R^{z\overline{z}}_{m,m}(\theta) \\ \frac{1}{2} R^{z\overline{z}}_{m,m}(\theta) & R^{\overline{z}\overline{z}}_{m,m}(\theta) \end{pmatrix},$$

then

$$R_{mm} = \left\langle \widetilde{R}_{mm} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \right\rangle.$$

So

$$(1 - \Gamma_{K_m}) R_{mm} \triangleq \left\langle (1 - \Gamma_{K_m}) \widetilde{R}_{mm} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \right\rangle.$$

By the definition of truncation operator Γ_{K_m},

$$(1 - \Gamma_{K_m}) \widetilde{R}_{mm} = \sum_{|k| > K_m} \widehat{\widetilde{R}}_{mm}(k) e^{i\langle k, \theta \rangle}, \quad \theta \in \mathbb{T}^n_{s_m}, \ \tau \in \Pi_m.$$

Since $\widetilde{R}_{mm} = \widetilde{R}_{mm}(\theta)$ is analytic in $\theta \in \mathbb{T}^n_{s_m}$,

$$\sup_{(\theta,\tau) \in \mathbb{T}^n_{s_{m+1}} \times \Pi_{m+1}} \|J(1 - \Gamma_{K_m}) \widetilde{R}_{mm} J\|^2_{h_N \to h_N} \le \sum_{|k| > K_m} \|J \widehat{\widetilde{R}}_{mm}(k) J\|^2_N e^{2|k| s_{m+1}}$$

$$
\begin{aligned}
&\le \|J\widetilde{R}_{mm}J\|^2_{\mathbb{T}^n_{s_m}\times\Pi_m} \sum_{|k|>K_m} e^{-2(s_m-s_{m+1})|k|} \\
&\le C^2(m)\varepsilon_m^{-1}e^{-2K_m(s_m-s_{m+1})} \quad \text{(by (5.5))} \\
&\le C^2(m)\varepsilon_m^2.
\end{aligned}
$$

That is,

$$\|J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m C(m).$$

Thus,

$$\|\varepsilon_m J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_m^2 C(m) \le \varepsilon_{m+1}C(m+1).$$

Similarly,

$$\|\varepsilon_m J(1-\Gamma_{K_m})\widetilde{R}_{mm}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \varepsilon_{m+1}C(m+1).$$

Case 2: Estimate of (6.12). Let

$$S_m = \begin{pmatrix} F^{zz}(\theta,\tau) & \frac{1}{2}F^{z\overline{z}}(\theta,\tau) \\ \frac{1}{2}F^{z\overline{z}}(\theta,\tau) & F^{\overline{z}\overline{z}}(\theta,\tau)\end{pmatrix}, \quad \mathscr{J} = \begin{pmatrix} 0 & -\mathbf{i}\,\mathrm{id} \\ \mathbf{i}\,\mathrm{id} & 0\end{pmatrix}.$$

Then we can write

$$F = \left\langle S_m(\theta)\begin{pmatrix} z \\ \overline{z}\end{pmatrix}, \begin{pmatrix} z \\ \overline{z}\end{pmatrix}\right\rangle = \langle S_m u, u\rangle, \quad u = \begin{pmatrix} z \\ \overline{z}\end{pmatrix}.$$

Then

$$\varepsilon_m^2\{R_{mm}, F\} = 4\varepsilon_m^2\langle \widetilde{R}_{mm}(\theta)\mathscr{J}S_m(\theta)u,u\rangle.$$

Noting $\mathbb{T}^n_{s_m}\times\Pi_m \supset \mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}$. By (5.6) with $l=m$, $v=m$,

$$\|\widetilde{R}_{mm}(\theta)\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le \|\widetilde{R}_{mm}(\theta)\|_{\mathbb{T}^n_{s_m}\times\Pi_m} \le C(m), \tag{9.1}$$

$$\|\widetilde{R}_{mm}(\theta)\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m). \tag{9.2}$$

Let $\widetilde{S}_m(\theta) = \mathscr{J}S_m(\theta)$. Then by Lemmas 7.1, 7.5 and 7.6, we have

$$\|J\widetilde{S}_m(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-6(n+1)/N}, \tag{9.3}$$

$$\|J\widetilde{S}_m(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_m^{-12(n+1)/N} \tag{9.4}$$

and

$$
\begin{aligned}
\|\widetilde{R}_{mm}\mathscr{J}S_m\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &= \|\widetilde{R}_{mm}\widetilde{S}_m\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \\
&\le C(m)C(m+1)\varepsilon_m^{-6(n+1)/N}.
\end{aligned}
$$

Set

$$[\widetilde{R}_{mm}, \widetilde{S}_m] = \widetilde{R}_{mm}\widetilde{S}_m + (\widetilde{R}_{mm}\widetilde{S}_m)^T.$$

Note that the vector field is linear. So, by Taylor formula, one has

$$(6.12) = \varepsilon_m^2 \langle \widetilde{R}_m^*(\theta)u, u\rangle,$$

where

$$\widetilde{R}_m^*(\theta) = 2^2\widetilde{R}_{mm}\widetilde{S}_m + \sum_{j=2}^{\infty} \frac{2^{j+1}\varepsilon_m^{j-1}}{j!} \underbrace{[\cdots[\widetilde{R}_{mm}, \widetilde{S}_m], \dots, \widetilde{S}_m]}_{(j-1)\text{-fold}} \widetilde{S}_m.$$

By (9.1) and (9.3),

$$\begin{aligned} \|J\widetilde{R}_m^*(\theta)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le \sum_{j=1}^{\infty} \frac{C(m)C(m+1)\varepsilon_m^{j-1}(\varepsilon_m^{-6(n+1)/N})^j}{j!} \\ &\le C(m)C(m+1)\varepsilon_m^{-6(n+1)/N}. \end{aligned}$$

By (9.2) and (9.4),

$$\|J\widetilde{R}_m^*(\theta)J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{-12(n+1)/N}.$$

Thus,

$$\|\varepsilon_m^2 J\widetilde{R}_m^* J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{2-6(n+1)/N} \le C(m+1)\varepsilon_{m+1}$$

and

$$\|\varepsilon_m^2 J\widetilde{R}_m^* J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m)C(m+1)\varepsilon_m^{2-12(n+1)/N} \le C(m+1)\varepsilon_{m+1}.$$

Case 3*: Estimate of* (6.11). By (6.8),

$$\{N_m, F\} = \langle [R_{mm}^{z\bar{z}}]z, \bar{z}\rangle - \Gamma_{K_m}R_{mm} - \omega\cdot\partial_\theta F \triangleq R_{mm}^*.$$

Thus,

$$(6.11) = \varepsilon_m^2 \int_0^1 (1-\tau)\{R_{mm}^*, F\}\circ X^\tau_{\varepsilon_m F}\, d\tau. \tag{9.5}$$

Note R_{mm}^* is a quadratic polynomial in z and $\bar{z}$. So we write

$$R_{mm}^* = \langle \mathscr{R}_m(\theta,\tau)u, u\rangle, \quad u = \begin{pmatrix} z \\ \bar{z} \end{pmatrix}.$$

By (5.3) and (5.4) with $l = \nu = m$, and using (9.3) and (9.4),

$$\begin{aligned} \|J\mathscr{R}_m J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le C(m)\varepsilon_m^{-6(n+1)/N}, \\ \|J\mathscr{R}_m J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} &\le C(m)\varepsilon_m^{-12(n+1)/N}, \end{aligned} \tag{9.6}$$

where $\|\cdot\|$ is the operator norm in $h_N \times h_N \to h_N \times h_N$. Recall $F = \langle S_m(\theta,\tau)u,u\rangle$. Set

$$[\mathscr{R}_m, \widetilde{S}_m] = \mathscr{R}_m\widetilde{S}_m + (\mathscr{R}_m\widetilde{S}_m)^T. \tag{9.7}$$

Using Taylor formula to (9.5), we get

$$\begin{aligned}(6.11) &= \frac{\varepsilon_m^2}{2!}\{\{R^*_{mm},F\},F\} + \cdots + \frac{\varepsilon_m^j}{j!}\underbrace{\{\cdots\{R^*_{mm},F\},\ldots,F\}}_{j\text{-fold}} + \cdots \\ &= \Big\langle\Big(\sum_{j=2}^{\infty}\frac{2^{j+1}\varepsilon_m^j}{j!}\underbrace{[\cdots[\mathscr{R}_m,\widetilde{S}_m],\ldots,\widetilde{S}_m]}_{(j-1)\text{-fold}}\widetilde{S}_m\Big)\widetilde{u},\widetilde{u}\Big\rangle \\ &\triangleq \langle\mathscr{R}^{**}(\theta,\tau)u,u\rangle.\end{aligned}$$

By (9.3),(9.6) and (9.7), we have

$$\begin{aligned}&\|J\mathscr{R}^{**}(\theta,\tau)J\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \\ \le& \sum_{j=2}^{\infty}\frac{2^{j+1}}{j!}\|J\mathscr{R}_m(\theta,\tau)J\|_{\mathbb{T}^n_{s_m}\times\Pi_m}\big(\|J\widetilde{S}_mJ\|_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}}\varepsilon_m\big)^j \\ \le& \sum_{j=2}^{\infty}\frac{C(m)}{j!}\Big(\varepsilon_m C(m+1)\varepsilon_m^{-6(n+1)/N}\Big)^j \\ \le& C(m+1)\varepsilon_m^{4/3} = C(m+1)\varepsilon_{m+1}.\end{aligned}$$

Similarly,

$$\|J\mathscr{R}^{**}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_{m+1}}\times\Pi_{m+1}} \le C(m+1)\varepsilon_{m+1}.$$

Case 4*: Estimate of* (6.13).

$$(6.13) = \sum_{l=m+1}^{\infty}\varepsilon_l(R_{lm}\circ X^1_{\varepsilon_mF}).$$

Write $R_{lm} = \langle\widetilde{R}_{lm}(\theta)u,u\rangle$. Then, by Taylor formula

$$R_{lm}\circ X^1_{\varepsilon_mF} = R_{lm} + \sum_{j=1}^{\infty}\frac{1}{j!}\langle\widetilde{R}_{lmj}u,u\rangle,$$

where

$$\widetilde{R}_{lmj} = 2^{j+1}\underbrace{[\cdots[\widetilde{R}_{lm},\widetilde{S}_m],\ldots]}_{(j-1)\text{-fold}}\widetilde{S}_m\varepsilon_m^j.$$

By (5.5), (5.6),

$$\|J\widetilde{R}_{lm}J\|_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l), \quad \|J\widetilde{R}_{lm}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_m} \le C(l).$$

Combing the last inequalities with (9.3) and (9.4), we have

$$\begin{aligned}\|J\widetilde{R}_{lmj}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\le \|J\widetilde{R}_{lm}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\cdot\big(\|J\widetilde{S}_mJ\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}4\varepsilon_m\big)^j\\ &\le C^2(m)\big(\varepsilon_m\varepsilon_m^{-6(n+1)/N}\big)^j,\end{aligned}$$

where we use $\|J^{-1}\|_{\mathbb{T}^n_{sl}\times\Pi_{m+1}}\le C$, and

$$\begin{aligned}\|J\widetilde{R}_{lmj}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}} &\le \|J\widetilde{R}_{lm}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\big(\|J\widetilde{S}_mJ\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}4\varepsilon_m\big)^j\\ &\quad+\|J\widetilde{R}_{lm}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\big(\|J\widetilde{S}_mJ\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}\varepsilon_m\big)^j\\ &\le C^2(m)\big(\varepsilon_m\varepsilon_m^{-12(n+1)/N}\big)^j.\end{aligned}$$

Thus, let

$$\overline{R}_{l,m+1}:=\widetilde{R}_{lm}+\sum_{j=1}^{\infty}\frac{1}{j!}\widetilde{R}_{lmj},$$

then

$$(6.13)=\sum_{l=m+1}^{\infty}\varepsilon_l\langle\overline{R}_{l,m+1}u,u\rangle$$

and

$$\begin{aligned}\|J\overline{R}_{l,m+1}J\|_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}&\le C^2(m)\le C(m+1),\\ \|J\overline{R}_{l,m+1}J\|^{\mathscr{L}}_{\mathbb{T}^n_{s_l}\times\Pi_{m+1}}&\le C^2(m)\le C(m+1).\end{aligned}$$

As a whole, the remainder R_{m+1} can be written as

$$C_{m+1}R_{m+1}=\sum_{l=m+1}^{\infty}\varepsilon_l\big(\langle R^{zz}_{l,\nu}(\theta)z,z\rangle+\langle R^{z\overline{z}}_{l,\nu}(\theta)z,\overline{z}\rangle+\langle R^{\overline{z}\overline{z}}_{l,\nu}(\theta)\overline{z},\overline{z}\rangle\big),\quad \nu=m+1,$$

where, for $p,q\in\{z,\overline{z}\}$, $R^{p,q}_{l\nu}$ satisfies (5.5) and (5.6) with $\nu=m+1$, $l\ge m+1$. This shows that Assumption $(\mathrm{A2})_\nu$ with $\nu=m+1$ holds true.

By (6.9),

$$\mu_j^{(m)}=\widehat{R}^{z\overline{z}}_{mmjj}(0).$$

In (5.5) and (5.6), taking $p=z$, $q=\overline{z}$, we have

$$\begin{aligned}|\mu_j^{(m)}|_{\Pi_m}&\le|R^{z\overline{z}}_{mmjj}(\theta,\tau)|/j\le C(m)/j,\\ |\mu_j^{(m)}|^{\mathscr{L}}_{\Pi_m}&\le|\partial_\tau R^{z\overline{z}}_{mmjj}(\theta,\tau)|/j\le C(m)/j.\end{aligned}$$

This shows that Assumption $(\mathrm{A1})_\nu$ with $\nu=m+1$ holds true.

10. Estimate of measure

In this section, C denotes a universal constant, which may be different in different places. Now let us return to (7.3).

$$Q_{kij}^{(m)} \triangleq \left\{ \tau \in \Pi_m \,\middle|\, \big| - \langle k, \omega_0 \rangle \tau + \lambda_i^{(m)} - \lambda_j^{(m)} \big| < \frac{(|i-j|+1)\gamma_m}{A_k} \right\}. \tag{10.1}$$

First let $i = j$, then $k \neq 0$. At this time, (10.1) becomes

$$Q_{kii}^{(m)} = \left\{ \tau \in \Pi_m \,\middle|\, |\langle k, \omega_0 \rangle \tau| < \frac{\gamma_m}{A_k} \right\}.$$

It follows

$$|\langle k, \omega_0 \rangle| < \frac{\gamma_m}{(|k|^{2n+3}+8)\tau}.$$

Recall $|\langle k, \omega_0 \rangle| > \gamma / |k|^{n+1}$. Then

$$\operatorname{mes} Q_{kii}^{(m)} = 0. \tag{10.2}$$

In the following, let $i \neq j$. If $Q_{kij}^{(m)} = \emptyset$, then $\operatorname{mes} Q_{kij}^{(m)} = 0$. So we assume $Q_{kij}^{(m)} \neq \emptyset$. Then $\exists\, \tau \in \Pi_m$ such that

$$\big| - \langle k, \omega_0 \rangle \tau + \lambda_i^{(m)} - \lambda_j^{(m)} \big| < \frac{|i-j|+1}{A_k} \gamma_m. \tag{10.3}$$

It follows from (5.2) and (5.3) that

$$\lambda_i^{(m)} - \lambda_j^{(m)} = i - j + O(\varepsilon_0 / i) + O(\varepsilon_0 / j). \tag{10.4}$$

Moreover,

$$\big| \lambda_i^{(m)} - \lambda_j^{(m)} \big| \geq \frac{1}{2} |i-j|. \tag{10.5}$$

By (10.3) and (10.5), one has

$$|\langle k, \omega_0 \rangle \tau| \geq \big| \lambda_i^{(m)} - \lambda_j^{(m)} \big| - \frac{|i-j|+1}{A_k} \gamma_m \geq \frac{1}{2}|i-j| - \frac{|i-j|+1}{A_k}\gamma_m \geq \frac{1}{4}|i-j|.$$

Recall $\omega = \omega_0 \tau$. So

$$4 |\langle k, \omega \rangle| \geq |i-j|. \tag{10.6}$$

Again by (10.3) and (10.4), we have that, when $\tau \in \Pi_m$ such that (10.3) holds true, the following inequality holds true:

$$\begin{aligned} | - \langle k, \omega \rangle + i - j| &\leq \frac{|i-j|+1}{A_k} \gamma_m + \frac{C_1 \varepsilon_0}{i} + \frac{C_2 \varepsilon_0}{j} \\ &\leq \frac{|i-j|+1}{A_k} \gamma_m + \frac{C_1 \varepsilon_0}{i_0} + \frac{C_2 \varepsilon_0}{j_0} \quad \text{if } i \geq i_0,\ j \geq j_0, \end{aligned}$$

where $C_1 > 0$, $C_2 > 0$ are constants.

Thus

$$Q_{kij}^{(m)} \subset \left\{ \tau \in \Pi_m \;\middle|\; |-\langle k, \omega \rangle + l| < \frac{|l|+1}{A_k}\gamma_m + \frac{C_1\varepsilon_0}{i_0} + \frac{C_2\varepsilon_0}{j_0} \right\} \triangleq \widetilde{Q}_{kl},$$

when $i \geq i_0$, $j \geq j_0$. By (10.6), one has

$$|l| \leq 4|\langle k, \omega \rangle| \leq C|k|.$$

Note that

$$-\langle k, \omega \rangle + l = -\langle k, \omega_0 \rangle \tau + l = \tau \left(-\langle k, \omega_0 \rangle + \frac{l}{\tau} \right), \quad \tau \in [1, 2].$$

Thus

$$\widetilde{Q}_{kl} \subset \left\{ \tau \in \Pi_m \;\middle|\; \left| -\langle k, \omega_0 \rangle + \frac{l}{\tau} \right| < \frac{|l|+1}{A_k}\gamma_m + \frac{C_1\varepsilon_0}{i_0} + \frac{C_2\varepsilon_0}{j_0} \right\} \triangleq \widetilde{Q}_{kl}^*.$$

Note

$$\left| \frac{d}{d\tau} \left(-\langle k, \omega_0 \rangle + \frac{l}{\tau} \right) \right| = \frac{|l|}{\tau^2} \geq \frac{1}{4}|l|.$$

It follows that

$$\operatorname{mes} \widetilde{Q}_{kl} \leq \operatorname{mes} \widetilde{Q}_{kl}^* \leq \frac{8}{|l|} \left(\frac{|l|+1}{A_k}\gamma_m + \frac{C_1\varepsilon_0}{i_0} + \frac{C_2\varepsilon_0}{j_0} \right).$$

Take

$$j_0 = i_0 = |k|^{n+1}\gamma_m^{-1/3}.$$

Then

$$\begin{aligned} \operatorname{mes} \bigcup_{1 \leq l \leq C|k|} \widetilde{Q}_{kl} &\leq \frac{C|k|\gamma_m}{A_k} + C \sum_{1 \leq |l| \leq C|k|} \frac{1}{|l|} \left(\frac{C_1\varepsilon_0}{i_0} + \frac{C_2\varepsilon_0}{j_0} \right) \\ &\leq \frac{C|k|\gamma_m}{A_k} + C\gamma_m^{1/3}\varepsilon_0 \frac{\log|k|}{|k|^{n+1}} \\ &\leq C\gamma_m^{1/3}\varepsilon_0 \frac{\log|k|}{|k|^{n+1}}. \end{aligned}$$

Thus,

$$\operatorname{mes} \bigcup_{\substack{i \geq i_0 \\ j \geq j_0 \\ |i-j| \leq C|k|}} Q_{kij}^{(m)} \leq C\gamma_m^{1/3}\varepsilon_0 \frac{\log|k|}{|k|^{n+1}}. \tag{10.7}$$

Now assume

$$i \leq i_0 \quad \text{or} \quad j \leq j_0 \quad \text{and} \quad |i-j| \leq C|k|.$$

By (10.3) and (10.5), we have

$$\left|\frac{d}{d\tau}\Big(\frac{-\langle k,\omega_0\rangle\tau+\lambda_i^{(m)}-\lambda_j^{(m)}}{\tau}\Big)\right|\geq\frac{\left|\lambda_i^{(m)}-\lambda_j^{(m)}\right|}{4}\geq\frac{|i-j|+1}{16},$$

$$\begin{aligned}\text{mes}\bigcup_{\substack{i\leq i_0\\|i-j|\leq C|k|}}Q_{kij}^{(m)}&\leq\sum_{\substack{1\leq i\leq i_0\\|i-j|\leq C|k|}}\frac{2(|i-j|+1)\gamma_m}{A_k}\cdot\frac{16}{|i-j|+1}\\&\leq Ci_0\frac{C|k|\gamma_m}{A_k}\leq C|k|^{n+2}\gamma_m^{2/3}\frac{1}{A_k}\leq\frac{C\gamma_m^{2/3}}{|k|^{n+1}}\end{aligned}\tag{10.8}$$

and

$$\text{mes}\bigcup_{\substack{j\leq j_0\\|i-j|\leq C|k|}}Q_{kij}^{(m)}\leq\frac{C\gamma_m^{2/3}}{|k|^{n+1}}.\tag{10.9}$$

Combining (10.2), (10.7) (10.8) and (10.9), we have

$$\text{mes}\bigcup_{|k|\leq K_m}\bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}Q_{kij}^{(m)}\leq C\gamma_m^{1/3}.$$

Let

$$\Pi_{m+1}^{+-}=\Pi_m\setminus\bigcup_{|k|\leq K_m}\bigcup_{i,j=1}^{\infty}Q_{kij}^{(m)}.$$

Then we have proved the following Lemma 10.1.

Lemma 10.1.

$$\text{mes}\,\Pi_{m+1}^{+-}\geq\text{mes}\,\Pi_m-C\gamma_m^{1/3}.$$

11. Proofs of theorem and corollaries

Proofs of Theorem 1.1 *and Corollary* 1.2. Let

$$\Pi_\infty=\bigcap_{m=1}^{\infty}\Pi_m\quad\text{and}\quad\Psi_\infty=\lim_{m\to\infty}\Psi_0\circ\Psi_1\circ\cdots\circ\Psi_m.$$

By (5.7) and (5.8), one has

$$\Psi_\infty\colon\mathbb{T}^n\times\Pi_\infty\to\mathbb{T}^n\times\Pi_\infty,\quad\|\Psi_\infty-\text{id}\,\|_{h_N\to h_N}\leq\varepsilon^{1/2},$$

and, by (5.9),

$$H_\infty=H\circ\Psi_\infty=\sum_{j=1}^{\infty}\lambda_j^\infty Z_j\overline{Z}_j,$$

where

$$\lambda_j^\infty = \lim_{m\to\infty} \lambda_j^{(m)}.$$

By (5.2) and (5.3), the limit λ_j^∞ does exist and

$$\lambda_j^\infty = j + O(\varepsilon/j) := \sqrt{j^2+\xi_j}.$$

Introduce a transformation $\mathscr{G}\colon Z = (Z_j \in \mathbb{C} : j \geq 1) \mapsto v(t,x)$ by

$$v(t,x) = \sum_{j=1}^{\infty} q_j(t)\sin jx, \quad Z_j = \frac{1}{\sqrt{2}}(q_j - \mathbf{i}p_j), \quad \overline{Z}_j = \frac{1}{\sqrt{2}}(q_j + \mathbf{i}p_j).$$

Let

$$\Phi = (\mathscr{S}\mathscr{T}\mathscr{G}\Psi_\infty \mathscr{G}^{-1}\mathscr{T}^{-1}\mathscr{S}^{-1})^{-1}.$$

Then Φ is a symplectic transformation and changes (1.3) subject to (1.4) into (1.5). Also, the transformation Φ changes the wave operator

$$\mathscr{L}_V\colon \mathscr{L}_V u(t,x) = (\partial_t^2 - \partial_x^2 + \varepsilon V(\omega t, x))u(t,x), \quad u(t,-\pi) = u(t,\pi) = 0$$

into

$$\mathscr{L}_M\colon \mathscr{L}_M v(t,x) = (\partial_t^2 - \partial_x^2 + \varepsilon M_\xi)v(t,x), \quad v(t,-\pi) = v(t,\pi) = 0,$$

which possesses the property of pure point spectra and zero Lyapunov exponent.

This completes the proofs of Theorem 1.1 and Corollary 1.2. □

Proof of Corollary 1.3. By Theorem 1.1, we have that if $u = u(t,x)$ is a solution to (1.6), then

$$v(t,x) := (\Phi(u,u_t))(t,x)$$

is the solution to (1.5) with

$$\begin{aligned} v(0,x) &= (\Phi(u,u_t))(0,x) = (\Phi(u_0,\widetilde{u}_0))(x), \\ v_t(0,x) &= \partial_t(\Phi(u,u_t))(0,x) = (\Phi_t(u_0,\widetilde{u}_0))(x). \end{aligned}$$

Write

$$v_0 = v(0,x) = \sum_{k\in\mathbb{N}} C_k \sin kx, \quad \widetilde{v}_0 = v_t(0,x) = \sum_{k\in\mathbb{N}} C_k' \sin kx.$$

By solving (1.5) directly, we have

$$v(t,x) = \sum_{k=1}^{\infty} \big(C_k'(\lambda_k+\varepsilon\xi_k)^{-1/2}\sin(\sqrt{\lambda_k+\varepsilon\xi_k}\,t) + C_k\cos(\sqrt{\lambda_k+\varepsilon\xi_k}\,t)\big)\sin kx.$$

Note $k^2(1-c\varepsilon) \leq \lambda_k + \varepsilon\xi_k \leq k^2(1+c\varepsilon)$. It follows

$$\begin{aligned}
\|v(t)\|^2_{\mathscr{H}^N} &= \sum_{k=1}^{\infty} \big(|C_k|^2 + (\lambda_k + \varepsilon\xi_k)^{-1}|C'_k|^2\big)k^{2N} \\
&\leq \sum_{k=1}^{\infty} \big(|C_k|^2 + (1+c\varepsilon)k^{-2}|C'_k|^2\big)k^{2N} \\
&= \|v_0\|^2_{\mathscr{H}^N} + (1+c\varepsilon)\|\widetilde{v}_0\|^2_{\mathscr{H}^{N-1}}.
\end{aligned}$$

Noting $\mathscr{S}$, $\mathscr{T}$, $\mathscr{G}$ are isometric maps, using $\|\Psi_\infty - \mathrm{id}\,\|_{h_N \to h_N} \leq \varepsilon^{1/2}$, we have

$$\begin{aligned}
\|u(t)\|^2_{\mathscr{H}^N} &\leq (1+c\sqrt{\varepsilon})\|v(t)\|^2_{\mathscr{H}^N} \leq (1+c\sqrt{\varepsilon})\big(\|v_0\|^2_{\mathscr{H}^N} + (1+c\varepsilon)\|\widetilde{v}_0\|^2_{\mathscr{H}^{N-1}}\big) \\
&\leq (1+c\sqrt{\varepsilon})\big(\|u_0\|^2_{\mathscr{H}^N} + \|\widetilde{u}_0\|^2_{\mathscr{H}^{N-1}}\big).
\end{aligned}$$

And

$$\begin{aligned}
\|v(t)\|^2_{\mathscr{H}^N} &= \sum_{k=1}^{\infty} \big(|C_k|^2 + (\lambda_k + \varepsilon\xi_k)^{-1}|C'_k|^2\big)k^{2N} \\
&\geq \sum_{k=1}^{\infty} \big(|C_k|^2 + (1-c\varepsilon)k^{-2}|C'_k|^2\big)k^{2N} \\
&= \|v_0\|^2_{\mathscr{H}^N} + (1-c\varepsilon)\|\widetilde{v}_0\|^2_{\mathscr{H}^{N-1}},
\end{aligned}$$

thus

$$\|u(t)\|^2_{\mathscr{H}^N} \geq (1-c\sqrt{\varepsilon})\|v(t)\|^2_{\mathscr{H}^N} \geq (1-c\sqrt{\varepsilon})\big(\|u_0\|^2_{\mathscr{H}^N} + \|\widetilde{u}_0\|^2_{\mathscr{H}^{N-1}}\big),$$

where c is a positive constant which might be different in different places. Note

$$v_t(t,x) = \sum_{k=1}^{\infty} \big(C'_k \cos(\sqrt{\lambda_k + \varepsilon\xi_k}\, t) - C_k\sqrt{\lambda_k + \varepsilon\xi_k}\sin(\sqrt{\lambda_k + \varepsilon\xi_k}\, t)\big)\sin kx.$$

Then

$$\begin{aligned}
\|v_t(t)\|^2_{\mathscr{H}^{N-1}} &= \sum_{k=1}^{\infty} \Big(\big(\sqrt{\lambda_k + \varepsilon\xi_k}\big)^2|C_k|^2 + |C'_k|^2\Big)k^{2(N-1)} \\
&\leq \sum_{k=1}^{\infty} \big((1+c\varepsilon)|C_k|^2k^{2N} + |C'_k|^2k^{2(N-1)}\big) \\
&\leq (1+c\varepsilon)\big(\|v_0\|^2_{\mathscr{H}^N} + \|\widetilde{v}_0\|^2_{\mathscr{H}^{N-1}}\big).
\end{aligned}$$

Thus, by the preceding proof, we have

$$(1-c\sqrt{\varepsilon})\big(\|u_0\|^2_{\mathscr{H}^N} + \|\widetilde{u}_0\|^2_{\mathscr{H}^{N-1}}\big) \leq \|u_t(t)\|^2_{\mathscr{H}^{N-1}} \leq (1+c\sqrt{\varepsilon})\big(\|u_0\|^2_{\mathscr{H}^N} + \|\widetilde{u}_0\|^2_{\mathscr{H}^{N-1}}\big).$$

This completes the proof of Corollary 1.3. $\square$

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