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A TAUBERIAN THEOREM FOR UNIFORMLY WEAKLY CONVERGENCE AND ITS APPLICATION TO FOURIER SERIES

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Abstract. In 1995, S. Mercourakis introduced the concept of uniformly weakly convergent sequences and characterized such sequences as those with the property that any of its subsequences is Cesàro-summable. In this paper, we present a Tauberian theorem for such kind of convergence. As a consequence, we prove that the uniformly pointwise convergence and the uniform convergence of a sequence of complex-valued functions coincide under a suitable Tauberian condition. This result affirmatively answers a question raised by S. Mercourakis concerning the Fourier series of a continuous function on the circle group T. In this paper, a result of Banach type is also established for uniformly weakly convergent sequences. Our result generalizes the work of Mercourakis.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $f_n, f \in X$. In the theory of mathematical analysis, pointwise convergence and uniform convergence are two important concepts in the literature (cf. [7] and [8]). They are exhaustively studied in many aspects, e.g., in the metric theory of functions (cf. [9]) and in Fourier series (cf. [1] and [11]). The notion of pointwise convergence was extended to the Banach space theory in the following setting for a long time (cf. [4]). We say that $f_n \to f$ weakly in X if $\lambda(f_n) \to \lambda(f)$ for every $\lambda \in X^*$, where X^* denotes the dual space of X consisting of all continuous linear functionals λ on X. This concept has shown its importance in the study of the classical Banach spaces, e.g., in the study of the Banach-Saks property (cf. [3, Chapter VII]). Corresponding to the uniform

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convergence, Mercourakis [6, Definition 2.1] introduced the concept of uniformly weakly convergence, which is defined as follows. We say that $f_n \longrightarrow f$ uniformly weakly in X if for each $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that

$$\sharp(\{n : |\lambda(f_n) - \lambda(f)| \geq \varepsilon\}) \leq N(\varepsilon) \qquad (\lambda \in X^*; \|\lambda\| \leq 1). \tag{1.1}$$

Here the notation $\sharp$ denotes the cardinality of a set. In [6, Theorem 2.6], Mercourakis characterized uniformly weakly convergent sequences as those obeying the property that any of its subsequences is Cesàro-summable in X. He also proved that $f_n \longrightarrow f$ uniformly weakly in X if and only if

$$\lim_{N\to\infty} \left\{ \sup_{k_1<\cdots<k_N} \left\| \frac{1}{N} \sum_{i=1}^{N} f_{k_i} - f \right\| \right\} = 0. \tag{1.2}$$

These results led Mercourakis to characterize the Banach-Saks and the weak Banach-Saks properties from the viewpoint of uniformly weakly convergence (cf. [6, Theorems 2.9 and 2.10]).

For $f_n \in C(\Omega)$, where Ω is a given compact Hausdorff space, we have the following implications:

$$\begin{aligned} \|\cdot\|_\infty - \text{convergence} &\Longrightarrow \text{uniformly weakly convergence} \\ &\Longrightarrow \text{weak convergence} \\ &\Longrightarrow \text{pointwise convergence.} \end{aligned} \tag{1.3}$$

It is known (see, for example, [3] and [6, p.91]) that the converse implications in (1.3) are false, in general. In [3, p. 66, Theorem 1], Banach proved that $f_n \to f$ weakly in $C(\Omega)$ if and only if $\sup_n \|f_n\|_\infty < \infty$ and $f_n \to f$ pointwise on Ω. This result has been extended by Mercourakis to uniformly weakly convergence (see [6, Proposition 2.2]). He proved that for a given uniformly bounded sequence, $f_n \to f$ uniformly weakly in $C(\Omega)$ if and only if $f_n \to f$ uniformly pointwise on Ω, that is, for each $\epsilon > 0$ there exists a natural number $N(\varepsilon)$ such that

$$\sharp(\{n : |f_n(\gamma) - f(\gamma)| \geq \varepsilon\}) \leq N(\varepsilon) \qquad (\gamma \in \Omega). \tag{1.4}$$

Mercourakis's result only deals with uniformly bounded sequences. We shall prove in Lemma 2.2 that it can be extended to the general case in a form of Banach type. We shall see its application later. As for the implication from weak convergence to uniformly weakly convergence, this part involves the Banach-Saks or the weak Banach-Saks property. We refer the readers to [3, pp. 109-113] and [6, pp. 101-103] for details.

In [6, p.103], Mercourakis asked a question of the implication from uniformly weakly convergence to norm convergence. His question reads as follows. Let

$s_n(f;t)$ denote the nth partial sum of the Fourier series of $f \in C(T)$, where $T = [-\pi, \pi]$. Suppose that $\{s_n(f)\}_{n=0}^{\infty}$ is uniformly bounded and converges uniformly pointwise on T to f. Does then $s_n(f)$ converge uniformly on T to f? This question is still open. The purpose of this paper is to answer this question affirmatively. To do so, we first establish a Tauberian theorem for uniformly weakly convergence (see Theorem 2.1). More precisely, we shall prove that under (1.5), uniformly weakly convergence implies norm convergence:

$$\lim_{n\to\infty} \|f_{n+1} - f_n\| = 0. \tag{1.5}$$

Such a condition is known as a Tauberian condition and the corresponding result is called a Tauberian theorem (see [2] and [5] for the definitions). With the help of Lemma 2.2, we deduce the second form of the aforementioned Tauberian theorem for $X = \ell^\infty(\Gamma)$ or $C(\Omega)$, in which the concept of uniformly pointwise convergence is involved. This result says that the $\|\cdot\|_\infty$ convergence coincides with the uniformly pointwise convergence under condition (1.5) (see Theorem 2.4). For $f_n = s_n(f;t)$, condition (1.5) is automatically satisfied and so Theorem 2.4 answers the question of Mercourakis affirmatively.

2. Main Results

The following result gives a Tauberian theorem for uniformly weakly convergence.

Theorem 2.1. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in a Banach space $(X, \|\cdot\|)$ and $f \in X$. Then $f_n \to f$ in X if and only if $f_n \to f$ uniformly weakly in X and (1.5) is satisfied.*

Proof. The "only if" part follows from the definitions. We prove the converse. Assume that $f_n \to f$ uniformly weakly in X and (1.5) is satisfied. By the uniform boundedness theorem (cf. [10, p.68]), we know that $\{\|f_n\|\}_{n=0}^{\infty}$ is bounded. It follows from [6, Theorem 2.6] that (1.2) is true. We have

$$\begin{aligned} \|f_n - f\| &\le \left\| \frac{1}{N} \sum_{k=n}^{n+N-1} f_k - f \right\| + \frac{1}{N} \sum_{k=n}^{n+N-1} \|f_k - f_n\| \\ &\le \sup_{k_1<\cdots<k_N} \left\| \frac{1}{N} \sum_{i=1}^{N} f_{k_i} - f \right\| + (N-1)\left\{ \sup_{k\ge n} \|f_{k+1} - f_k\| \right\}. \end{aligned}$$

By (1.2) and (1.5), we conclude that $f_n \to f$ in X. This completes the proof. ■

The condition (1.5) can not be removed from Theorem 2.1. For instance, consider $X = c_0(\mathbb{N})$ (or $\ell^2(\mathbb{N})$) and $f_n = e_n$, where $\mathbb{N}$ denotes the set of all nonnegative integers and e_n is the sequence with 1 at the nth position and 0 otherwise. We have that $f_n \to 0$ uniformly weakly in X but $f_n \nrightarrow 0$ in X. Even for the case $X = C[0,1]$, (1.5) is still necessary. A counterexample is given by

$$f_n(x) = \begin{cases} 2(n+1)(n+2)(x - \dfrac{1}{n+2}) & \text{on } (\dfrac{1}{n+2}, \dfrac{2n+3}{2(n+1)(n+2)}), \\ -2(n+1)(n+2)(x - \dfrac{1}{n+1}) & \text{on } [\dfrac{2n+3}{2(n+1)(n+2)}, \dfrac{1}{n+1}), \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we assume that Γ is a nonempty set and Ω is a compact Hausdorff space. In order to get the second form of Theorem 2.1, we need the following generalization of [6, Proposition 2.2]. This is a result of Banach type.

Lemma 2.2. *$f_n \to f$ uniformly weakly in $\ell^\infty(\Gamma)$ (respectively, $C(\Omega)$) if and only if $\sup_n \|f_n\|_\infty < \infty$ and $f_n \to f$ uniformly pointwise on Γ (respectively, Ω).*

Proof. We know that any uniformly weakly convergent sequence is bounded, so we can easily deduce the "only if" part by using the fact that $(1.1) \Longrightarrow (1.4)$. The if part of the case $C(\Omega)$ follows from [6, Proposition 2.2]. As for the case $\ell^\infty(\Gamma)$, by [6, Theorems 1.8 & 2.6], we find that for uniformly bounded sequences, uniformly weakly convergence $\Longleftrightarrow$ (1.2) $\Longleftrightarrow$ uniformly pointwise convergence. This leads us to the conclusion. ∎

The condition $\sup_n \|f_n\|_\infty < \infty$ in Lemma 2.2 is necessary. The following example displays this fact: let $f_n(\gamma) = n+1$ for $\gamma = 1/(n+1)$ and 0 otherwise. Then $f_n \to 0$ uniformly pointwise on $[0,1]$, but $f_n \nrightarrow 0$ uniformly weakly in $\ell^\infty([0,1])$.

For uniformly pointwise convergent sequences, we show below that the condition $\sup_n \|f_n\|_\infty < \infty$ can be derived from (1.5).

Lemma 2.3. *Let $X = \ell^\infty(\Gamma)$ or $C(\Omega)$ and $f_n, f \in X$. If $f_n \to f$ uniformly pointwise and (1.5) holds, then $\sup_n \|f_n\|_\infty < \infty$.*

Proof. We prove the case $X = \ell^\infty(\Gamma)$ and leave $X = C(\Omega)$ to the readers. Without loss of generality, we assume $f = 0$. Since $f_n \to 0$ uniformly pointwise on Γ, there exists a positive integer N such that $\sharp(\{n : |f_n(\gamma)| \geq 1\}) \leq N$ for all $\gamma \in \Gamma$. This implies that for any n and any γ, one of $|f_n(\gamma)|, |f_{n+1}(\gamma)|, \cdots, |f_{n+N}(\gamma)|$ is less than 1, say $|f_m(\gamma)|$, and so

$$|f_n(\gamma)| \leq \sum_{k=n}^{m-1} |f_{k+1}(\gamma) - f_k(\gamma)| + |f_m(\gamma)| \leq N(\sup_{k\geq 0} \|f_{k+1} - f_k\|_\infty) + 1.$$

Taking supremum over n and γ gives $\sup_n \|f_n\|_\infty < \infty$. This completes the proof. ■

Putting Theorem 2.1 and Lemmas 2.2-2.3 together, we get the second form of Theorem 2.1 for $X = \ell^\infty(\Gamma)$ or $C(\Omega)$.

Theorem 2.4. *Let $X = \ell^\infty(\Gamma)$ or $C(\Omega)$ and $f_n, f \in X$. Then $f_n \to f$ in X if and only if $f_n \to f$ uniformly pointwise and* (1.5) *holds.*

For $f \in C(T)$, we know that $\lim\limits_{n\to\infty} \|s_{n+1}(f) - s_n(f)\|_\infty = 0$. This can be proved by using the Riemann-Lebesgue theorem (see [11, Vol. I, p.45]). Hence, the condition (1.5) with $f_n = s_n(f)$ holds. As a consequence of Theorem 2.4, we conclude that $s_n(f) \to f$ uniformly on T if and only if $s_n(f) \to f$ uniformly pointwise on T. This answers the question of Mercourakis affirmatively. Moreover, the condition of uniformly boundedness required there for $s_n(f), n \geq 0$, is not necessary.

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