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λ PROPERTY FOR BOCHNER-ORLICZ SEQUENCE SPACES WITH ORLICZ NORM

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Abstract. We give the sufficient and necessary conditions of Bochner-Orlicz sequence spaces equipped with Orlicz norm that have the λ property and uniform λ property, respectively. The results show that the λ property can not be lifted from X to $l_M(X)$.

1. Introduction

Let X be a Banach space and let $S(X)$ and $B(X)$ be the unit sphere and the unit ball of X, and let $ExtB(X)$ be the set of all extreme points of $B(X)$. For $x \in B(X)$, we associate the number $\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e+(1-\lambda)y, y \in B(X), e \in ExtB(X)\}$. We call x a λ point if $\lambda(x) > 0$; we say that X has λ property if $\lambda(x) > 0$ for all $x \in B(X)$; we say that X has uniform λ property if $\lambda(X) > 0$ where $\lambda(X) = \inf\{\lambda(x) : x \in B(X)\}$. Banach spaces with these types of λ property were studied in [1], and ones of Orlicz spaces were studied in [2-4]. For Bochner-Orlicz sequence space with Luxemburg norm, the criterion of uniform λ property was given in [10]. But until now, it has not seen for the λ property and uniform λ property of Bochner-Orlicz sequence space with Orlicz norm. In this paper, we investigate them and give their criteria. The results says that it is not as usual as X need only have the corresponding one, which shows that λ property can not be lifted from X to $l_M(X)$.

In the sequel, let $\Re$ be the set of all real numbers. A function $M : \Re \to \Re_+$ is called a N-function if M is convex and even, $M(u) > 0$ as $u > 0$, $\lim_{u\to 0} \frac{M(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. And its complementary function is defined in Young's sense

$$N(v) = max\{u|v| - M(u) : u \in \Re\}$$

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which is also a N-function [8, 5]. We use p and q stand for the right derivatives of M and N, respectively. An interval $[a, b]$ is called a structural affine interval (SAI)of M provided that M is affine on $[a, b]$ and it is not affine on $[a-\epsilon, b]$ or $[a, b+\epsilon]$, for all $\epsilon > 0$. Let $\{[a_{i'}, b_{i'}]\}_{i'=1}^{\infty}$ be all SAI of M and denote $S_M = R \setminus [\bigcup\limits_{i'=1}^{\infty}(a_{i'}, b_{i'})]$.

For $x = (x(1), x(2), ...)$, $x(i) \in X$, its modular is defined by $\rho_M(x) = \sum_{i=1}^{\infty} M(\|x(i)\|)$. The Orlicz sequence space $l_M(X)$ is generated as follows

$$l_M(X) = \{x : \exists\lambda > 0, \rho_M(\lambda x) < \infty\},$$

endowed with the Orlicz norm

$$\|x\|_M = \inf_{k>0} \frac{1}{k}\{1 + \rho_M(kx)\},$$

$l_M(X)$ is a Banach space if X is a Banach space and we assume that $ExtB(X) \neq \emptyset$ in this paper.

2. Main Results

Lemma 1. ([10]) *There exist $e \in ExtB(X)$ such that for all $\lambda < \lambda(x)$ we have $y \in B(X)$ such that $x = \lambda e + (1-\lambda)y$.*

Proof. Please refer the proof of [10]. ∎

Lemma 2. ([5]) *Let $ExtB(X) \neq \emptyset$. If $x = \alpha y + (1-\alpha)z$ for $y, z \in B(X), \alpha \in (0, 1)$, then $\lambda(x) \geq \alpha\lambda(y)$. Consequently, $\lambda(\theta) = \frac{1}{2}$ and $\lambda(x) \geq \max\{\frac{1-\|x\|}{2}, \lambda(\frac{x}{\|x\|})\|x\|\}$.*

Lemma 3. ([10]) *Either X or l_M is isometric to a subspace of $l_M(X)$.*

Lemma 4. ([6]) *In $l_M(X)$, for $x = (x(1), x(2)...)$, we have*

$$\begin{aligned}
\|x\|_M &= \frac{1}{k}\{1 + \rho_M(kx)\}, \forall k \in k(x) = [k^\star, k^{\star\star}] \\
&= \sup\{\sum_{i=1}^{\infty} x(i)y(i) : \rho_N(y) \leq 1, y(i) \in X^\star\} \\
&= \sup\{\sum_{i=1}^{\infty} \|x(i)\|\|y(i)\| : \rho_N(y) \leq 1, y(i) \in X^\star\}
\end{aligned}$$

where $k^\star = \inf\{k > 0 : \rho_N(p(kx)) \geq 1\}$, $k^{\star\star} = \sup\{k > 0 : \rho_N(p(kx)) \leq 1\}$.

Proof. Please refer [4]. ■

Lemma 5. ([6]) *For* $x \in l_M, y \in l_N$, *where* $x = (x(1), x(2)...), x(i) \in \Re$ *and* $y = (y(1), y(2)...)$, $y(i) \in \Re$, *we have that* $\|x\|_M = \sum_{i=1}^{\infty} x(i)y(i)$ *if and only if for all* i, $p_-(|x(i)|) \le |y(i)| \le p(|x(i)|)$, $x(i)y(i) \ge 0$, *where* p_- *is the left hand derivative of* M.

Lemma 6. ([9]) *In* $l_M(X)$, *for* $x = (x(1), x(2)...), \|x\|_M = 1$. $x \in ExtB(l_M(X))$ *if and only if* (1) *(i)* $\mu\{i : \|x(i)\|_X \neq 0\} \le 1$ *or (ii)* $k\|x(i)\|_X \in S_M, \forall\, i \in N$ *and* $\forall\, k \in K(x)$, (2) $\frac{x(i)}{\|x(i)\|_X} \in ExtB(X) \quad (\forall\ \|x(i)\|_X \neq 0)$.

Lemma 7. $l_M(X)$ *has* λ *property if* X *has uniform* λ *property.*

Proof. For $x \in S\Big(l_M(X)\Big)$. If $x \in ExtB\Big(l_M(X)\Big)$, then $\lambda(x) = 1$. For $x \in B\Big(l_M(X)\Big) \backslash ExtB\Big(l_M(X)\Big)$, and $k \in k(x)$, set

$$I(i') = \{i : a_{i'} < k\|x(i)\| \le \frac{a_{i'} + b_{i'}}{2}\} \quad J(i') = \{i : \frac{a_{i'} + b_{i'}}{2}\} < k\|x(i)\| < b_{i'}\}.$$

Split the set of all positive integers into parts:

$$I(x,k) = \bigcup_{i'=1}^{\infty} I(i'), \qquad J(x,k) = \bigcup_{i'=1}^{\infty} J(i'), \qquad \tilde{S}(x,k) = \{i : k\|x(i)\| \in S_M\},$$

we discuss in several steps.

A. $\mu\{i : \frac{x(i)}{\|x(i)\|} \notin ExtB(X)\} = 0$.

A-1. $\mu\{i : k\|x(i)\| \in S_M\} > 0$, for some $k \in k(x) = [k^\star, k^{\star\star}]$.
Define

$$ky(i) = \begin{cases} a_{i'} \dfrac{x(i)}{\|x(i)\|} & i \in I \\ b_{i'} \dfrac{x(i)}{\|x(i)\|} & i \in J \\ kx(i) & i \in \tilde{S}_M \end{cases} \tag{1}$$

then $k(y) = \{k\}$. In fact for all $\varepsilon > 0$, we have

$$\begin{aligned} \rho_N(p((1+\varepsilon)ky)) \ = \ & \sum_{i \in I} N(p((1+\varepsilon)a_{i'})) + \sum_{i \in J} N(p((1+\varepsilon)b_{i'})) \\ & + \sum_{i \in \tilde{S}} N(p((1+\varepsilon)k\|x(i)\|)) \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i\in I} N(p(k\|x(i)\|)) + \sum_{i\in J} N(p(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p((1+\varepsilon)k\|x(i)\|)) \\
&= \sum_{i\in I} N(p(k\|x(i)\|)) + \sum_{i\in J} N(p(k\|x(i)\|)) + \sum_{i\in\tilde{S}} N(p(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p((1+\varepsilon)k\|x(i)\|)) - \sum_{i\in\tilde{S}} N(p(k\|x(i)\|)) \\
&> \sum_{i\in I} N(p(k\|x(i)\|)) + \sum_{i\in J} N(p(k\|x(i)\|)) + \sum_{i\in\tilde{S}} N(p(k\|x(i)\|)) \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_N(p((1-\varepsilon)ky)) &= \sum_{i\in I} N(p((1-\varepsilon)a_{i'})) + \sum_{i\in J} N(p((1-\varepsilon)b_{i'})) \\
&\quad + \sum_{i\in\tilde{S}} N(p((1-\varepsilon)k\|x(i)\|)) \\
&\leq \sum_{i\in I} N(p_{\_}(k\|x(i)\|)) + \sum_{i\in J} N(p_{\_}(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p((1-\varepsilon)k\|x(i)\|)) \\
&= \sum_{i\in I} N(p_{\_}(k\|x(i)\|)) + \sum_{i\in J} N(p_{\_}(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p_{\_}(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p((1-\varepsilon)k\|x(i)\|)) - \sum_{i\in\tilde{S}} N(p_{\_}(k\|x(i)\|)) \\
&< \sum_{i\in I} N(p_{\_}(k\|x(i)\|)) + \sum_{i\in J} N(p_{\_}(k\|x(i)\|)) \\
&\quad + \sum_{i\in\tilde{S}} N(p_{\_}(k\|x(i)\|)) \\
&= 1.
\end{aligned}
$$

Hence, from $k(\frac{y}{\|y\|}) = \|y\|k(y) = \{k\|y\|\}$ and Lemma 6, we have $\frac{y}{\|y\|} \in ExtB\Big(l_M(X)\Big)$.

Set

$$z = 2x - y.$$

Then, if $i \in \tilde{S}$, we have $y(i) = x(i)$, moreover $z(i) = y(i) = x(i)$.
If if $i \in I$, we have $a_{i'} < k\|x(i)\| \leq \frac{a_{i'}+b_{i'}}{2}$, moreover

$$\begin{aligned} a_{i'} < k\|x(i)\| < k\|z(i)\| &= \|2kx(i) - ky(i)\| \\ &= \|2kx(i) - a_{i'}\frac{x(i)}{\|x(i)\|}\| \\ &= |2k\|x(i)\| - a_{i'}| \\ &\leq 2\frac{a_{i'} + b_{i'}}{2} - a_{i'} \\ &= b_{i'} \end{aligned}$$

If if $i \in J$, we have $k\|y(i)\| = b_{i'}\frac{x(i)}{\|x(i)\|}$, so $\frac{a_{i'}+b_{i'}}{2} < k\|x(i)\| < b_{i'}$. Moreover

$$\begin{aligned} k\|z(i)\| &= \|2kx(i) - ky(i)\| \\ &= \|2kx(i) - b_{i'}\frac{x(i)}{\|x(i)\|}\| \\ &= |2k\|x(i)\| - b_{i'}| \\ &> 2\frac{a_{i'} + b_{i'}}{2} - b_{i'} \\ &= a_{i'}, \end{aligned}$$

and $k\|z(i)\| = 2k\|x(i)\| - b_{i'} < 2b_{i'} - b_{i'} = b_{i'}$. Summarily, we know that $k\|x(i)\|$,$k\|y(i)\|$ and $k\|z(i)\|$ are in the same SAI of M.

On the other hand

$$\begin{aligned} 1 = \|x\|_M &= \frac{1}{k}\{1 + \rho_M(kx)\} \\ &= \frac{1}{k}\{1 + \rho_M(k\frac{y+z}{2})\} \\ &= \frac{1}{k}\{1 + \frac{1}{2}\rho_M(ky) + \frac{1}{2}\rho_M(kz)\} \\ &= \frac{1}{2}[\frac{1}{k}\{1 + \rho_M(ky)\}] + \frac{1}{2}[\frac{1}{k}\{1 + \rho_M(kz)\}] \\ &\geq \frac{1}{2}[\|y\|_M + \|z\|_M], \end{aligned}$$

so $\|y\|_M + \|z\|_M \leq 2$. Since $\|\frac{y+z}{2}\|_M\| = \|x\|_M = 1$, we get $\|y\|_M + \|z\|_M \geq \|y+z\|_M = 2$, thus $\|y\|_M + \|z\|_M = \|y+z\|_M = 2$. Noticing

$$x = \frac{y+z}{2} = \frac{y+z}{\|y\|_M + \|z\|_M} = \frac{\|y\|_M}{\|y\|_M + \|z\|_M}\frac{y}{\|y\|_M} + \frac{\|z\|_M}{\|y\|_M + \|z\|_M}\frac{z}{\|z\|_M}$$

by Lemma 2, we have

$$\lambda(x) \geq \frac{\|y\|_M}{2}\lambda(\frac{y}{\|y\|_M}) \geq \frac{\|y\|_M}{\|y\|_M} = \frac{\|y\|_M}{2} > 0.$$

A-2. $\mu\{i : k\|x(i)\| \in S_M\} = 0$, for all $k \in k(x) = [k^\star, k^{\star\star}]$.
In this case, $\rho_N(p_-(kx)) = \rho_N(p(kx)) = 1$.

A-2-1. $\inf_{a_{i'}<k\|x(i)\|<b_{i'}} \frac{b_{i'}}{a_{i'}} = 1$.

Without loss of generality, assume $\lim_{i'\to\infty} \frac{b_{i'}}{a_{i'}} = 1$. Let $I = \bigcup_{i'=1}^{\infty} I(2i'-1)$, $J = \bigcup_{i'=1}^{\infty} I(2i')$, and define

$$ky(i) = \begin{cases} a_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in I \\ b_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in J \end{cases} \tag{2}$$

Then $k(y) = \{k\}$, so $\frac{y}{\|y\|} \in ExtB\Big(l_M(X)\Big)$. In fact, for all $\varepsilon > 0$, take i_0 satisfying $\frac{b_{i'}}{a_{i'}} \le \frac{2+2\varepsilon}{2+\varepsilon} < \frac{2-\varepsilon}{2-2\varepsilon}$, for $i \ge i_0$, thus we have

$$\begin{aligned}
&\rho_N(p((1+\varepsilon)ky)) \\
=\ & \sum_{i\in I} N(p((1+\varepsilon)a_{i'})) + \sum_{i\in J} N(p((1+\varepsilon)b_{i'})) \\
\ge\ & \sum_{i\in I\setminus I(2i'_0-1)} N(p((1+\varepsilon)a_{i'})) + \sum_{i\in I(2i'_0-1)} N(p((1+\frac{\varepsilon}{2})b_{2i'_0-1})) \\
& +\sum_{i\in J} N(p((1+\varepsilon)b_{i'})) \\
\ge\ & \sum_{i\in I\setminus I(2i'_0-1)} N(p((kx(i))) + \sum_{i\in I(2i'_0-1)} N(p((1+\frac{\varepsilon}{2})b_{2i'_0-1})) \\
& +\sum_{i\in J} N(p((kx(i))) \\
=\ & \sum_{i\in I} N(p((kx(i))) + \sum_{i\in J} N(p((kx(i))) \\
& + \sum_{i\in I(2i'_0-1)} [N(p((1+\frac{\varepsilon}{2})b_{2i'_0-1})) - N(p(kx(2i'_0-1)))] \\
>\ & \sum_{i\in I} N(p((kx(i))) + \sum_{i\in J} N(p((kx(i))) \\
=\ & 1
\end{aligned}$$

and

$$\rho_N(p((1-\varepsilon)ky))$$

$$
\begin{aligned}
&= \sum_{i\in I} N(p((1-\varepsilon)a_{i'})) + \sum_{i\in J} N(p((1-\varepsilon)b_{i'})) \\
&\le \sum_{i\in I} N(p((1-\varepsilon)a_{i'})) + \sum_{i\in J\setminus J(2i'_0)} N(p((1-\varepsilon)b_{i'})) \\
&\quad + \sum_{i\in J(2i'_0)} N(p((1-\frac{\varepsilon}{2})a_{2i'_0})) \\
&\le \sum_{i\in I} N(p_-((kx(i))) + \sum_{i\in J} N(p_-(kx(i))) + \sum_{i\in J(2i'_0)} [N(p((1-\frac{\varepsilon}{2})a_{2i'_0})) \\
&\quad - N(p_-(kx(2i'_0)))] \\
&< \sum_{i\in I} N(p_-((kx(i))) + \sum_{i\in J} N(p_-(kx(i))) \\
&= 1.
\end{aligned}
$$

Set

$$z = 2x - y.$$

Similarly as A-1, we have $\|y\|_M + \|z\|_M = \|y+z\|_M = 2$ and

$$x = \frac{y+z}{2} = \frac{y+z}{\|y\|_M + \|z\|_M} = \frac{\|y\|_M}{\|y\|_M + \|z\|_M}\frac{y}{\|y\|_M} + \frac{\|z\|_M}{\|y\|_M + \|z\|_M}\frac{z}{\|z\|_M}$$

and

$$\lambda(x) \ge \frac{\|y\|_M}{2} > 0.$$

A-2-2. $\inf_{a_{i'} < k\|x(i)\| < b_{i'}} \frac{b_{i'}}{a_{i'}} = \alpha > 1 + \delta$, for some $\delta > 0$.
A-2-2-1. $k^\star = k^{\star\star}$

Claim. $\mu I \ge 1$ and $\mu J \ge 1$.
If suppose $\mu I = 0$. Noticing

$$\frac{a_{i'} + b_{i'}}{2} = \frac{a_{i'}}{2}(1 + \frac{b_{i'}}{a_{i'}}) > \frac{a_{i'}}{2}(1 + 1 + \delta) = (1 + \frac{\delta}{2})a_{i'},$$

so for all i, $k\|x(i)\| > \frac{a_{i'}+b_{i'}}{2} > (1+\frac{\delta}{2})a_{i'}$, thus

$$1 = \rho_N(p(kx)) = \sum_{i=1}^{\infty} N(p(a_{i'})) \le \sum_{i=1}^{\infty} N(p(\frac{1}{1+\frac{\delta}{2}}kx(i))) \le \rho_N(p(kx)) = 1,$$

hence $\frac{1}{1+\frac{\delta}{2}}k \in k(x)$, a contradiction with that $k^\star = k^{\star\star}$.

If suppose $\mu J = 0$. Noticing

$$\frac{a_{i'} + b_{i'}}{2} = \frac{b_{i'}}{2}(1 + \frac{a_{i'}}{b_{i'}}) < \frac{b_{i'}}{2}(1 + \frac{1}{1+\delta}) = b_{i'}\frac{2+\delta}{2(1+\delta)},$$

so for all i, $a_{i'} < k\|x(i)\| \le \frac{a_{i'}+b_{i'}}{2} < b_{i'}\frac{2+\delta}{2(1+\delta)}$, thus $a_{i'} < k\|x(i)\| < \frac{2(1+\delta)}{2+\delta}k\|x(i)\|$ $< b_{i'}$

$$1 = \rho_N(p(kx)) = \sum_{i=1}^{\infty} N(p(a_{i'})) \le \sum_{i=1}^{\infty} N(p(\frac{2(1+\delta)}{2+\delta}kx(i))) \le \rho_N(p(kx)) = 1,$$

hence $\frac{2(1+\delta)}{2+\delta}k \in k(x)$, a contradiction with that $k^\star = k^{\star\star}$.

Define

$$ky(i) = \begin{cases} a_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in I \\ b_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in J \end{cases} \tag{3}$$

Then $k(y) = \{k\}$, so $\frac{y}{\|y\|} \in ExtB\Big(l_M(X)\Big)$. In fact, for all $\varepsilon > 0$, we have

$$\begin{aligned} &\rho_N(p((1+\varepsilon)ky)) \\ &= \sum_{i\in I} N(p((1+\varepsilon)a_{i'})) + \sum_{i\in J} N(p((1+\varepsilon)b_{i'})) \\ &\ge \sum_{i\in I} N(p(kx(i))) + \sum_{i\in J} N(p(kx(i))) + \sum_{i\in J} N(p((1+\varepsilon)b_{i'})) - \sum_{i\in J} N(p(kx(i))) \\ &> \sum_{i\in I} N(p(kx(i))) + \sum_{i\in J} N(p(kx(i))) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} &\rho_N(p((1-\varepsilon)ky)) \\ &= \sum_{i\in I} N(p((1-\varepsilon)a_{i'})) + \sum_{i\in J} N(p((1-\varepsilon)b_{i'})) \\ &\ge \sum_{i\in I} N(p\_(kx(i))) + \sum_{i\in J} N(p\_(kx(i))) + \sum_{i\in I} N(p((1-\varepsilon)a_{i'})) - \sum_{i\in I} N(p\_(kx(i))) \\ &< \sum_{i\in I} N(p\_(kx(i))) + \sum_{i\in J} N(p\_(kx(i))) \\ &= 1 \end{aligned}$$

Set

$$z = 2x - y.$$

Similarly as A-1, we have $\|y\|_M + \|z\|_M = \|y + z\|_M = 2$ and

$$x = \frac{y+z}{2} = \frac{y+z}{\|y\|_M + \|z\|_M} = \frac{\|y\|_M}{\|y\|_M + \|z\|_M}\frac{y}{\|y\|_M} + \frac{\|z\|_M}{\|y\|_M + \|z\|_M}\frac{z}{\|z\|_M}$$

and

$$\lambda(x) \geq \frac{\|y\|_M}{2} > 0.$$

A-2-2-2. $k^\star < k^{\star\star}$

Denote

$$k_\star = \sup\{k \in [k^\star, k^{\star\star}] : \mu I(x,k) = 0\}, \;\; k_{\star\star} = \inf\{k \in [k^\star, k^{\star\star}] : \mu J(x,k) = 0\}$$

then $k_\star \leq k_{\star\star}$. Otherwise $k_\star > k_{\star\star}$, take $k_\star > k > k_{\star\star}$ From $k_\star > k$, we get $\mu I(x,k) = 0$. From $k > k_{\star\star}$, we get $\mu J(x,k) = 0$. By $\{i : x(i) \neq \theta\} = I(x,k) \bigcup J(x,k) = \emptyset$, it follows a contradiction with that $\|x\|_M = 1$.

A-2-2-2-1. $k_\star < k_{\star\star}$.

Take $k_\star < k < k_{\star\star}$. From $k_\star < k$, we see $\mu I \geq 1$. From $k < k_{\star\star}$, we see $\mu J \geq 1$. Define y as in (3), by making that same argument, we can have that $k(y) = \{k\}$, $\frac{y}{\|y\|} \in ExtB\Big(l_M(X)\Big)$ and $\lambda(x) \geq \frac{\|y\|_M}{2} > 0$.

A-2-2-2-2. $k_\star = k_{\star\star}$.

Take $k_\star = k = k_{\star\star}$, then for all $i \in suppx$, $k\|x(i)\| = \frac{a_{i'}+b_{i'}}{2}$. In fact,it is enough to see that for such each i

$$\forall h < k_\star = k, h\|x(i)\| > \frac{a_{i'}+b_{i'}}{2} \quad so \quad k\|x(i)\| \geq \frac{a_{i'}+b_{i'}}{2},$$

$$\forall h > k_{\star\star} = k, h\|x(i)\| \leq \frac{a_{i'}+b_{i'}}{2} \quad so \quad k\|x(i)\| \leq \frac{a_{i'}+b_{i'}}{2}.$$

Noticing $x \notin ExtB\Big(l_M(X)\Big)$ and $\mu\tilde{S} = 0$, by Lemma 6, we get that there exist h and j with $h \neq j$ and $h, j \notin \tilde{S}$. Let $J = \{h\}$ and $I = \{i \in suppx : i \neq h\}$, and define

$$ky(i) = \begin{cases} a_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in I \\ b_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in J \end{cases} \tag{4}$$

then we have that $k(y)=\{k\}$, by Lemma 6, $\frac{y}{\|y\|}\in ExtB\Big(l_{_M}(X)\Big)$. In fact

$$\begin{aligned}
&\rho_{_N}(p((1+\varepsilon)ky))\\
=\ & N(p((1+\varepsilon)b_{h'}))+\sum_{i\neq h}N(p((1+\varepsilon)a_{i'}))\\
\geq\ & N(p((1+\varepsilon)b_{h'}))+\sum_{i\neq h}N(p(kx(i)))\\
=\ & N(p((kx(h)))+\sum_{i\neq h}N(p(kx(i)))+N(p((1+\varepsilon)b_{h'}))-N(p((kx(h)))\\
>\ & N(p((kx(h)))+\sum_{i\neq h}N(p(kx(i)))\\
=\ & 1
\end{aligned}$$

and

$$\begin{aligned}
&\rho_{_N}(p((1-\varepsilon)ky))\\
=\ & N(p((1-\varepsilon)b_{h'}))+\sum_{i\neq h}N(p((1-\varepsilon)a_{i'}))\\
\leq\ & N(p_{_}(kx(h)))+\sum_{i\neq h}N(p_{_}(kx(i)))+\sum_{i\neq h}N(p((1-\varepsilon)a_{i'}))-\sum_{i\neq h}N(p_{_}(kx(i)))\\
<\ & N(p_{_}(kx(h)))+\sum_{i\neq h}N(p_{_}(kx(i)))\\
=\ & 1.
\end{aligned}$$

Set

$$z=2x-y.$$

Then

$$\begin{aligned}
k\|z(h)\| &= \|2kx(h)-ky(h)\|\\
&= \|2kx(h)-b_{h'}\frac{x(h)}{\|x(h)\|}\|\\
&= |2k\|x(h)\|-b_{h'}|\\
&= 2\frac{a_{h'}+b_{h'}}{2}-b_{h'}\\
&= a_{h'}
\end{aligned}$$

for $i\neq h$

$$\begin{aligned} k\|z(i)\| &= \|2kx(i) - ky(i)\| \\ &= \|2kx(i) - a_{i'}\frac{x(i)}{\|x(i)\|}\| \\ &= |2k\|x(i)\| - a_{i'}| \\ &= 2\frac{a_{i'} + b_{i'}}{2} - a_{i'} \\ &= b_{i'} \end{aligned}$$

thus for all i,$k\|x(i)\|$,$k\|y(i)\|$ and $k\|z(i)\|$ are in the same SAI of M. Similarly as A-1, we have $\|y\|_M + \|z\|_M = \|y+z\|_M = 2$ and

$$x = \frac{y+z}{2} = \frac{y+z}{\|y\|_M + \|z\|_M} = \frac{\|y\|_M}{\|y\|_M + \|z\|_M}\frac{y}{\|y\|_M} + \frac{\|z\|_M}{\|y\|_M + \|z\|_M}\frac{z}{\|z\|_M}$$

and

$$\lambda(x) \geq \frac{\|y\|_M}{2} > 0.$$

B $\mu\{i : \frac{x(i)}{\|x(i)\|} \notin ExtB(X)\} \geq 1$.

From $inf\{\lambda(\frac{x(i)}{\|x(i)\|}) : x(i) \neq 0\} \geq \lambda(X) > 0$, for all $0 < \lambda < inf\{\lambda(\frac{x(i)}{\|x(i)\|}) : x(i) \neq 0\}$, by Lemma 1, take $e_i \in ExtB(X), z_i \in B(X)$ with $\frac{x(i)}{\|x(i)\|} = \lambda e_i + (1-\lambda)z_i$ for all $i \in suppx$. Then

$$x(i) = \|x(i)\|\frac{x(i)}{\|x(i)\|} = \|x(i)\|(\lambda e_i + (1-\lambda)z_i) \quad i = 1, 2, \cdots.$$

Set

$$y(i) = \|x(i)\|e_i \qquad z(i) = \|x(i)\|z_i \quad i = 1, 2, \cdots$$

then

$$\|y\|_M = \|x\|_M = 1 \qquad \|z\|_M = \|x\|_M = 1.$$

By A, we get $\lambda(y) > 0$. By Lemma 2, we get $\lambda(x) \geq \lambda(X)\lambda(y) > 0$. ■

Theorem 1. *$l_M(X)$ has λ property if and only if X has uniform λ property.*

Proof. Sufficiency. It follows by Lemma 7. Necessity. Otherwise, suppose that X fails to have the uniform λ property, then there exist $x_i \in S(X)$ so that

$$\lambda(x_i) < \frac{1}{i}, \qquad i = 1, 2, \cdots.$$

Since M is a N-function there exist positive numbers $t_i \in S_M$ so that $t_i \searrow 0$(see [8]). If necessary passing to a subsequence, assume

$$0 < N(p(t_i)) < \frac{1}{2^i}.$$

Since $0 < N(p(t_1)) < \frac{1}{2}$, take positive integer m_1 such that $0 < m_1 N(p(t_1)) \le \frac{1}{2}$. Since $0 < N(p(t_2)) < \frac{1}{4}$, take positive integer m_2 such that $\frac{1}{2} < m_1 N(p(t_1)) + m_2 N(p(t_2)) \le \frac{1}{2} + \frac{1}{4}$. In such a way, since $0 < N(p(t_n)) < \frac{1}{2^n}$, take positive integer m_n such that

$$\sum_{i=1}^{n-1} \frac{1}{2^i} < \sum_{i=1}^{n} m_i N(p(t_i)) \le \sum_{i=1}^{n} \frac{1}{2^i} \qquad n = 1, 2, \cdots$$

hence $\sum_{i=1}^{\infty} m_i N(p(t_i)) = 1$. Define

$$x = (\cdots, \overbrace{t_i x_i, \cdots, t_i x_i}^{m_i}, \cdots),$$

then

$$\rho_N(p(x)) = \sum_{i=1}^{\infty} N(p(\|x(i)\|)) = \sum_{i=1}^{\infty} m_i N(p(t_i \|x_i\|)) = 1.$$

Hence $1 \in k(x)$ and $\|x\|_M \in \|x\|_M k(x) = k(\frac{x}{\|x\|_M})$. Since $l_M(X)$ has λ property we have $\lambda(\frac{x}{\|x\|_M}) > 0$. For $\lambda(\frac{x}{\|x\|_M}) > \lambda > 0$, there exist $y \in ExtB\Big(l_M(X)\Big)$ and $z \in B\Big(l_M(X)\Big)$ satisfying

$$\frac{x}{\|x\|_M} = \lambda y + (1 - \lambda) z.$$

By [8], we have

$$\begin{aligned}
1 &= \|\{\frac{\|x(i)\|}{\|x\|_M}\}\|_M \\
&= \sum_{i=1}^{\infty} \frac{\|x(i)\|}{\|x\|_M} p(\|x(i)\|) \\
&= \sum_{i=1}^{\infty} (\|\lambda y(i) + (1-\lambda) z(i)\|) p(\|x(i)\|) \\
&\le \sum_{i=1}^{\infty} (\lambda \|y(i)\| + (1-\lambda)\|z(i)\|) p(\|x(i)\|) \\
&\le \lambda \sum_{i=1}^{\infty} \|y(i)\| p(\|x(i)\|) + (1-\lambda) \sum_{i=1}^{\infty} \|z(i)\|) p(\|x(i)\|) \\
&= 1
\end{aligned}$$

thus

$$\sum_{i=1}^{\infty} \|y(i)\| p(\|x(i)\|) = 1 = \|y\|_{M} \qquad \sum_{i=1}^{\infty} \|z(i)\|) p(\|x(i)\|) = 1 = \|z\|_{M}.$$

By Lemma 5, we have for all i

$$p_{\text{-}}(h\|y(i)\|) \leq p(\|x(i)\|) \leq p(h\|y(i)\|) \quad \forall h \in k(y)$$

$$p_{\text{-}}(k\|z(i)\|) \leq p(\|x(i)\|) \leq p(k\|z(i)\|) \quad \forall k \in k(z).$$

Since $\|x(i)\| = t_{i'} \in S_{M}$, we have for all i

$$h\|y(i)\| = \|x(i)\| = k\|z(i)\|.$$

Moreover

$$\begin{aligned} 1 = \| \frac{x}{\|x\|_{M}} \|_{M} &= \| \{ \frac{\|x(i)\|}{\|x\|_{M}} \} \|_{M} = \| \frac{h}{\|x\|_{M}} \{ \|y(i)\| \} \|_{M} \\ &= \frac{h}{\|x\|_{M}} \| \{ \|y(i)\| \} \|_{M} = \frac{h}{\|x\|_{M}} \end{aligned}$$

we have $h = \|x\|_{M}$. Similarly $k = \|x\|_{M}$. Thus we have for all i

$$\|y(i)\| = \frac{\|x(i)\|}{\|x\|_{M}} = \|z(i)\|.$$

From

$$\frac{x(i)}{\|x\|_{M}} = \lambda y(i) + (1-\lambda) z(i),$$

we have

$$\frac{t_{i'} x_{i'}}{\|x\|_{M}} = \lambda y(i) + (1-\lambda) z(i),$$

$$\begin{aligned} x_{i'} &= \lambda \frac{\|x\|_{M}}{t_{i'}} y(i) + (1-\lambda) \frac{\|x\|_{M}}{t_{i'}} z(i) \\ &= \lambda \frac{h}{t_{i'}} y(i) + (1-\lambda) \frac{k}{t_{i'}} z(i) \\ &= \lambda \frac{h\|y(i)\|}{t_{i'}} \frac{y(i)}{\|y(i)\|} + (1-\lambda) \frac{k\|z(i)\|}{t_{i'}} \frac{z(i)}{\|z(i)\|} \\ &= \lambda \frac{y(i)}{\|y(i)\|} + (1-\lambda) \frac{z(i)}{\|z(i)\|}. \end{aligned}$$

By Lemma 6 and $y \in ExtB\Big(l_M(X)\Big)$, we have that for all i, $\frac{y(i)}{\|y(i)\|} \in ExtB(X)$, hence

$$\lambda(x_i) \geq \lambda,$$

a contradiction with that $\lambda(x_i) < \frac{1}{i}, \qquad i = 1, 2, \cdots.$ ■

Theorem 2. *$l_M(X)$ has uniform λ property if and only if (i)X has uniform λ property and (ii)* $\sup_{0<b_i\leq 1} \frac{b_i}{a_i} < \infty$.

Proof. Necessity. By Theorem 1, (i) follows. By Lemma 3 and [5], similarly to [10], (ii) follows.

Sufficiency. For $x \in S\Big(l_M(X)\Big)$. If $x \in ExtB\Big(l_M(X)\Big)$, then $\lambda(x) = 1$. For $x \in B\Big(l_M(X)\Big)\backslash ExtB\Big(l_M(X)\Big)$, by the proof of Lemma 7, we have

$$\lambda(x) \geq \frac{\|y\|_M}{2}\lambda(X)$$

where y is defined (in the formula (1) in Lemma 7) by

$$ky(i) = \begin{cases} a_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in I \\ b_{i'}\dfrac{x(i)}{\|x(i)\|} & i \in J \\ kx(i) & i \in \tilde{S}_M \end{cases}$$

Since $a_{i'} < k\|x(i)\| < b_{i'}$, we have

$$N(p(a_{i'})) = N(p_{\_}(k\|x(i)\|)) \leq \rho_N(p_{\_}(kx)) \leq 1$$

so $a_{i'} \leq q(N^{-1}(1)$. Since $p(s) \longrightarrow \infty \ (s \longrightarrow \infty)$, we get $b_{i'} \leq c$ for some $c > 0$. Set $c_M = \sup\{\frac{b_i}{a_i} : 0 < b_i \leq c\}$, then $c_M < \infty$. In fact

$$\begin{aligned} c_M &\leq \sup\{\frac{b_i}{a_i} : 0 < b_i \leq 1\} + \sup\{\frac{b_i}{a_i} : 1 < b_i \leq c\} \\ &\leq \sup\{\frac{b_i}{a_i} : 0 < b_i \leq 1\} + \frac{c}{1} \\ &< \infty. \end{aligned}$$

Hence if $i \in \tilde{S}_M$, $y(i) = x(i)$, so $\|y(i)\| = \|x(i)\|$. If $i \in J$, $ky(i) = b_i\frac{x(i)}{\|x(i)\|}$, so $k\|y(i)\| = b_i > k\|x(i)\|$, moreover $\|y(i)\| > \|x(i)\|$. If $i \in I$, $ky(i) = a_i\frac{x(i)}{\|x(i)\|}$, so $k\|y(i)\| = a_i = \frac{a_i}{b_i}b_i > \frac{a_i}{b_i}k\|x(i)\| \geq \frac{1}{c_M}k\|x(i)\|$, moreover $\|y(i)\| > \frac{1}{c_M}\|x(i)\|$.

Since $\frac{1}{c_M} \le 1$, we have for all i, $\|y(i)\| > \frac{1}{c_M}\|x(i)\|$. By Lemma 4, we have $\|y\|_M \ge \frac{1}{c_M}\|x\|_M = \frac{1}{c_M}$. Hence

$$\lambda(x) \ge \frac{\|y\|_M}{2}\lambda(X) \ge \frac{1}{2c_M}\lambda(X). \qquad \blacksquare$$

References

1. R. M. Aron and R. H. Lohman, A geometric function determined by extreme points of the unit ball of a normed space, *Pacific J. Math.*, **127** (1987), 209-231.
2. A. S. Granero, λ-property in Orlicz Spaces, *Bull. Polish Aca. Sci. Math.*, **37** (1989), 421-431.
3. S. Chen, H. Sun and C. Wu, λ-property of Orlicz spaces, *Bull. Polish Acad. Sci. Math.*, **39** (1991), 63-69.
4. S. Chen and H. Sun, λ-property of Orlicz sequence spaces, *Ann. Polan. Math.*, **59** (1994), 239-249.
5. S. Chen, *Geometry of Orlicz spaces*, Dissertation, Warszawa, Poland, 1996.
6. Y. Liu, Z. Shi and P. Zhang, Rotundity of Orlicz-Bochner spaces with Luxemburg norm, *J. ShangHai Univ.* (*English Edition*), **7(4)** (2003), 322-326.
7. H. Hudzik, Strongly extreme points in Kothe-Bochner spaces, *Rocky Mountain J. Math.*, **23** (1993), 899-909.
8. M. A. Krasnoselskii and Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen, 1961.
9. Z. Shi, Y. Liu and P. Zhang, Rotundity of Orlicz-Bochner sequence spaces with Orlicz norm, *Functional space theory and its applications*, 2004, pp. 232-242. (United Kingdom).
10. R.Shi and P. Zhang, Orlicz-Bochner sequence spaces that have the uniform lamda-property, *Comment. Math.*, (2004), Ser. 1, 215-227, (Poland),

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