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Searching for Structures Inside of the Family of Bounded Derivatives Which are not Riemann Integrable

Pablo Jiménez-Rodríguez

Abstract. We construct a non-separable Banach space every nonzero element of which is a bounded derivative that is not Riemann integrable. This in particular improves a result presented in [3], where the corresponding space was found to be separable.

1. Introduction and preliminaries

When introduced to a basic course in integration, the Fundamental Theorem of Calculus might hint that the concepts of *antiderivative* and *integral* are synonymous. In fact, an antiderivative is usually denoted with an integral symbol. Nevertheless, in 1881, Volterra [9] gave an example that went against that intuition (the reader can also check the monograph in [5]). Indeed, the function

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable everywhere but its differential is unbounded in any open interval containing 0, so in particular it can not be Riemann integrable (in fact, since the function f is not absolutely continuous, its differential is not Lebesgue integrable either, see for example [7, Theorem 20.8]).

It might then seem reasonable to think that if f has a bounded derivative, then f could be recovered from f' via the Fundamental Theorem of Calculus. Yet, Volterra focused on the function

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

to give an example of a differentiable function with bounded derivative, but whose differential was again not Riemann integrable.

As usual, if A is a set and $f\colon A \to \mathbb{R}$ is a bounded function, we may consider the sup norm of f, $\|f\|_\infty = \sup\{|f(a)| : a \in A\}$. ℓ_∞ denotes the space formed by all such elements

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when considered such norm in the case where $A = \mathbb{N}$. We will also denote by $B[0,1]$ the space obtained when $A = [0,1]$.

Following the same line of work that motivated the study of the function g, in [3] the authors managed to construct a Banach space (using the sup norm) of differentiable functions, every non zero element of which fulfilled the same properties as g: it is a differentiable function and its derivative is bounded but not Riemann integrable. The idea for that result is to consider the closure in the sup norm of a sequence of functions, everyone of which is a perturbation of the function g supported on one member of a countable family of pairwise disjoint open intervals. Therefore, in particular the Banach space that was constructed is separable.

Given a cardinal number κ, we say that a subset M of a topological vector space is *κ-lineable* if we can find a vector space V of dimension κ so that $V \subseteq M \cup \{0\}$. Analogously, we say that M is *κ-spaceable* if this space is closed. Typically, the elements of the set M will be described in terms of certain property, usually against the intuition (what in mathematical folklore it has been given the name of *pathological property*). Those terms were coined in the beginning of the 21^{st} century in [2] (see also [8]), and since its appearance it has proven to be a fruitful field. The interested reader may access to the majority of the results in this trend (up to its publication in 2016) in the monograph [1].

Hence, the result from [3] appears as follows:

Theorem 1.1. [3] *The set of differentiable functions whose derivative is bounded but is not Riemann integrable is spaceable.*

In order to find another (more accessible) example of a bounded derivative which is not Riemann integrable, R. A. Gordon attempted an idea which was similar to Volterra's construction: he started with a countable family of pairwise disjoint intervals, (for whom he asked some further properties) and then he considered an infinite sum of polygonal functions, each of whose support was contained in one of the mentioned intervals (see [4]).

In the present paper we will carry on with Gordon's construction in order to improve the result in [3] by showing that in fact the Banach space contained in the set of differentiable functions whose derivative is bounded but is not Riemann integrable can be obtained to be non separable (in fact isometric to ℓ_∞).

2. Our construction

Theorem 2.1. *The set*

$$\{f\colon [0,1] \to \mathbb{R} \textit{ such that } f \textit{ is a bounded derivative, but is not Riemann integrable}\}$$

is spaceable. Moreover, there exists a linear isometry from ℓ_∞ into $B[0,1]$ verifying the following: the image of every non zero element of ℓ_∞ is a bounded, non-Riemann integrable function which is (also) a derivative.

Proof. Let $\mathcal{O} = \bigcup_{n=1}^{\infty}(a_n, b_n)$ be an open dense set in the interval $(0,1)$, where the intervals (a_n, b_n) are pairwise disjoint and the inequality

$$\sum_{n=1}^{\infty}(b_n - a_n) < \frac{1}{2}$$

is satisfied.

Without loss of generality, we may assume that $a_i \neq b_j$ for all positive integers $i \neq j$. For every $n \in \mathbb{N}$, consider $\mathcal{O}_n \subseteq (a_n, b_n)$ open and dense, verifying similar properties as $\mathcal{O}$, that is, we can write $\mathcal{O}_n = \bigcup_{k=1}^{\infty}(a_k^{(n)}, b_k^{(n)})$, where $(a_k^{(n)}, b_k^{(n)})$ are pairwise disjoint.

For every k, n choose $c_k^{(n)} < d_k^{(n)}$ so that $a_k^{(n)} < c_k^{(n)} < d_k^{(n)} < b_k^{(n)}$ and verifying

$$\frac{c_k^{(n)} + d_k^{(n)}}{2} = \frac{a_k^{(n)} + b_k^{(n)}}{2} \quad \text{and} \quad d_k^{(n)} - c_k^{(n)} = (b_k^{(n)} - a_k^{(n)})^2.$$

Notice that, with this choice of intermediate points, we have

$$c_k^{(n)} - a_k^{(n)} = \frac{(b_k^{(n)} - a_k^{(n)}) - (b_k^{(n)} - a_k^{(n)})^2}{2} > \frac{b_k^{(n)} - a_k^{(n)}}{4}. \tag{2.1}$$

For every k, we consider the function $h_k\colon (\ell_\infty, \|\cdot\|_\infty) \to (B[0,1], \|\cdot\|_\infty)$ by, given $\{x_s\}_{s=1}^{\infty} \in \ell_\infty$ and $t \in [0,1]$,

$$h_k(\{x_s\}_{s=1}^{\infty})(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq c_k^{(n)} \text{ or } d_k^{(n)} \leq t \leq 1,\ n \in \mathbb{N}, \\ x_k & \text{if } t = (3c_k^{(n)} + d_k^{(n)})/4,\ n \in \mathbb{N}, \\ -x_k & \text{if } t = (c_k^{(n)} + 3d_k^{(n)})/4,\ n \in \mathbb{N}, \\ \text{linear} & \text{elsewhere.} \end{cases}$$

Notice that h_k is a linear operator, since the equation of the line joining the points $(a_1, b_1 + c_1)$ with $(a_2, b_2 + c_2)$ is the sum of the equations of the lines joining the points (a_1, b_1) with (a_2, b_2), and (a_1, c_1) with (a_2, c_2) (similarly to what happened in the construction of the isometry in [6]). Define next

$$\begin{array}{rccc} h\colon & (\ell_\infty, \|\cdot\|_\infty) & \longrightarrow & (B[0,1], \|\cdot\|_\infty) \\ & \{x_s\}_{s=1}^{\infty} & \longmapsto & \sum_{k=1}^{\infty} h_k(\{x_s\}_{s=1}^{\infty}). \end{array}$$

Notice that h is well-defined since, for every $t \in [0,1]$, $h_k(\{x_s\}_{s=1}^{\infty})(t) \neq 0$ for at most one value of k. Also, it is clear that $\|h(\{x_s\}_{s=1}^{\infty})\|_\infty = \|\{x_s\}_{s=1}^{\infty}\|_\infty$, so h is an isometry.

We claim now that, for every $\{x_s\}_{s=1}^{\infty} \in \ell_\infty \setminus \{0\}$, $h(\{x_s\}_{s=1}^{\infty})$ is not Riemann integrable. To simplify the notation, for a function f and an interval I we will denote the quantity

$$\omega(f, I) = \sup\{f(t) : t \in I\} - \inf\{f(t) : t \in I\}.$$

Denote, furthermore, $J_m = \{1, 2, \dots, m\}$ and $E = [0,1] \setminus \mathcal{O}$, let $\{t_0 = 0 < t_1 < \cdots < t_m = 1\}$ be a partition of the interval $[0,1]$ and define the sets

$$S = \{i \in J_m : (t_{i-1}, t_i) \cap E = \emptyset\} \quad \text{and} \quad T = J_m \setminus S.$$

Note that if $i \in S$, then $(t_{i-1}, t_i) \subseteq \mathcal{O}$. We would like to show that if $i \in T$, then $(a_{n_0}, b_{n_0}) \subseteq (t_{i-1}, t_i)$ for some n_0.

Indeed, we remark that, if $(t_{i-1}, t_i) \cap E = \{t\}$, we would have $(t_{i-1}, t) \cup (t, t_i) \subseteq \mathcal{O} = \bigcup_{n=1}^{\infty} (a_n, b_n)$ and therefore we would have $a_n = b_{\widetilde{n}}$ for some n, $\widetilde{n}$, which we have already stated does not happen. Therefore, it must be $\#[(t_{i-1}, t_i) \cap E] \geq 2$.

Suppose then $u < v$ are two points in $(t_{i-1}, t_i) \cap E$. Since $\mathcal{O}$ is dense in $(0,1)$, we can find a point $q \in (u, v) \cap \mathcal{O}$. Let $n_0 \in \mathbb{N}$ so that $q \in (a_{n_0}, b_{n_0})$. Since $u \notin (a_{n_0}, b_{n_0})$ and $(a_{n_0}, b_{n_0}) \cap (u, v) \neq \emptyset$, it must be $u < a_{n_0}$. Following a similar argument, it must be $v > b_{n_0}$ and therefore $(a_{n_0}, b_{n_0}) \subseteq (u, v) \subseteq (t_{i-1}, t_i)$.

We may conclude

$$\sum_{i \in S} (t_i - t_{i-1}) \leq m(\mathcal{O}) < \frac{1}{2},$$

so that

$$\sum_{i \in T} (t_i - t_{i-1}) \geq \frac{1}{2}.$$

Hence,

$$\begin{aligned}
\sum_{i \in J_m} \omega(h(\{x_s\}_{s=1}^{\infty}), [t_{i-1}, t_i])(t_i - t_{i-1}) &\geq \sum_{i \in T} \omega(h(\{x_s\}_{s=1}^{\infty}), [t_{i-1}, t_i])(t_i - t_{i-1}) \\
&\geq 2\|\{x_s\}_{s=1}^{\infty}\|_\infty \sum_{i \in T} (t_i - t_{i-1}) \geq \|\{x_s\}_{s=1}^{\infty}\|_\infty \neq 0.
\end{aligned}$$

Since we have this bound for any partition of the interval $[0,1]$ we consider, we can conclude that the function $h(\{x_s\}_{s=1}^{\infty})$ is not Riemann integrable.

To finish, we claim that $h(\{x_s\}_{s=1}^{\infty})$ is a derivative. Indeed, we define the function F on $[0,1]$ by

$$F(t) = \sum_{k=1}^{\infty} \int_0^t h_k(\{x_s\}_{s=1}^{\infty})(\zeta)\, d\zeta.$$

We will see that $F' = h$.

Note that F is well-defined since, again, for each $t \in [0,1]$, $\int_0^t h_k(\{x_s\}_{s=1}^\infty)(\zeta)\,d\zeta \neq 0$ for at most one value of $k \in \mathbb{N}$. Let first $t \in (a_l^{(n)}, b_l^{(n)})$. Then,

$$F(t) - F(a_l^{(n)}) = \int_{a_l^{(n)}}^t h_l(\{x_s\}_{s=1}^\infty)(\zeta)\,d\zeta.$$

Since $h_l(\{x_s\}_{s=1}^\infty)$ is continuous, we can use the Fundamental Theorem of Calculus to write

$$F'(t) = h_l(\{x_s\}_{s=1}^\infty)(t) = h(\{x_s\}_{s=1}^\infty)(t). \tag{2.2}$$

Let next

$$t \in (0,1) \setminus \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty (a_k^{(n)}, b_k^{(n)})$$

and let us show that

$$\lim_{t' \to t^+} \frac{F(t') - F(t)}{t' - t} = 0 = h(\{x_s\}_{s=1}^\infty)(t).$$

If $t = a_k^{(n)}$ for some k, n, then the result is trivial, since

$$\int_0^{t'} h_l(\{x_s\}_{s=1}^\infty)(\zeta)\,d\zeta = 0$$

for every $l \in \mathbb{N}$ and t' in the interval $[a_k^{(n)}, c_k^{(n)}]$.

Assume then $t \neq a_k^{(n)}$ for every k, n and let $t < t' < 1$. If $t' \notin [c_k^{(n)}, d_k^{(n)}]$ for every k, n, then $F(t') = F(t) = 0$. If we want, finally, to see the case where $t' \in [c_k^{(n)}, d_k^n]$ for some some k, n, we shall use the inequality (2.1) in the form

$$b_k^{(n)} - a_k^{(n)} < 4(c_k^{(n)} - a_k^{(n)}).$$

Therefore,

$$\begin{aligned}
|F(t') - F(t)| = |F(t')| &= \left| \int_{c_k^{(n)}}^{t'} h_k(\{x_s\}_{s=1}^\infty)(\zeta)\,d\zeta \right| \leq \int_{c_k^{(n)}}^{t'} |h_k(\{x_s\}_{s=1}^\infty)(\zeta)|\,d\zeta \\
&\leq \|\{x_l\}_{l=1}^\infty\|_\infty (t' - c_k^{(n)}) \leq \|\{x_l\}_{l=1}^\infty\|_\infty (d_k^{(n)} - c_k^{(n)}) \\
&= \|\{x_l\}_{l=1}^\infty\|_\infty (b_k^{(n)} - a_k^{(n)})^2 < 16\|\{x_l\}_{l=1}^\infty\|_\infty (c_k^{(n)} - a_k^{(n)})^2.
\end{aligned}$$

Let us remember now that we had chosen

$$t \in (0,1) \setminus \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty (a_k^{(n)}, b_k^{(n)}),$$

so in particular, using the fact that $t < t' \in [c_k^{(n)}, d_k^n] \subseteq [a_k^{(n)}, b_k^n]$, we can conclude $t < a_k^{(n)}$ and $c_k^{(n)} < t'$. Having that in mind, we can conclude

$$16\|\{x_l\}_{l=1}^\infty\|_\infty (c_k^{(n)} - a_k^{(n)})^2 \leq 16\|\{x_l\}_{l=1}^\infty\|_\infty (t' - t)^2.$$

Hence,

$$\lim_{t' \to t^+} \left| \frac{F(t') - F(t)}{t' - t} \right| \leq \lim_{t' \to t^+} 16\|\{x_l\}_{l=1}^{\infty}\|_{\infty}(t' - t) = 0.$$

Analogously, one can prove that

$$\lim_{t' \to t^-} \left| \frac{F(t') - F(t)}{t' - t} \right| = 0,$$

so we can deduce that $F'(t) = 0 = h(\{x_s\}_{s=1}^{\infty})(t)$, for

$$t \in (0,1) \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (a_k^{(n)}, b_k^{(n)}),$$

that is, together with the conclusion in (2.2),

$$F'(t) = h(\{x_s\}_{s=1}^{\infty})(t) \quad \text{for every } t \in (0,1). \qquad \square$$

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Pablo Jiménez-Rodríguez
Department of Mathematical Sciences, Mathematics and Computer Science Building, Summit Street, Kent OH 44242, USA
E-mail address: pjimene1@kent.edu