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ALGEBRA OF CALDERÓN-ZYGMUND OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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Abstract. Applying orthonormal wavelets, Meyer proved that all Calderón-Zygmund operators satisfying $T(1) = T^*(1) = 0$ form an algebra. In this article the same result is proved on spaces of homogeneous type introduced by Coifman and Weiss [5]. Since there is no such an orthonormal wavelet on the general setting, we apply the discrete Calderón reproducing formula developed in [13] to approach.

1. Introduction

We begin by recalling the definitions necessary for introducing the Calderón-Zygmund operator and spaces of homogeneous type.

To generalize the Hilbert transform and the Riesz transforms, Calderón and Zygmund developed a class of singular integral operators called convolution operators, which commute with translations. Notice that the Riesz transforms $R_j, 1 \le j \le n$, are defined by $R_j = D_j(-\Delta)^{-1/2}$, where $D_j = -i\partial/\partial x_j$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. We have $R_j(1) = (R_j)^*(1) = 0$, where $(R_j)^*$ is the transpose of R_j, and all Calderón-Zygmund convolution operators have this property. The collection of these operators is a commutative algebra of Calderón-Zygmund convolution operators and $T(1) = T^*(1) = 0$ for every T in this collection. However, there are a lot of non-convolution operators such as the Calderón commutators, the Cauchy integral on Lipschitz curves, the double layer potential on Lipschitz surfaces, the multilinear operators of Coifman and Meyer (see [1], [2], [4], [9], [7]). Coifman and Meyer

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introduced the following generalized Calderón-Zygmund singular integral operators which include all non-convolution operators mentioned above and, of course, the Calderón-Zygmund singular integral convolution operators.

Definition 1.1 ([4]). *Let* $T : \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{D}'(\mathbb{R}^n)$ *be a continuous linear operator associated to a kernel* $K(x,y)$, *a continuous function defined on* $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\}$. *We say that* T *is a Calderón-Zygmund singular integral operator if there exist a constant* C *and an exponent* $\varepsilon \in (0,1]$ *such that the following conditions are satisfied*:

$$|K(x,y)| \cdot \ C|x-y|^{-n}; \tag{1.2}$$

$$|K(x,y) - K(x',y)| \cdot \ C|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon} \quad \text{for all } |x-x'| \cdot \ \frac{1}{2}|x-y|; \tag{1.3}$$

$$|K(x,y) - K(x,y')| \cdot \ C|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon} \quad \text{for all } |y-y'| \cdot \ \frac{1}{2}|x-y|; \tag{1.4}$$

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) d(y) \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^n) \text{ and } x \notin \ \text{supp}(f).$$

Definition 1.5 ([4]). *A Calderón-Zygmund singular integral operator* T *is said to be a Calderón-Zygmund operator if* T *can be extended to a bounded operator on* $L^2(\mathbb{R}^n)$. *The norm of such an operator is defined by*

$$\|T\|_{CZ} = \|T\|_{2,2} + \inf\{C : (1.2), (1.3), \text{ and } (1.4) \text{ hold}\}.$$

Meyer introduced a class of Calderón-Zygmund operators and proved that this class forms an algebra of Calderón-Zygmund operators. To state Meyer's result, we need to explain the definition of $T(1) = 0$. By Calderón-Zygmund operator theory, if T is a Calderón-Zygmund operator, then T is also a bounded operator on L^p for all $1 < p < \infty$, and from L^∞ to $BMO(\mathbb{R}^n)$, where a locally integrable function $f \in BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty,$$

where the supremum is taken over all cubes Q whose sides are parallel to the axes and $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$. See [4] for details.

If T is a Calderón-Zygmund operator and, hence T^* is a Calderón-Zygmund operator as well. Then by a remarkable duality argument between the Hardy space H^1 and BMO proved by C. Fefferman [10], for any function $f \in H^1, T(1)$ can be well defined by

$$\langle T(1), f \rangle = \langle 1, T^*(f) \rangle$$

since T and T^* are bounded from H^1 into L^1, and therefore, $T(1) = 0$ means that $\int T^* f(x)dx = 0$ for all $f \in H^1$. Similarly, $T^*(1) = 0$ means that $\int Tf(x)dx = 0$ for all $f \in H^1$.

We now can state Meyer's result as follows.

Theorem 1.6. *Let $\mathcal{A}$ be the collection of Calderón-Zygmund operators satisfying $T(1) = T^*(1) = 0$. Then $\mathcal{A}$ is an algebra.*

The idea of the proof of Theorem 1.6 is to introduce a non-commutative algebra of matrices acting on ℓ^2. Meyer considered the matrix representations of operators in the collection $\mathcal{A}$ with respect to an orthonormal wavelet basis, and showed that these matrices representing such operators belong to the non-commutative algebra of matrices on ℓ^2 mentioned above. See [17] for more details.

The purpose of this paper is to generalize Meyer's result to more general setting, namely spaces of homogeneous type introduced by Coifman and Weiss [5]. Spaces of homogeneous type include the Euclidean space, the n-torus in $\mathbb{R}^n$, the C^∞-compact Riemann manifolds, the boundaries of bounded Lipschitz domains in $\mathbb{R}^n$, and the Lipschitz manifolds introduced recently by Triebel [18], which include various kind of fractals. See [6] and [19] for more examples.

A quasi-metric d on a set X is a function $d : X \times X \mapsto [0, \infty]$ satisfying:

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) there exists a constant $A < \infty$ such that

$$d(x, y) \cdot \quad A\big(d(x, z) + d(z, y)\big) \quad \text{for all } x, y, \text{ and } z \in X.$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X : d(y, x) < r\}, r > 0$, form a base. However, the balls themselves need not to be open when $A > 1$. In the sequel we always use A to denote this constant.

Definition 1.7 ([5])**.** *A space of homogeneous type (X, d, μ) is a set X together with a quasi-metric d and a nonnegative measure μ on X satisfying*

(i) $\mu(B(x, r)) < \infty$ *for all $x \in X$ and all $r > 0$;*

(ii) *there exists a constant $C < \infty$ such that*

$$\mu(B(x, 2r)) \cdot \quad C\mu(B(x, r)) \quad \text{for all } x \in X \text{ and all } r > 0.$$

Here μ is assumed to be defined on a σ-algebra which contains all Borel sets and all balls $B(x, r)$.

Macias and Segovia [16] have shown that one can replace d by another quasi-metric ρ such that there exist $C < \infty$ and some $\theta, 0 < \theta < 1$,

$$\rho(x,y) \approx \inf\{\mu(B) : B \text{ is a ball containing } x \text{ and } y\},$$

$$|\rho(x,y) - \rho(x',y)| \le C\rho(x,x')^{\theta}[\rho(x,y) + \rho(x',y)]^{1-\theta} \quad \text{for all } x, x', \text{ and } y \in X,$$

where the expression $a \approx b$ means, as usual, that there are constants C_1 and C_2 (independent of the main parameters involved) such that $C_1 \le a/b \le C_2$. We also preserve θ to denote this constant.

We now can introduce Calderón-Zygmund operator theory on spaces of homogeneous type.

Definition 1.8 ([6]). *Let C_0^{η} denote the collection of all continuous functions with compact support such that $\|f\|_{\eta} = \sup \frac{|f(x)-f(y)|}{\rho(x,y)^{\eta}} < \infty$. Let $T : C_0^{\eta}(X) \mapsto (C_0^{\eta})'(X), \eta > 0$, be a continuous linear operator. We say that T is a Calderón-Zygmund singular integral operator if there exist a continuous function $K(x,y)$, a constant C, and an exponent $\varepsilon \in (0, \theta]$ satisfying*

$$|K(x,y)| \le C\rho(x,y)^{-1}; \tag{1.9}$$

$$|K(x,y) - K(x',y)| \le C\rho(x,x')^{\varepsilon}\rho(x,y)^{-1-\varepsilon} \quad \text{for all } \rho(x,x') \le \frac{\rho(x,y)}{2A}; \tag{1.10}$$

$$|K(x,y) - K(x,y')| \le C\rho(y,y')^{\varepsilon}\rho(x,y)^{-1-\varepsilon} \quad \text{for all } \rho(y,y') \le \frac{\rho(x,y)}{2A}; \tag{1.11}$$

$$T(f)(x) = \int_X K(x,y)f(y)d\mu(y) \quad \text{for all } f \in C_0^{\eta}(X) \text{ and } x \notin \text{supp}(f).$$

Definition 1.12 ([6]). *A Calderón-Zygmund singular integral operator T defined in Definition 1.8 is said to be a Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(X)$. The norm of such an operator is defined by*

$$\|T\|_{CZ} = \|T\|_{2,2} + \inf\{C : (1.9), (1.10) \text{ and } (1.11) \text{ hold}\}.$$

Again, by the Calderón-Zygmund operator theory on spaces of homogeneous type, any Calderón-Zygmund operator is also bounded on $L^p, 1 < p < \infty$, and bounded from L^{∞} to BMO, where space BMO on spaces of homogeneous type is defined by similar way as in $\mathbb{R}^n$ with replacing cubes Q on $\mathbb{R}^n$ by balls B on X. See [6] for more details. Moreover, if T is a Calderón-Zygmund operator on spaces of homogeneous type, then $T(1) = 0$ and $T^*(1) = 0$ have the same meaning as mentioned above for $\mathbb{R}^n$.

We now are able to state our main theorem.

Theorem 1.13. *Let $\mathcal{A}$ be the collection of Calderón-Zygmund operators on spaces of homogeneous type, which satisfy $T(1) = T^*(1) = 0$. Then $\mathcal{A}$ is an algebra.*

There are no Fourier transform, translation and dilation on spaces of homogeneous type, so the orthonormal wavelet is not available. Hence, the idea used in [17] doesn't work for this more general setting. A new idea to prove theorem 1.13 is to use the discrete Calderón reproducing formula developed in [13]. To state such a discrete Calderón reproducing formula, we will suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. These hypotheses allow us to construct an approximation to the identity (see [15]).

Definition 1.14. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called an approximation to the identity if the kernels $S_k(x, y)$ of S_k are functions from $X \times X$ into $\mathbb{C}$ such that there exist constant C, and some $0 < \varepsilon \cdot \ \theta$ satisfying, for all $k \in \mathbb{Z}$ and all $x, x', y,$ and $y' \in X$,

(i) $|S_k(x,y)| \cdot \ C\dfrac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,x_0))^{1+\varepsilon}}$,

(ii) $|S_k(x,y) - S_k(x',y)| \cdot \ C\Big(\dfrac{\rho(x,x')}{2^{-k}+\rho(x,x_0)}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,x_0))^{1+\varepsilon}}$,
for $\rho(x,x') \cdot \ \dfrac{1}{2A}(2^{-k}+\rho(x,x_0))$,

(iii) $|S_k(x,y) - S_k(x,y')| \cdot \ C\Big(\dfrac{\rho(y,y')}{2^{-k}+\rho(x,x_0)}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,x_0))^{1+\varepsilon}}$,
for $\rho(y,y') \cdot \ \dfrac{1}{2A}(2^{-k}+\rho(x,x_0))$,

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]|$
$\cdot \ C\Big(\dfrac{\rho(x,x')}{2^{-k}+\rho(x,x_0)}\Big)^{\varepsilon}\Big(\dfrac{\rho(y,y')}{2^{-k}+\rho(x,x_0)}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,x_0))^{1+\varepsilon}}$,
for $\rho(x,x') \cdot \ \dfrac{1}{2A}(2^{-k}+\rho(x,x_0))$ and $\rho(y,y') \cdot \ \dfrac{1}{2A}(2^{-k}+\rho(x,x_0))$,

(v) $\displaystyle\int_X S_k(x,y)\,d\mu(y) = 1$,

(vi) $\displaystyle\int_X S_k(x,y)\,d\mu(x) = 1$.

The existence of the above approximation to the identity has been established in [8] (condition (iv) is not stated there but can be easily established by the same arguments).

To state the discrete Calderón type reproducing formula on spaces of homogeneous type, we recall the following result given by Christ [3], which is an analogue of the Euclidean dyadic cubes.

Theorem 1.15. *There exist a collection of open subsets* $\{Q_\tau^k \subset X : k \in \mathbb{Z}, \tau \in I_k\}$, *where* I_k *denotes some (possibly finite) index set depending on* k, *and constants* $\delta \in (0,1)$, $\alpha > 0$, *and* $C > 0$ *such that*

(i) $\mu(X \backslash \bigcup_\tau Q_\tau^k) = 0$ *for all* $k \in \mathbb{Z}$;

(ii) *if* $j \geq k$, *then either* $Q_{\tau'}^j \subset Q_\tau^k$ or $Q_{\tau'}^j \cap Q_\tau^k = \phi$;

(iii) *for each* (k, τ) *and each* $j < k$, *there is a unique* τ' *such that* $Q_\tau^k \subset Q_{\tau'}^j$;

(iv) *diameter* $(Q_\tau^k) \cdot \;\; C\delta^k$;

(v) *each* Q_τ^k *contains some ball* $B(z_\tau^k, \alpha\delta^k)$.

We fix such a collection of open subsets and call all Q_τ^k in Theorem 1.15 the "dyadic cubes" in X. Without loss of generality, we may assume $\delta = \frac{1}{2}$ in Theorem 1.15. Let i be a fixed large positive integer, and denote by y_τ^{k+i} the point in Q_τ^{k+i}. The discrete Calderón type reproducing formula on spaces of homogeneous type can be stated as follows.

Theorem 1.16 ([13]). *Suppose that* $\{S_k\}_{k\in\mathbb{Z}}$ *is an approximation to the identity defined above. Set* $D_k = S_k - S_{k-1}$. *Then there exist two families of operators* $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ *and* $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$ *such that, for all fixed* $y_\tau^{k+i} \in Q_\tau^{k+i}$ *and all* $f \in L^2(X)$,

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_{k+i}} \mu(Q_\tau^{k+i}) \widetilde{D}_k(x, y_\tau^{k+i}) D_k(f)(y_\tau^{k+i}) \\ &= \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_{k+i}} \mu(Q_\tau^{k+i}) D_k(x, y_\tau^{k+i}) \widetilde{\widetilde{D}}_k(f)(y_\tau^{k+i}), \end{aligned}$$

where the series converge in $L^2(X)$. *Moreover,* $\widetilde{D}_k(x,y)$, *the kernel of* $\widetilde{D}_k$, *satisfy the following estimates: for* $0 < \varepsilon' < \varepsilon$, *there exists a constant* $C > 0$ *depending* *on ε and ε' such that*

$$|\widetilde{D}_k(x,y)| \cdot \quad C\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,y))^{1+\varepsilon'}},$$

$$|\widetilde{D}_k(x,y)-\widetilde{D}_k(x',y)| \cdot \quad C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\varepsilon'}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,y))^{1+\varepsilon'}}$$

$$\text{for } \rho(x,x') \cdot \quad \frac{1}{2A}\big(2^{-k}+\rho(x,y)\big),$$

$$\int_X \widetilde{D}_k(x,y)d\mu(x)=\int_X \widetilde{D}_k(x,y)d\mu(y)=0 \qquad \text{for all } k\in\mathbb{Z}.$$

$\widetilde{\widetilde{D}}_k(x,y)$*, the kernel of* $\widetilde{\widetilde{D}}_k$*, satisfy the same conditions above but with interchanging the positions of x and y.*

Suppose that $\{D_k\}$, $\{\widetilde{D}_k\}$, and $\{\widetilde{\widetilde{D}}_k\}, k\in\mathbb{Z}$, are families of operators given by the discrete Calderón reproducing formula in Theorem 1.16. Let T be a Calderón-Zygmund operator. Then we obtain the following matrix representation of T with respect to all these families $\{D_k\},\{\widetilde{D}_k\}$, and $\{\widetilde{\widetilde{D}}_k\}$, $k\in\mathbb{Z}$.

$$\begin{aligned} T(f)(x) = &\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}} \langle T\mu(Q_{\tau'}^{k'+i})^{\frac12}D_{k'}(\cdot,y_{\tau'}^{k'+i}),\mu(Q_\tau^{k+i})^{\frac12}D_k(y_\tau^{k+i},\cdot)\rangle \\ &\cdot\mu(Q_\tau^{k+i})^{\frac12}\widetilde{D}_k(x,y_\tau^{k+i})\mu(Q_{\tau'}^{k'+i})^{\frac12}\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i}). \end{aligned} \tag{1.17}$$

It is easy to see that $K(x,y)$, the kernel of T, can be written as

$$\begin{aligned} K(x,y) = &\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}} \langle T\mu(Q_{\tau'}^{k'+i})^{\frac12}D_{k'}(\cdot,y_{\tau'}^{k'+i}),\mu(Q_\tau^{k+i})^{\frac12}D_k(y_\tau^{k+i},\cdot)\rangle \\ &\cdot\mu(Q_\tau^{k+i})^{\frac12}\widetilde{D}_k(x,y_\tau^{k+i})\mu(Q_{\tau'}^{k'+i})^{\frac12}\widetilde{\widetilde{D}}_{k'}(y_{\tau'}^{k'+i},y). \end{aligned}$$

To introduce the non-commutative matrices algebra, we need the following definition.

Definition 1.18. A matrix $A=(\alpha(\lambda,\lambda'))_{(\lambda,\lambda')\in\Lambda\times\Lambda}$ belongs to $\mathcal{M}_\varepsilon$ if there exists a constant $C>0$ such that, for all $(\lambda,\lambda')\in\Lambda\times\Lambda$,

$$|\alpha(\lambda,\lambda')| \cdot \quad C\omega_\varepsilon(\lambda,\lambda'),$$

where $\Lambda = \{(k, y_\tau^{k+i}) : k \in \mathbb{Z}, \tau \in I_{k+i},\ y_\tau^{k+i}$ is the center of ball $Q_\tau^{k+i}\}$ and, there is an $\varepsilon' < \varepsilon$,

$$\omega_\varepsilon(\lambda, \lambda') = \sqrt{\mu(Q_\tau^{k+i})\mu(Q_{\tau'}^{k'+i})}\, 2^{-|k-k'|\varepsilon'} \frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')} + \rho(y_\tau^{k+i}, y_{\tau'}^{k'+i}))^{1+\varepsilon}},$$

where $a \wedge b$ denotes the minimum of a and b.

Now we have

Proposition 1.19. *For any* $0 < \varepsilon \leq \theta$, $\mathcal{M}_\varepsilon$ *is an algebra.*

We say that an operator $T \in \mathcal{OPM}_\varepsilon$ if the matrix of T with respect to $\{D_k\}$ as mentioned in (1.17) belongs to $\mathcal{M}_\varepsilon$, and say that a Calderón-Zygmund operator $T \in \mathcal{A}_\varepsilon$ if T is a Calderón-Zygmund operator with the regularity exponent ε in Definition 1.8, and $T(1) = T^*(1) = 0$.

The following theorem together with Proposition 1.19 shows the main Theorem 1.13.

Theorem 1.20. *If* $0 < \varepsilon \leq \theta$ *and* $T \in \mathcal{OPM}_\varepsilon$, *then* $T \in \mathcal{A}_{\varepsilon'}$, *and, conversely, if* $T \in \mathcal{A}_\varepsilon$, *then* $T \in OPM_{\varepsilon'}$, *for all* $\varepsilon' < \varepsilon$.

2. The Proof of Main Theorem

The proof of Proposition 1.19. It suffices to show that

$$\sum_\Lambda \omega_\varepsilon(\lambda_0, \lambda)\omega_\varepsilon(\lambda, \lambda_1) \leq C\omega_\varepsilon(\lambda_0, \lambda_1). \tag{2.1}$$

To establish this inequality, by symmetry we may consider only the cases where $k_0 \leq k_1 \leq k, k_0 \leq k \leq k_1$, and $k \leq k_0 \leq k_1$. Denote I_1, I_2, and I_3 three partial sums in (2.1) corresponding to these three cases, respectively.

Notice that

$$\frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y_\tau^{k+i}, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \leq C\frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \quad \text{for } y \in Q_\tau^{k+i},$$

and similarly

$$\frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y_\tau^{k+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \leq C\frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \quad \text{for } y \in Q_\tau^{k+i}.$$

Thus,

$$\sum_{y_\tau^{k+i}} \mu(Q_\tau^{k+i}) \frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y_\tau^{k+i}, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y_\tau^{k+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}$$
$$\cdot \quad C \sum_{y_\tau^{k+i}} \int_{Q_\tau^{k+i}} \frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y)$$
$$= C \int_X \frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y).$$

In the first case $k_0 \cdot \; k_1 \cdot \; k$, we get

$$I_1 \cdot \; C\mu(Q_{\tau_0}^{k_0+i})^{1/2} \mu(Q_{\tau_1}^{k_1+i})^{1/2} \sum_{k_0 \;\; k_1 \;\; k} 2^{(k_0-k)\varepsilon'} 2^{(k_1-k)\varepsilon'}$$
$$\cdot \int_X \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y).$$

We write

$$\int_X \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y)$$
$$= \int_{\rho(y, y_{\tau_0}^{k_0+i}) \geq \frac{1}{2A}\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})} + \int_{\rho(y, y_{\tau_0}^{k_0+i}) < \frac{1}{2A}\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})}$$
$$= I_1^1 + I_1^2.$$

Since $\rho(y, y_{\tau_0}^{k_0+i}) < \frac{1}{2A}\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})$ implies $\rho(y, y_{\tau_1}^{k_1+i}) \geq \frac{1}{2A}\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})$,

$$I_1^1 \cdot \; C \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \cdot \int_X \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y)$$
$$\cdot \quad C \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}.$$

To estimate I_1^2, consider first that $2^{-k_0} \geq \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})$. Then

$$I_1^2 \cdot \; 2^{k_0} \int_X \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \, d\mu(y)$$
$$\cdot \quad C 2^{k_0}$$
$$\cdot \quad C \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}.$$

If $2^{-k_0} < \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})$, then

$$
\begin{aligned}
I_1^2 \cdot \; & C\frac{2^{-k_1\varepsilon}}{\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})^{1+\varepsilon}} \cdot \int_X \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}}\, d\mu(y) \\
\cdot \; & C\frac{2^{-k_1\varepsilon}}{\rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i})^{1+\varepsilon}} \\
\cdot \; & C\frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}
\end{aligned}
$$

since $k_0 \cdot \; k_1$. Both estimates show

$$
I_1^2 \cdot \; C\frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}
$$

and hence, together with the estimate on I_1^1,

$$
\begin{aligned}
& \int_X \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}\, d\mu(y) \\
& \quad \cdot \; C\frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
I_1 \cdot \; & C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}\frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \sum_{k_0 \; k_1 \; k} 2^{(k_0+k_1-2k)\varepsilon'} \\
\cdot \; & C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2} \cdot 2^{-|k_0-k_1|\varepsilon'} \frac{2^{-(k_0\wedge k_1)\varepsilon}}{(2^{-(k_0\wedge k_1)} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
= \; & C\omega_\varepsilon(\lambda_0, \lambda_1)
\end{aligned}
$$

since $k_0 \cdot \; k_1$.

Similarly, for the case $k_0 \cdot \; k \cdot \; k_1$,

$$
\begin{aligned}
& \int_X \frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}\, d\mu(y) \\
& \quad = \int_X \frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y, y_{\tau_0}^{k_0+i}))^{1+\varepsilon}} \cdot \frac{2^{-k\varepsilon}}{(2^{-k} + \rho(y, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}\, d\mu(y) \\
& \quad \cdot \; C\frac{2^{-k_0\varepsilon}}{(2^{-k_0} + \rho(y_{\tau_0}^{k_0+i}, y_{\tau_1}^{k_1+i}))^{1+\varepsilon}},
\end{aligned}
$$

and thus,

$$
\begin{aligned}
I_2 &\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2} \\
&\qquad\cdot \sum_{k_0\le k\le k_1} 2^{(k_0-k)\varepsilon'}2^{(k-k_1)\varepsilon'}\frac{2^{-k_0\varepsilon}}{(2^{-k_0}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}(k_1-k_0)2^{(k_0-k_1)\varepsilon'}\frac{2^{-k_0\varepsilon}}{(2^{-k_0}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}\cdot 2^{-|k_0-k_1|\varepsilon''}\frac{2^{-(k_0\wedge k_1)\varepsilon}}{(2^{-(k_0\wedge k_1)}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&\le C\omega_\varepsilon(\lambda_0,\lambda_1),
\end{aligned}
$$

where we use the facts that $k_0 \le k_1$ and $(k_1-k_0)2^{(k_0-k_1)\varepsilon'} \le C2^{-|k_0-k_1|\varepsilon''}$ for any $\varepsilon''<\varepsilon'$.

Finally, if $k\le k_0\le k_1$, by an easier estimates

$$
\begin{aligned}
&\int_X \frac{2^{-(k\wedge k_0)\varepsilon}}{(2^{-(k\wedge k_0)}+\rho(y,y_{\tau_0}^{k_0+i}))^{1+\varepsilon}}\cdot\frac{2^{-(k\wedge k_1)\varepsilon}}{(2^{-(k\wedge k_1)}+\rho(y,y_{\tau_1}^{k_1+i}))^{1+\varepsilon}}\,d\mu(y) \\
&\qquad\le C\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_3 &\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2} \\
&\qquad\cdot \sum_{k\le k_0\le k_1} 2^{(k-k_0)\varepsilon'}2^{(k-k_1)\varepsilon'}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}2^{-(k_0+k_1)\varepsilon'} \\
&\qquad\cdot \sum_{\substack{k\le k_0 \\ \rho(y_{\tau_0}^{k_0+i},\, y_{\tau_1}^{k_1+i})\ge 2^{-k}\ge 2^{-k_0}}} \frac{2^{-k(\varepsilon-2\varepsilon')}}{(\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&\quad+C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}\,2^{-(k_0+k_1)\varepsilon'}\sum_{\substack{k\le k_0 \\ 2^{-k_0}\le\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i})<2^{-k}}}\frac{2^{-k(\varepsilon-2\varepsilon')}}{2^{-k(1+\varepsilon)}} \\
&\quad+C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}2^{-(k_0+k_1)\varepsilon'}\sum_{\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i})\le 2^{-k_0}<2^{-k}}\frac{2^{-k(\varepsilon-2\varepsilon')}}{2^{-k(1+\varepsilon)}} \\
&\le C\mu(Q_{\tau_0}^{k_0+i})^{1/2}\mu(Q_{\tau_1}^{k_1+i})^{1/2}2^{-|k_1-k_0|\varepsilon'}\frac{2^{-k_0\varepsilon}}{(2^{-k_0}+\rho(y_{\tau_0}^{k_0+i},y_{\tau_1}^{k_1+i}))^{1+\varepsilon}} \\
&= C\omega_\varepsilon(\lambda_0,\lambda_1).
\end{aligned}
$$

We remark that the estimate in (2.1) is very useful and it will be used often in the proof of Theorem 1.20.

The proof of Theorem 1.20. We first prove the converse in Theorem 1.20. Suppose that $T \in A_\varepsilon$. It is sufficient to show the following estimate:

$$\begin{aligned} &\big|\; TD_{k'}(\cdot, y_{\tau'}^{k'+i}), D_k(y_\tau^{k+i}, \cdot)\rangle\big| \\ &\cdot\;\; C2^{-|k-k'|\varepsilon''}\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')} + \rho(y_\tau^{k+i}, y_{\tau'}^{k'+i}))^{1+\varepsilon'}}, \end{aligned} \tag{2.2}$$

where $0 < \varepsilon'' < \varepsilon'$. If $E_k(x, y)$, the kernel of E_k appeared in the discrete Calderón reproducing formula, satisfies

(i) $E_k(x,y) = 0$ if $\rho(x,y) \geq 2^{-k}$ and $\|E_k(x,y)\|_\infty \cdot\;\; 2^k$,

(ii) $|E_k(x,y) - E_k(x',y)| \cdot\;\; C\rho(x,x')^\varepsilon 2^{k(1+\varepsilon)}$,

(iii) $|E_k(x,y) - E_k(x,y')| \cdot\;\; C\rho(y,y')^\varepsilon 2^{k(1+\varepsilon)}$,

(iv) $\displaystyle\int E_k(x,y)\,d\mu(y) = \int E_k(x,y)\,d\mu(x) = 0$,

then the same estimate as (2.2) for E_k instead of D_k is easy to prove. See [15] for details. To deal with a general D_k whose kernel satisfies the conditions of Definition 1.14, we use the discrete Calderón reproducing formula:

$$\begin{aligned} f(x) &= \sum_j \sum_\tau \mu(Q_\tau^{j+i})\widetilde{E}_j(\cdot, y_\tau^{j+i})E_k(f)(y_\tau^{j+i}) \\ &= \sum_j \sum_\tau \mu(Q_\tau^{j+i})E_j(\cdot, y_\tau^{j+i})\widetilde{\widetilde{E}}_j(f)(y_\tau^{j+i}), \end{aligned}$$

where $E_j(x,y)$, the kernel of E_j, satisfies the conditions (i)–(iv) mentioned above. Now we obtain

$$\begin{aligned} &\big|\; TD_{k'}(\cdot, y_{\tau'}^{k'+i}), D_k(y_\tau^{k+i}, \cdot)\rangle\big| \\ &= \Big|\Big\langle \sum_j \sum_{\tau''} \mu(Q_{\tau''}^{j+i})TE_j(\cdot, y_{\tau''}^{j+i})\widetilde{\widetilde{E}}_j(D_{k'}(\cdot, y_{\tau'}^{k'+i})(y_{\tau''}^{j+i}), \\ &\qquad\qquad \sum_{j'} \sum_{\tau'''} \mu(Q_{\tau'''}^{j'+i})E_{j'}(\cdot, y_{\tau'''}^{j'+i})\widetilde{\widetilde{E}}_{j'}(D_k(y_\tau^{k+i}, \cdot)(y_{\tau'''}^{j'+i})\Big\rangle\Big| \\ &\cdot\;\; \sum_j \sum_{\tau''} \sum_{j'} \sum_{\tau'''} |\widetilde{\widetilde{E}}_j(D_{k'}(\cdot, y_{\tau'}^{k'+i})(y_{\tau''}^{j+i})\widetilde{\widetilde{E}}_{j'}(D_k(y_\tau^{k+i}, \cdot)(y_{\tau'''}^{j'+i})| \\ &\qquad\qquad \cdot\Big|\Big\langle TE_j(\cdot, y_{\tau''}^{j+i}), E_{j'}(y_{\tau'''}^{j'+i}, \cdot)\Big\rangle\Big|\mu(Q_{\tau''}^{j+i})\mu(Q_{\tau'''}^{j'+i}) \end{aligned}$$

$$
\begin{aligned}
\cdot\ & C\sum_{j}\sum_{\tau''}\sum_{j'}\sum_{\tau'''}2^{-|j-k'|\varepsilon''}\frac{2^{-(j\wedge k')\varepsilon'}}{(2^{-(j\wedge k')}+\rho(y_{\tau'}^{k'+i},y_{\tau''}^{j+i}))^{1+\varepsilon'}}\\
&\quad\cdot 2^{-|j'-k|\varepsilon''}\frac{2^{-(j'\wedge k)\varepsilon'}}{(2^{-(j'\wedge k)}+\rho(y_{\tau}^{k+i},y_{\tau'''}^{j'+i}))^{1+\varepsilon'}}\\
&\quad\cdot 2^{-|j-j'|\varepsilon''}\frac{2^{-(j\wedge j')\varepsilon'}}{(2^{-(j\wedge j')}+\rho(y_{\tau''}^{j+i},y_{\tau'''}^{j'+i}))^{1+\varepsilon'}}\mu(Q_{\tau''}^{j+i})\mu(Q_{\tau'''}^{j'+i})\\
= & C\sum_{j}\sum_{\tau''}\sum_{j'}\sum_{\tau'''}\mu(Q_{\tau'}^{k'+i})^{-1/2}\mu(Q_{\tau''}^{j+i})^{-1/2}\omega_{\varepsilon'}(\lambda_1,\lambda_2)\mu(Q_{\tau''}^{j+i})^{-1/2}\mu(Q_{\tau'''}^{j'+i})^{-1/2}\\
&\quad\cdot\omega_{\varepsilon'}(\lambda_2,\lambda_3)\mu(Q_{\tau'''}^{j'+i})^{-1/2}\mu(Q_{\tau}^{k+i})^{-1/2}\omega_{\varepsilon'}(\lambda_3,\lambda_4)\mu(Q_{\tau''}^{j+i})\mu(Q_{\tau'''}^{j'+i}),
\end{aligned}
$$

where $\lambda_1=(k',y_{\tau'}^{k'+i})$, $\lambda_2=(j,y_{\tau''}^{j+i})$, $\lambda_3=(j',y_{\tau'''}^{j'+i})$, and $\lambda_4=(k,y_{\tau}^{k+i})$. We hence have

$$
\begin{aligned}
& \big|\ TD_{k'}(\cdot,y_{\tau'}^{k'+i}),D_k(y_{\tau}^{k+i},\cdot)\big\rangle\big|\\
\cdot\ & C\sum_{j}\sum_{\tau''}\sum_{j'}\sum_{\tau'''}\mu(Q_{\tau'}^{k'+i})^{-1/2}\mu(Q_{\tau}^{k+i})^{-1/2}\omega_{\varepsilon'}(\lambda_1,\lambda_2)\omega_{\varepsilon'}(\lambda_2,\lambda_3)\omega_{\varepsilon'}(\lambda_3,\lambda_4)\\
\cdot\ & C\mu(Q_{\tau'}^{k'+i})^{-1/2}\mu(Q_{\tau}^{k+i})^{-1/2}\omega_{\varepsilon'}(\lambda_1,\lambda_4)\\
= & C2^{-|k-k'|\varepsilon''}\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\rho(y_{\tau}^{k+i},y_{\tau'}^{k'+i}))^{1+\varepsilon'}}.
\end{aligned}
$$

The last inequality follows again from estimate (2.1).

We now return to the proof that if $0<\varepsilon\cdot\ \theta$ and $T\in\mathcal{OPM}_{\varepsilon}$, then $T\in\mathcal{A}_{\varepsilon''}$ where $\varepsilon''<\varepsilon$. As mentioned before, $K(x,y)$, the kernel of T, can be written as

$$
\begin{aligned}
K(x,y)=\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\quad & T\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}D_{k'}(\cdot,y_{\tau'}^{k'+i}),\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}D_k(y_{\tau}^{k+i},\cdot)\big\rangle\\
& \cdot\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}\widetilde{D}_k(x,y_{\tau}^{k+i})\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}\widetilde{\widetilde{D}}_{k'}(y_{\tau'}^{k'+i},y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& |K(x,y)|\\
\cdot\ & C\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\big|\ T\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}D_{k'}(\cdot,y_{\tau'}^{k'+i}),\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}D_k(y_{\tau}^{k+i},\cdot)\big\rangle\big|\\
&\quad\cdot|\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}\widetilde{D}_k(x,y_{\tau}^{k+i})\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}\widetilde{\widetilde{D}}_{k'}(y_{\tau'}^{k'+i},y)|
\end{aligned}
$$

$$\cdot\ C\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\mu(Q_{\tau'}^{k'+i})\mu(Q_{\tau}^{k+i})2^{-|k-k'|\varepsilon'}$$
$$\cdot\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(y_{\tau}^{k+i},y_{\tau'}^{k'+i}))^{1+\varepsilon}}|\widetilde{D}_k(x,y_{\tau}^{k+i})|\,|\widetilde{\widetilde{D}}_{k'}(y_{\tau'}^{k'+i},y)|$$
$$\cdot\ C\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\int_{Q_{\tau'}^{k'+i}}\int_{Q_{\tau}^{k+i}}2^{-|k-k'|\varepsilon'}\cdot\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,v))^{1+\varepsilon}}$$
$$\cdot\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}\frac{2^{-k'\varepsilon'}}{(2^{-k'})+\rho(v,y))^{1+\varepsilon'}}d\mu(u)d\mu(v)$$
$$\cdot\ C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}\int_X\int_X\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,v))^{1+\varepsilon}}$$
$$\cdot\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}\frac{2^{-k'\varepsilon'}}{(2^{-k'})+\rho(v,y))^{1+\varepsilon'}}d\mu(u)d\mu(v)$$
$$\cdot\ C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(x,y))^{1+\varepsilon}}$$
$$\cdot\ C\rho(x,y)^{-1}.$$

To show that $K(x,y)$ satisfies the estimate of (1.10), for any fixed $k\in\mathbb{Z}$, set

$$\mathcal{T}_1=\Big\{\tau:\rho(x,x')\cdot\ \frac{2^{-k}+\rho(x,y_{\tau}^{k+i})}{4A}\Big\}$$

and

$$\mathcal{T}_2=\Big\{\tau:\frac{2^{-k}+\rho(x,y_{\tau}^{k+i})}{4A}<\rho(x,x')\Big\}.$$

It is easy to check that if $u\in Q_{\tau}^{k+i}$ with $\tau\in\mathcal{T}_1$, then, for $\varepsilon''<\varepsilon'<\varepsilon$,

$$|\widetilde{D}_k(x,y_{\tau}^{k+i})-\widetilde{D}_k(x',y_{\tau}^{k+i})|\cdot\ C\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}.$$

If $u\in Q_{\tau}^{k+i}$ with $\tau\in\mathcal{T}_2$, then $\frac{1}{4A}(2^{-k}+\rho(x,y_{\tau}^{k+i})<\rho(x,x')$ and hence

$$|\widetilde{D}_k(x,y_{\tau}^{k+i})-\widetilde{D}_k(x',y_{\tau}^{k+i})|\cdot\ |\widetilde{D}_k(x,y_{\tau}^{k+i})|+|\widetilde{D}_k(x',y_{\tau}^{k+i})|$$
$$\cdot\ C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\right)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}$$
$$+C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x',u)}\right)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x',u))^{1+\varepsilon'}}.$$

We now write

$$
\begin{aligned}
&|K(x,y)-K(x',y)| \\
&\cdot\ C\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\Big|\ T\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}D_{k'}(\cdot,y_{\tau'}^{k'+i}),\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}D_k(y_{\tau}^{k+i},\cdot)\rangle\Big| \\
&\qquad\cdot\Big|\mu(Q_{\tau}^{k+i})^{\frac{1}{2}}[\widetilde{D}_k(x,y_{\tau}^{k+i})-\widetilde{D}_k(x',y_{\tau}^{k+i})]\mu(Q_{\tau'}^{k'+i})^{\frac{1}{2}}\widetilde{\widetilde{D}}_{k'}(y_{\tau'}^{k'+i},y)\Big| \\
&=C\bigg(\sum_{k\in\mathbb{Z}}\sum_{\tau\in\mathcal{T}_1}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}+\sum_{k\in\mathbb{Z}}\sum_{\tau\in\mathcal{T}_2}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}\bigg) \\
&:=C(J_1+J_2).
\end{aligned}
$$

For $\tau\in\mathcal{T}_1$ and $\varepsilon''<\varepsilon'<\varepsilon$,

$$
\begin{aligned}
J_1\cdot\ & C\sum_{k\in\mathbb{Z}}\sum_{\tau\in\mathcal{T}_1}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}}2^{-|k-k'|\varepsilon'}\int_{Q_{\tau'}^{k'+i}}\int_{Q_{\tau}^{k+i}}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,v))^{1+\varepsilon}} \\
&\qquad\cdot\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(v,y))^{1+\varepsilon'}}dudv \\
\cdot\ & C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}\int_X\int_X\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,v))^{1+\varepsilon}} \\
&\qquad\cdot\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(v,y))^{1+\varepsilon'}}dudv \\
\cdot\ & C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}\int_X\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,y))^{1+\varepsilon}} \\
&\qquad\cdot\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}du.
\end{aligned}
$$

For $\frac{1}{2A}\rho(x,y)\cdot\ \rho(x,u)$ and $\rho(x,u)<\frac{1}{2A}\rho(x,y)$, the last integral is dominated by

$$
C\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,y))^{1+\varepsilon'}}+C2^{k\varepsilon''}\rho(x,x')^{\varepsilon''}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(x,y))^{1+\varepsilon}},
$$

and hence

$$
\begin{aligned}
J_1\cdot\ & C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,y))^{1+\varepsilon'}} \\
&+C\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}}2^{-|k-k'|\varepsilon'}2^{k\varepsilon''}\rho(x,x')^{\varepsilon''}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(x,y))^{1+\varepsilon}} \\
\cdot\ & C\rho(x,x')^{\varepsilon''}\rho(x,y)^{-(1+\varepsilon'')}.
\end{aligned}
$$

To deal with J_2, consider first $\rho(x,x') < \dfrac{1}{4A^2}\rho(x,y)$. Using the estimate for $u \in Q_\tau^{k+i}$ with $\tau \in \mathcal{T}_2$, we have

$$J_2 \cdot \; C\sum_{k\in\mathbb{Z}}\sum_{\tau\in\mathcal{T}_2}\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'+i}} 2^{-|k-k'|\varepsilon'}\int_{Q_{\tau'}^{k'+i}}\int_{Q_\tau^{k+i}}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\rho(u,v))^{1+\varepsilon}}$$
$$\cdot\Big\{\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x,u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x,u))^{1+\varepsilon'}}$$
$$+\Big(\frac{\rho(x,x')}{2^{-k}+\rho(x',u)}\Big)^{\varepsilon''}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(x',u))^{1+\varepsilon'}}\Big\}\frac{2^{-k\varepsilon'}}{(2^{-k}+\rho(v,y))^{1+\varepsilon'}}dudv,$$

which implies

$$J_2 \cdot \;\; C\rho(x,x')^{\varepsilon''}\rho(x,y)^{-(1+\varepsilon'')} \quad \text{for } \rho(x,x') < \frac{1}{4A^2}\rho(x,y).$$

This inequality together with the estimate on J_1 yields

$$|K(x,y)-K(x',y)| \cdot \;\; C\rho(x,x')^{\varepsilon''}\rho(x,y)^{-(1+\varepsilon'')} \quad \text{for } \rho(x,x') < \frac{1}{4A^2}\rho(x,y),$$

which together with the estimate of (1.9) on $K(x,y)$ shows that $K(x,y)$ satisfies the estimate (1.10).

The proof of the estimate for $|K(x,y)-K(x,y')|$ is the same.

To see $T^*(1)=0$, it suffices to show that T is bounded from H^1 to H^1. In order to do so, we need

Theorem 2.3. ([14]). *For* $\frac{1}{1+\varepsilon} < p \cdot \; 1$,

$$\|f\|_{H^p} \approx \Bigg\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{\tau\in I_{k+i}}\Big(|D_k(f)(y_\tau^{k+i})|\chi_{Q_\tau^{k+i}}\Big)^2\Big\}^{1/2}\Bigg\|_p,$$

where D_k is the same as the one in the discrete Calderón reproducing formula.

To show the H^1 boundedness of T, applying Theorem 2.3 and the representation of T, we write

$$D_j(Tf)(y_{\tau''}^{j+i}) = \sum_k\sum_\tau\sum_{k'}\sum_{\tau'} T\mu(Q_{\tau'}^{k'+i})^{\frac12}D_{k'}(\,\cdot, y_{\tau'}^{k'+i}), \mu(Q_\tau^{k+i})^{\frac12}D_k(y_\tau^{k+i},\cdot)\rangle$$
$$\cdot\,\mu(Q_\tau^{k+i})^{\frac12}\mu(Q_{\tau'}^{k'+i})^{\frac12}D_j\widetilde{D}_k(y_{\tau''}^{j+i},y_\tau^{k+i})\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i}).$$

Using the estimate

$$|D_j\widetilde{D}_k(y_{\tau''}^{j+i},y_\tau^{k+i})| \cdot \;\; C2^{-|j-k|\varepsilon''}\frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)}+\rho(y_{\tau''}^{j+i},y_\tau^{k+i}))^{1+\varepsilon'}}$$

and estimate (2.1) again, we obtain

$$
\begin{aligned}
&D_j(Tf)(y_{\tau''}^{j+i})\\
&\cdot\ C\sum_k\sum_\tau\sum_{k'}\sum_{\tau'}\omega_{\varepsilon'}(\lambda_1,\lambda_2)\mu(Q_\tau^{k'+i})^{1/2}\mu(Q_{\tau''}^{j+i})^{-1/2}\ \omega_{\varepsilon'}(\lambda_1,\lambda_3)|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\\
&\cdot\ C\sum_{k'}\sum_{\tau'}\omega_{\varepsilon'}(\lambda_2,\lambda_3)\mu(Q_\tau^{k'+i})^{1/2}\mu(Q_{\tau''}^{j+i})^{-1/2}|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|.
\end{aligned}
$$

Here we use notation $\lambda_1=(k,y_\tau^{k+i})$, $\lambda_2=(k',y_{\tau'}^{k'+i})$, and $\lambda_3=(j,y_{\tau''}^{j+i})$. Thus,

$$
\begin{aligned}
&|D_j(Tf)(y_{\tau''}^{j+i})|\chi_{Q_{\tau''}^{j+i}}(x)\\
&\quad\cdot\ C\sum_{k'}\sum_{\tau'}\mu(Q_{\tau'}^{k'+i})2^{-|k'-j|\varepsilon''}\frac{2^{-(k'\wedge j)\varepsilon'}}{(2^{-(k'\wedge j)}+\rho(y_{\tau'}^{k'+i},y_{\tau''}^{j+i}))^{1+\varepsilon'}}\\
&\qquad\qquad\cdot|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau''}^{j+i}}(x)\\
&\quad\cdot\ C\sum_{k'}\sum_{\tau'}\mu(Q_{\tau'}^{k'+i})2^{-|k'-j|\varepsilon''}\frac{2^{-(k'\wedge j)\varepsilon'}}{(2^{-(k'\wedge j)}+\rho(y_{\tau'}^{k'+i},x))^{1+\varepsilon'}}\\
&\qquad\qquad\cdot|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau''}^{j+i}}(x).
\end{aligned}
$$

By an estimate in [12],

$$
\begin{aligned}
&\sum_{\tau'}\frac{2^{-(k'\wedge j)\varepsilon'}}{(2^{-(k'\wedge j)}+\rho(y_{\tau'}^{k'+i},x))^{1+\varepsilon'}}|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\\
&\quad\cdot\ C2^{k'\wedge j}2^{(k'-(k'\wedge j))/r}\Big\{M\Big(\sum_{\tau'}|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{1/r}(x),
\end{aligned}
$$

where $1/(1+\varepsilon')<r<1$, we obtain

$$
\begin{aligned}
|D_j(Tf)(y_{\tau''}^{j+i})|\chi_{Q_{\tau''}^{j+i}}(x)\cdot\ &C\sum_{k'}2^{-k'}2^{-|k'-j|\varepsilon''}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}\\
&\cdot\Big\{M\Big(\sum_{\tau'}|\widetilde{\widetilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{1/r}(x)\chi_{Q_{\tau''}^{j+i}}(x).
\end{aligned}
$$

Since $1/(1+\varepsilon'')<1/(1+\varepsilon')<r$ implies

$$
\begin{cases}
\sup_j\sum_{k'}2^{-k'}2^{-|k'-j|\varepsilon''}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}<\infty\\
\sup_{k'}\sum_j 2^{-k'}2^{-|k'-j|\varepsilon'}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}<\infty,
\end{cases}
$$

$$
\begin{aligned}
&\Big\{\sum_j\sum_{\tau''}|D_j(Tf)(y_{\tau''}^{j+i})|^2\chi_{Q_{\tau''}^{j+i}}(x)\Big\}^{1/2}\\
&\quad\cdot\ C\Big\{\sum_j\sum_{\tau''}\Big[\sum_{k'}2^{-k'}2^{-|k'-j|\varepsilon'}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}\\
&\quad\cdot\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{1/r}\chi_{Q_{\tau''}^{j+i}}(x)\Big]^2\Big\}^{1/2}\\
&\quad\cdot\ C\Big\{\sum_j\sum_{\tau''}\sum_{k'}2^{-k'}2^{-|k'-j|\varepsilon'}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}\\
&\quad\cdot\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{2/r}\chi_{Q_{\tau''}^{j+i}}(x)\Big\}^{1/2}\\
&\quad\cdot\ C\Big\{\sum_{k'}\sum_j 2^{-k'}2^{-|k'-j|\varepsilon'}2^{(k'\wedge j)}2^{(k'-(k'\wedge j))/r}\\
&\quad\cdot\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{2/r}\Big\}^{1/2}\\
&\quad\cdot\ C\Big\{\sum_{k'}\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{2/r}\Big\}^{1/2}.
\end{aligned}
$$

This shows, by $1/r > 1$ and Fefferman-Stein's vector valued maximal inequality [11],

$$
\begin{aligned}
&\Big\|\Big\{\sum_j\sum_{\tau''}|D_j(Tf)(y_{\tau''}^{j+i})|^2\chi_{Q_{\tau''}^{j+i}}(x)\Big\}^{1/2}\Big\|_1\\
&\quad\cdot\ C\Big\|\Big\{\sum_{k'}\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|^2\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{2/r}\Big\}^{1/2}\Big\|_1\\
&\quad=C\Big\|\Big\{\sum_{k'}\Big\{M\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^r\Big\}^{2/r}\Big\}^{r/2}\Big\|_{1/r}^{1/r}\\
&\quad\cdot\ C\Big\|\Big\{\sum_{k'}\Big(\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|\chi_{Q_{\tau'}^{k'+i}}\Big)^2\Big\}^{r/2}\Big\|_{1/r}^{1/r}\\
&=C\Big\|\Big\{\sum_{k'}\sum_{\tau'}|\tilde{\tilde{D}}_{k'}(f)(y_{\tau'}^{k'+i})|^2\chi_{Q_{\tau'}^{k'+i}}\Big\}^{1/2}\Big\|_1\\
&\cdot\ C\|f\|_{H^1},
\end{aligned}
$$

where the last inequality follows from a result in [14].

The proof of $T(1) = 0$ is similar and we leave details to the reader.

Remark. The above proof can be applied to the H^p boundedness of T for $\frac{1}{1+\varepsilon} < p < 1$. It seems that the method used here is new even though for the case of $\mathbb{R}^n$.

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