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RANK-ONE OPERATORS IN REFLEXIVE $\mathcal{A}$-SUBMODULES OF OPERATOR ALGEBRAS

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Abstract. In this paper, we first show that any reflexive $\mathcal{A}$-submodule $\mathcal{U}$ of a unital operator algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ is precisely of the following form:

$$\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq \phi(E) \quad \forall E \in \text{Lat}\mathcal{A}\},$$

where ϕ is an order homomorphism of Lat$\mathcal{A}$ into itself. Furthermore we investigate the density of the rank-one submodule of a reflexive $\mathcal{A}$-submodule in the w^*-topology and in certain pointwise approximation, and obtain several equivalent conditions by means of the order homomorphism ϕ.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ the set of all finite-rank operators in $\mathcal{B}(\mathcal{H})$. Suppose that $\mathcal{A}$ is a unital operator algebra in $\mathcal{B}(\mathcal{H})$ and ϕ is an order homomorphism of Lat$\mathcal{A}$ into itself (i.e., $E \leq F$ implies $\phi(E) \leq \phi(F)$), where Lat$\mathcal{A}$ is the complete lattice of all invariant projections for $\mathcal{A}$. Then the set $\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq \phi(E)$ for all $E \in$ Lat$\mathcal{A}\}$ is clearly a weakly closed two-sided $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$. In this paper, submodules always mean two-sided submodules.

It became apparent that many interesting classes of non-selfadjoint operator algebras arise as just such a module. J.A. Erdos and S.C. Power in [3] proved that any weakly closed $\mathcal{A}$-submodule of $\mathcal{B}(\mathcal{H})$ for a nest algebra $\mathcal{A}$ is of the above form. In [5], Han Deguang proved that this is also true for any reflexive algebra $\mathcal{A}$, which is σ -weakly generated by rank-one operators in itself. Han Deguang [6] also showed that any reflexive $\mathcal{A}$-submodule for a reflexive algebra $\mathcal{A}$ is of the above

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form. One purpose of this paper is to show that the hypothesis that $\mathcal{A}$ is reflexive is not needed, and any reflexive $\mathcal{A}$-submodule for a unital operator algebra $\mathcal{A}$ is precisely of the above form.

In [2] J.A. Erdos showed that if Lat$\mathcal{A}$ is a nest then the set of finite sums of rank-one operators in $\mathcal{A}$ is w^*-dense (s-dense) in $\mathcal{A}$. In [13] W.E. Longstaff asked whether the same conclusion holds for the more general case of completely distributive lattices, and showed that, in the opposite direction, complete distributivity is a necessary condition for this. Subsequently, M.S. Lambrou [9] showed that complete distributivity of the invariant subspace lattices implies a condition somewhat weaker than the strong density. C.Laurie and W.E. Longstaff [11] proved that the answer is affirmative if the additional requirement of commutativity is imposed on the invariant subspace lattice. The main purpose of this paper is to investigate the density of the rank-one submodule of a reflexive $\mathcal{A}$-submodule in the w^*-topology and in the sense of M.S. Lambrou [9]. The results about w^*-density are new even in reflexive algebras.

The terminology and notation of this paper concerning reflexive subspaces and pre-annihilators may be found in [8] and the formal definition and some properties of complete distributivity may be found in [10]. In what follows, we always assume that $\mathcal{A}$ is a unital operator algebra in $\mathcal{B}(\mathcal{H})$. Set

$$\text{HomLat}\mathcal{A} = \{\phi : \phi \text{ is an order homomorphism from Lat}\mathcal{A} \text{ into itself}\}.$$

Given ϕ in HomLat$\mathcal{A}$, there is associated a weakly closed $\mathcal{A}$-submodule given by

$$\mathcal{U}_\phi = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq \phi(E) \text{ for any } E \in \text{Lat}\mathcal{A}\}.$$

To each ϕ in HomLat$\mathcal{A}$ there is naturally associated $\phi_\sim$ in HomLat$\mathcal{A}$ given by

$$\phi_\sim(E) = \vee\{F \in \text{Lat}\mathcal{A} : \phi(F) \not\geq E\}, \quad \forall E \in \text{Lat}\mathcal{A}$$

(with the convention that $\phi_\sim(0) = 0$). The above two definitions are not new, but can be found in J.A. Erdos [4] pages 582 and 592, respectively (where they are called Op and $\phi^\vee$). Observe that HomLat$\mathcal{A}$ has a natural partial ordering given by $\phi \leq \psi$ if and only if $\phi(E) \leq \psi(E)$ for any $E \in \text{Lat}\mathcal{A}$. It follows that $\phi \leq \psi$ implies $\phi_\sim \geq \psi_\sim$.

2. Reflexive $\mathcal{A}$-Submodules

Theorem 1. *Suppose that $\mathcal{A}$ is a unital operator algebra in $\mathcal{B}(\mathcal{H})$ and $\mathcal{U}$ is an $\mathcal{A}$-submodule. Then $\mathcal{U}$ is reflexive if and only if there exists $\phi \in$ HomLat$\mathcal{A}$ such that*

$$\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : (I - \phi(E))TE = 0 \textit{ for any } E \in \textit{Lat}\mathcal{A}\}.$$

Proof. Sufficiency. Let $\mathcal{U}$ be determined by an order homomorphism ϕ from Lat$\mathcal{A}$ into itself. Suppose that $T \in \mathcal{B}(\mathcal{H})$ and $Tx \in [\mathcal{U}x]$ for any $x \in \mathcal{H}$. Thus for any $E \in$ Lat$\mathcal{A}$,

$$TE \subseteq [\mathcal{U}E] = [\phi(E)\mathcal{U}E] = \phi(E)[\mathcal{U}E] \subseteq \phi(E).$$

So $T \in \mathcal{U}$ and it follows from the definition of reflexivity that $\mathcal{U}$ is reflexive.

Necessity. For any $E \in$ Lat$\mathcal{A}$, let $\phi(E) = [\mathcal{U}E]$. Since $\mathcal{U}$ is a two-sided $\mathcal{A}$-submodule, $\phi(E)$ is invariant under $\mathcal{A}$ and $\phi(E) \in$ Lat$\mathcal{A}$. It is plainly seen that ϕ is an order homomorphism. Set

$$\mathcal{U}_\phi = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq \phi(E), \forall E \in \text{Lat}\mathcal{A}\}.$$

It is obvious that $\mathcal{U} \subseteq \mathcal{U}_\phi$. Conversely, let $T \in \mathcal{U}_\phi$. For any $x \in \mathcal{H}$, denote by E the orthogonal projection onto $[\mathcal{A}x]$. Then $E \in$ Lat$\mathcal{A}$, $x \in E$ and

$$Tx \in TE \subseteq \phi(E) = [\mathcal{U}E] = [\mathcal{U}[\mathcal{A}x]] = [\mathcal{U}x].$$

From the reflexivity of $\mathcal{U}$, it follows that $T \in \mathcal{U}$. Accordingly, $\mathcal{U}_\phi \subseteq \mathcal{U}$ and $\mathcal{U} = \mathcal{U}_\phi$. ■

A reflexive $\mathcal{A}$-submodule $\mathcal{U}$ is said to be determined by an order homomorphism ϕ if $\mathcal{U} = \mathcal{U}_\phi$, that is,

$$\mathcal{U} = \{T \in \mathcal{B}(\mathcal{H}) : TE \subseteq \phi(E), \forall E \in \text{Lat}\mathcal{A}\}.$$

If $\mathcal{U}$ is a reflexive $\mathcal{A}$-submodule, it follows from the proof of Theorem 1 that $\mathcal{U} = \mathcal{U}_\tau$, where $\tau(E) = [\mathcal{U}E]$ for any $E \in$ Lat$\mathcal{A}$.

For non-zero vectors $x, y \in \mathcal{H}$, the rank-one operator $x \otimes y$ is defined by the equation

$$(x \otimes y)z = < z, y > x, \quad \forall z \in \mathcal{H}.$$

The following lemma is in J.A. Erdos [4, Lemma 6.2]. We include the brief proof the part we shall need.

Lemma 2. *Suppose that $\mathcal{U}_\phi$ is a reflexive $\mathcal{A}$-submodule determined by $\phi \in$ HomLat$\mathcal{A}$. Then a rank-one operator $x \otimes y$ is in $\mathcal{U}_\phi$ if and only if for some $E \in$ Lat$\mathcal{A}$, $x \in E$ and $y \in \phi_\sim(E)^\perp$.*

Proof. Suppose that there exists $E \in$ Lat$\mathcal{A}$ such that $x \in E$ and $y \in \phi_\sim(E)^\perp$. For any $F \in$ Lat$\mathcal{A}$, if $\phi(F) \geq E$, then

$$(x \otimes y)F = E(x \otimes y)\phi_\sim(E)^\perp F \subseteq E \subseteq \phi(F);$$

if $\phi(F) \not\geq E$, it follows from the definition of $\phi_\sim(E)$ that $F \leq \phi_\sim(E)$. Thus

$$(x \otimes y)F = E(x \otimes y)\phi_\sim(E)^\perp F = (0) \subseteq \phi(F).$$

Accordingly, $x \otimes y \in \mathcal{U}_\phi$.

Conversely, suppose that $x \otimes y \in \mathcal{U}_\phi$. Let

$$E = \wedge\{F \in \text{Lat}\mathcal{A} : Fx = x\}.$$

Clearly, $E \in \text{Lat}\mathcal{A}$ and $x \in E$. For any $F \in \text{Lat}\mathcal{A}$ and $\phi(F) \not\geq E$, it follows from the definition of E that $\phi(F)x \neq x$. Since $x \otimes y \in \mathcal{U}_\phi$, we have

$$(x \otimes y)Fy = \phi(F)(x \otimes y)Fy$$

and

$$\| Fy \|^2 x = \| Fy \|^2 \phi(F)x.$$

So $Fy = 0$. From the definition of $\phi_\sim(E)$, it follows that $\phi_\sim(E)y = 0$ and $y \in \phi_\sim(E)^\perp$. ■

We define an equivalence relation in HomLat$\mathcal{A}$. For $\phi, \psi \in$ HomLat$\mathcal{A}$, $\phi \sim \psi$ if and only if $\mathcal{U}_\phi = \mathcal{U}_\psi$. Thus HomLat$\mathcal{A}/\sim$ consists of all equivalence classes $[\phi]$. From Theorem 1, there exists an one-to-one correspondence between HomLat$\mathcal{A}/\sim$ and reflexive $\mathcal{A}$-submodules.

Proposition 3. *Let $\mathcal{A}$ be a unital algebra in $\mathcal{B}(\mathcal{H})$ and $\phi \in$ HomLat$\mathcal{A}$. Then $\tau \leq \phi$ and $\tau_\sim = \phi_\sim$, where $\tau(E) = [\mathcal{U}_\phi E]$ for any $E \in$ Lat$\mathcal{A}$.*

Proof. It follows from the definition of $\mathcal{U}_\phi$ that

$$\tau(E) = [\mathcal{U}_\phi E] \subseteq \phi(E) \quad \text{for any } E \in \text{Lat}\mathcal{A}.$$

So $\tau \leq \phi$.

Since $\tau \leq \phi$, we have $\tau_\sim \geq \phi_\sim$. So it suffices to show that $\tau_\sim \leq \phi_\sim$. If otherwise, then there exists $E \in \text{Lat}\mathcal{A}$ such that $\tau_\sim(E) \not\leq \phi_\sim(E)$. It follows from the definition of $\tau_\sim$ that there exists $F \in \text{Lat}\mathcal{A}$ such that $\tau(F) \not\geq E$ and $F \not\leq \phi_\sim(E)$. Thus we can choose non-zero vectors x, y such that $x \in E$ and $x \notin \tau(F)$, $y \in \phi_\sim(E)^\perp$ and $y \notin F^\perp$. From Lemma 2, it follows that $x \otimes y \in \mathcal{U}_\phi$. Since $(I - \tau(F))(x \otimes y)F \neq 0$, $x \otimes y \notin \mathcal{U}_\tau$. However it follows from the proof of Theorem 1 that $\mathcal{U}_\tau = \mathcal{U}_\phi$. This is a contradiction. Accordingly, $\tau_\sim \leq \phi_\sim$ and $\tau_\sim = \phi_\sim$. ■

Corollary 4. *If $\phi \sim \psi$ in HomLat$\mathcal{A}$, then $\phi_\sim = \psi_\sim$.*

Lemma 5. *Let $\phi \in$ HomLat$\mathcal{A}$. For any $T \in \mathcal{B}(\mathcal{H})$ and $E \in$ Lat$\mathcal{A}$, $ET(I - \phi_\sim(E)) \in \mathcal{U}_\phi$.*

Proof. For any $F \in \text{Lat}\mathcal{A}$, we consider two separate cases.

(1) $\phi(F) \geq E$. Thus

$$ET(I - \phi_\sim(E))F \subseteq E \subseteq \phi(F);$$

(2) $\phi(F) \ngeq E$. From the definition of $\phi_\sim(E)$, it follows that $F \leq \phi_\sim(E)$. Thus

$$ET(I - \phi_\sim(E))F = (0) \subseteq \phi(F).$$

This shows that the operator $ET(I - \phi_\sim(E))$ is in $\mathcal{U}_\phi$. ■

Let $\mathcal{R}_\phi$ denote the subspace generated by the rank-one operators in $\mathcal{U}_\phi$. Note that $\mathcal{R}_\phi$ is a two-sided $\mathcal{A}$-submodule; we call $\mathcal{R}_\phi$ the rank-one submodule of $\mathcal{U}$.

Theorem 6. *Suppose that $\mathcal{A}$ is a unital algebra in $\mathcal{B}(\mathcal{H})$ and $\mathcal{U}_\phi$ is a reflexive $\mathcal{A}$-submodule determined by $\phi \in$ HomLat$\mathcal{A}$. Then the following statements are equivalent*:

(1) *$\mathcal{R}_\phi$ is w^*-dense in $\mathcal{U}_\phi$*;

(2) $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$;

(3) $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$ *for some $\psi \in$ HomLat$\mathcal{A}$.*

Proof. 1)⇒2) Suppose that $X \in (\mathcal{U}_\phi)_\perp \subseteq \mathcal{C}_1(\mathcal{H})$. For any $T \in \mathcal{B}(\mathcal{H})$ and $E \in$ Lat$\mathcal{A}$, it from follows Lemma 5 that $ET\phi_\sim(E)^\perp \in \mathcal{U}_\phi$. Thus

$$tr(\phi_\sim(E)^\perp XET) = tr(XET\phi_\sim(E)^\perp) = 0 \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

Hence

$$\phi_\sim(E)^\perp XE = 0 \quad \forall E \in \text{Lat}\mathcal{A}$$

and

$$X \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}.$$

Conversely, suppose that $X \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$. For any rank-one operator $x \otimes y \in \mathcal{U}_\phi$, it follows from Lemma 2 that there exists $E \in$ Lat$\mathcal{A}$ such that $x \in E$ and $y \in \phi_\sim(E)^\perp$. So

$$\begin{aligned}\text{tr}(X(x \otimes y)) &= \text{tr}(XE(x \otimes y)\phi_\sim(E)^\perp) \\ &= \text{tr}(\phi_\sim(E)^\perp XE(x \otimes y)) = 0.\end{aligned}$$

Owing to the w^*-continuity of the linear map tr$(X\cdot)$ and hypothesis 1), we have

$$\text{tr}(XA) = 0 \quad \forall A \in \mathcal{U}_\phi,$$

that is, $X \in (\mathcal{U}_\phi)_\perp$. Therefore $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$.

2)⇒1) Since $\mathcal{U}_\phi \supseteq \mathcal{R}_\phi$, $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim} = (\mathcal{U}_\phi)_\perp \subseteq (\mathcal{R}_\phi)_\perp$. We will show $(\mathcal{U}_\phi)_\perp = (\mathcal{R}_\phi)_\perp$, and it suffices to show that $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim} \supseteq (\mathcal{R}_\phi)_\perp$. Suppose that $X \in (\mathcal{R}_\phi)_\perp$. For any $T \in \mathcal{F}(\mathcal{H})$ and $E \in$ Lat$\mathcal{A}$, it follows from Lemma 5 that $ET\phi_\sim(E)^\perp \in \mathcal{R}_\phi$. Thus

$$\text{tr}(\phi_\sim(E)^\perp XET) = \text{tr}(XET\phi_\sim(E)^\perp) = 0 \quad \forall T \in \mathcal{F}(\mathcal{H}).$$

Since $\mathcal{F}(\mathcal{H})$ is w^*-dense in $\mathcal{B}(\mathcal{H})$, it follows from the w^*-continuity of the map tr$(\phi_\sim(E)^\perp XE\cdot)$ that

$$\text{tr}(\phi_\sim(E)^\perp XET) = 0 \qquad \forall T \in \mathcal{B}(\mathcal{H}).$$

Hence

$$\phi_\sim(E)^\perp XE = 0 \qquad \forall E \in \text{Lat}\mathcal{A}$$

and $X \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$. So $(\mathcal{R}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim} = (\mathcal{U}_\phi)_\perp$ and

$$(\mathcal{R}_\phi)^{w^*} = [(\mathcal{R}_\phi)_\perp]^\perp = [(\mathcal{U}_\phi)_\perp]^\perp = \mathcal{U}_\phi.$$

2)⇒3) Obviously.

Before giving the proof 3)⇒2), we summarize a useful result from the proofs of 1)⇒2) and 2)⇒1), that is,

$$(\mathcal{R}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim} \quad \text{for any } \phi \in \text{HomLat}\mathcal{A}.$$

Now we can give the proof of 3)⇒ 2) and this will complete the proof of Theorem 6.

3)⇒2) Since $\mathcal{U}_\phi \supseteq \mathcal{R}_\phi$,

$$(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi \subseteq \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim} = (\mathcal{R}_\phi)_\perp.$$

It suffices to show the converse inclusion. First we prove the following assertion:

$$(\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi)^\perp = \mathcal{U}_{\psi_\sim}.$$

In fact, suppose that $T \in (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi)^\perp$. For any $E \in \text{Lat}\mathcal{A}$ and $x, y \in \mathcal{H}$, it follows from Lemma 5 that $E(x \otimes y)\psi_\sim(E)^\perp \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$. Thus

$$\begin{aligned} 0 &= \text{tr}(TE(x \otimes y)\psi_\sim(E)^\perp) \\ &= \text{tr}(\psi_\sim(E)^\perp TE(x \otimes y)) \\ &= \langle \psi_\sim(E)^\perp TEx, y\rangle, \end{aligned}$$

so $\psi_\sim(E)^\perp TE = 0$ and $T \in \mathcal{U}_{\psi_\sim}$.

Conversely, let $T \in \mathcal{U}_{\psi_\sim}$. For any rank-one operator $x \otimes y$ in $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$, it follows from Lemma 2 that there is $E \in \text{Lat}\mathcal{A}$ such that $x \in E$ and $y \in \psi_\sim(E)^\perp$. Hence

$$\begin{aligned} \text{tr}(Tx \otimes y) &= \text{tr}(TE(x \otimes y)\psi_\sim(E)^\perp) \\ &= \text{tr}(\psi_\sim(E)^\perp TE(x \otimes y)) = 0. \end{aligned}$$

Since $\mathcal{U}_\phi$ is a reflexive subspace, it follows from [8, Theorem 2.1] that $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$ is the $\|\cdot\|_1$-closed linear span of rank-one operators in itself. Accordingly,

$$\text{tr}(TS) = 0, \qquad \forall S \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$$

and $T \in (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi)^\perp$. Thus the assertion is proved.

Since $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$, we have that

$$\mathcal{U}_\phi = [(\mathcal{U}_\phi)_\perp]^\perp = [\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi]^\perp = \mathcal{U}_{\psi_\sim}.$$

So $\phi \sim \psi_\sim$ and it follows from Corollary 4 that $\phi_\sim = (\psi_\sim)_\sim$. Now we will prove that

$$\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi \supseteq \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{(\psi_\sim)_\sim} = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}.$$

Suppose that $T \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{(\psi_\sim)_\sim}$. For any $E \in \mathrm{Lat}\mathcal{A}$, it follows from the definitions that

$$(\psi_\sim)_\sim(E) = \vee\{G \in \mathrm{Lat}\mathcal{A} : \psi_\sim(G) \not\geq E\} \tag{1}$$

and

$$\psi_\sim(G) = \vee\{F \in \mathrm{Lat}\mathcal{A} : \psi(F) \not\geq G\} \tag{2}$$

For $G \in \mathrm{Lat}\mathcal{A}$ and $\psi_\sim(G) \not\geq E$, if $\psi(E) \not\geq G$, it follows from equation (2) that $\psi_\sim(G) \geq E$. This contradiction shows that $\psi(E) \geq G$. Thus equation (1) tells us that $(\psi_\sim)_\sim(E) \leq \psi(E)$. Therefore

$$TE \subseteq (\psi_\sim)_\sim(E) \subseteq \psi(E)$$

and $T \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$. Hence

$$(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$$

and this concludes the proof. ∎

Let $\mathcal{S}$ be a subspace of $\mathcal{B}(\mathcal{H})$. We denote Ref$\mathcal{S}$ by

$$\mathrm{Ref}\mathcal{S} = \{T \in \mathcal{B}(\mathcal{H}) : Tx \in [\mathcal{S}x], \forall x \in \mathcal{H}\}.$$

Corollary 7. *$\mathcal{R}_\phi$ is w^*-dense in $\mathcal{U}_\phi$ if and only if $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap$ Ref$(\mathcal{U}_\phi)_\perp$.*

Proof. Suppose that $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathrm{Ref}(\mathcal{U}_\phi)_\perp$. From the definition of $(\mathcal{U}_\phi)_\perp$, it follows that $(\mathcal{U}_\phi)_\perp$ is a two-sided $\mathcal{A}$-submodule. Thus it is routine to show that Ref$(\mathcal{U}_\phi)_\perp$ is also a two-sided $\mathcal{A}$-submodule. Since Ref$(\mathcal{U}_\phi)_\perp$ is reflexive, it follows from Theorems 1 and 6 that $\mathcal{R}_\phi$ is w^*-dense in $\mathcal{U}_\phi$.

Conversely, let $\mathcal{R}_\phi$ be w^*-dense in $\mathcal{U}_\phi$. From Theorem 6, it follows that $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi$ for some $\psi \in$ HomLat$\mathcal{A}$. Certainly, $(\mathcal{U}_\phi)_\perp \subseteq$ Ref$(\mathcal{U}_\phi)_\perp$ and $(\mathcal{U}_\phi)_\perp \subseteq \mathcal{C}_1(\mathcal{H}) \cap$ Ref$(\mathcal{U}_\phi)_\perp$. Since $(\mathcal{U}_\phi)_\perp \subseteq \mathcal{U}_\psi$ and $\mathcal{U}_\psi$ is reflexive, so Ref$(\mathcal{U}_\phi)_\perp \subseteq \mathcal{U}_\psi$. Thus

$$(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi \subseteq \mathcal{C}_1(\mathcal{H}) \cap \mathrm{Ref}(\mathcal{U}_\phi)_\perp \subseteq \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\psi = (\mathcal{U}_\phi)_\perp,$$

hence $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathrm{Ref}(\mathcal{U}_\phi)_\perp$. ∎

The following corollary gives a necessary condition on the w^*-density of $\mathcal{R}_\phi$ by means of the property of ϕ.

Corollary 8. *Let $\phi \in$ HomLat$\mathcal{A}$. If $\mathcal{R}_\phi$ is w^*-dense in $\mathcal{U}_\phi$, then $\phi_\sim = \phi_{\sim^3}$, where $\phi_{\sim^3} = [(\phi_\sim)_\sim]_\sim$.*

Proof. By virtue of Theorem 6, it follows that $(\mathcal{U}_\phi)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}$. From the assertion in the proof of 3)⇒2) of Theorem 6, we have

$$\mathcal{U}_\phi = [(\mathcal{U}_\phi)_\perp]^\perp = [\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\phi_\sim}]^\perp = \mathcal{U}_{(\phi_\sim)_\sim}.$$

Thus $\phi \sim (\phi_\sim)_\sim$ and it follows from Corollary 4 that $\phi_\sim = \phi_{\sim^3}$. ∎

Corollary 9. *Let $\mathcal{U}$ be a reflexive $\mathcal{A}$-submodule. If the rank-one submodule of $\mathcal{U}$ is w^*-dense in $\mathcal{U}$, then $\tau = (\tau_\sim)_\sim$, where $\tau(E) = [\mathcal{U}E]$ for any $E \in$ Lat$\mathcal{A}$.*

Proof. From the proof of Theorem 1, we have $\mathcal{U} = \mathcal{U}_\tau$. Thus, it follows from Theorem 6 and the assertion in 3)⇒2) that $(\mathcal{U}_\tau)_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$ and

$$\mathcal{U}_\tau = [(\mathcal{U}_\tau)_\perp]^\perp = (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim})^\perp = \mathcal{U}_{(\tau_\sim)_\sim}.$$

By virtue of Proposition 3, it follows that $\tau \leq (\tau_\sim)_\sim$. From the last paragraph of the proof of 3)⇒2) in Theorem 6, we have $(\tau_\sim)_\sim \leq \tau$. Thus $\tau = (\tau_\sim)_\sim$. ∎

Remark 10. Let $\mathcal{U} = \mathcal{A}$ be a unital reflexive algebra. It follows from [9] that Lat$\mathcal{A}$ is completely distributive if and only if

$$E = E_\sharp = \vee\{F \in \mathrm{Lat}\mathcal{A} : F_- \not\geq E\} \quad \forall E \in \mathrm{Lat}\mathcal{A},$$

where $F_- = \vee\{G \in \mathrm{Lat}\mathcal{A} : G \not\geq F\}$.

In this case, $\tau(E) = [\mathcal{A}E] = E$, $\tau_\sim(E) = \vee\{G \in \mathrm{Lat}\mathcal{A} : \tau(G) \not\geq E\} = \vee\{G \in \mathrm{Lat}\mathcal{A} : G \not\geq E\} = E_-$ and $(\tau_\sim)_\sim(E) = \vee\{F \in \mathrm{Lat}\mathcal{A} : \tau_\sim(F) \not\geq E\} = \vee\{F \in \mathrm{Lat}\mathcal{A} : F_- \not\geq E\} = E_\sharp$. Thus Lat$\mathcal{A}$ is completely distributive if and only if $\tau = (\tau_\sim)_\sim$, where $\tau(E) = [\mathcal{A}E] = E$ for any $E \in$ Lat$\mathcal{A}$. So we can consider the conditions $\phi_\sim = \phi_{\sim^3}$ and $\tau = (\tau_\sim)_\sim$ in Corollary 8 and Corollary 9 are the generalizations of completely distributivity.

Property B. Let $\phi \in$ HomLat$\mathcal{A}$. We say that ϕ has Property B if and only if $(\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\phi)^\perp$ is a reflexive subspace.

By virtue of [8, Theorem 2.1], we know that ϕ has Property B if and only if $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_\phi$ is the $\|\cdot\|$1-closed linear span of rank-one operators in itself.

Theorem 11. *Let $\mathcal{U}$ be a reflexive $\mathcal{A}$-submodule. Then the rank-one submodule of $\mathcal{U}$ is w^*-dense in $\mathcal{U}$ if and only if $\tau_\sim$ has Property B and $\tau = (\tau_\sim)_\sim$, where $\tau(E) = [\mathcal{U}E]$ for any $E \in$ Lat$\mathcal{A}$.*

Proof. From the proof Theorem 1, $\mathcal{U} = \mathcal{U}_\tau$. Let $\mathcal{R}_\tau$ denote the rank-one submodule of $\mathcal{U}$.

Necessity. Suppose that $\mathcal{R}_\tau^{w^*} = \mathcal{U}$ Corollary 9 shows that $\tau = (\tau_\sim)_\sim$. It follows from Theorem 6 that $\mathcal{U}_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$. Thus

$$\mathcal{U} = (\mathcal{U}_\perp)^\perp = (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim})^\perp,$$

so $\tau_\sim$ has Property B.

Sufficiency. Set $\mathcal{S} = (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim})^\perp$. Thus $\mathcal{S}$ is a reflexive subspace and

$$\mathcal{S}_\perp = [(\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim})^\perp]_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}.$$

Now we will prove that $\mathcal{S} = \mathcal{U}$.

Let $T \in \mathcal{S}$. For any $x, y \in \mathcal{H}$ and $E \in$ Lat$\mathcal{A}$, it follows from Lemma 2 that $E(x \otimes y)(\tau_\sim)_\sim(E)^\perp \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$. Thus

$$\begin{aligned} 0 &= \mathrm{tr}(TE(x \otimes y)(\tau_\sim)_\sim(E)^\perp) \\ &= \mathrm{tr}((\tau_\sim)_\sim(E)^\perp TE(x \otimes y)) \\ &= \langle (\tau_\sim)_\sim(E)^\perp TEx, y\rangle, \end{aligned}$$

so $(\tau_\sim)_\sim(E)^\perp TE = 0$ and $T \in \mathcal{U}_{(\tau_\sim)_\sim} = \mathcal{U}_\tau = \mathcal{U}$.

Conversely, let $T \in \mathcal{U}$. Consider any rank-one operator $x \otimes y \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$. It follows from Lemma 2 that there exists $E \in$ Lat$\mathcal{A}$ such that $x \in E$ and $y \in (\tau_\sim)_\sim(E)^\perp = \tau(E)^\perp$. Thus

$$\begin{aligned} \mathrm{tr}(T(x \otimes y)) &= \mathrm{tr}(TE(x \otimes y)\tau(E)^\perp) \\ &= \mathrm{tr}(\tau(E)^\perp TE(x \otimes y)) = 0. \end{aligned}$$

So the map tr$(T\cdot)$ annihilates every rank-one operator in $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$. Since $\mathcal{S}$ is reflexive and $\mathcal{S}_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$, it follows from [8, Theorem 2.1] that $\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$ is the $\|\cdot\|_1$-closed linear span of rank-one operators it contains. Accordingly,

$$\mathrm{tr}(TS) = 0, \qquad \forall S \in \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$$

and $T \in (\mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim})^\perp = \mathcal{S}$. Hence $\mathcal{S} = \mathcal{U}$ and $\mathcal{U}_\perp = \mathcal{S}_\perp = \mathcal{C}_1(\mathcal{H}) \cap \mathcal{U}_{\tau_\sim}$, so it follows from Theorem 6 that $\mathcal{R}_\tau$ is w^*-dense in $\mathcal{U}$. ∎

Corollary 12. *Suppose that $\mathcal{L}$ is a subspace lattice. Then the rank-one subalgebra of Alg$\mathcal{L}$ is w^*-dense in Alg$\mathcal{L}$ if and only if $\mathcal{L}$ is completely distributive* *and the homomorphism $E \to E_-$ from $\mathcal{L}$ into $\mathcal{L}$ has Property B, where $E_- = \vee\{F \in \mathcal{L} : F \not\geq E\}$.*

Proof. Sufficiency. [12, Theorem 6.1] shows that a completely distributive subspace lattice is reflexive, that is, $\mathcal{L} = \text{LatAlg}\mathcal{L}$. Thus it follows from Remark 10 and Theorem 11 that the rank-one subalgebra of Alg$\mathcal{L}$ is w^*-dense in Alg$\mathcal{L}$.

Necessity. If the rank-one subalgebra is w^*-dense in Alg$\mathcal{L}$, it follows from [13, Theorem 3.1] that $\mathcal{L}$ is completely distributive. So $\mathcal{L} = \text{LatAlg}\mathcal{L}$, thus from Theorem 11, it follows that the homomorphism $E \mapsto E_-$ from $\mathcal{L}$ into $\mathcal{L}$ has Property B. ∎

Condition A. Let $\mathcal{U}$ be a reflexive $\mathcal{A}$-submodule. If $x \in \mathcal{H}, A \in \mathcal{U}$ and $\epsilon > 0$ are given, there is an F equal to a finite sum of rank-one operators of $\mathcal{U}$ such that $\| Ax - Fx \| < \epsilon$.

The conclusion of Theorem 13 below for reflexive algebras is in [9], and the proof is a slight modification of it.

Theorem 13. *Suppose that $\mathcal{U}$ is a reflexive $\mathcal{A}$-submodule and $\tau(E) = [\mathcal{U}E]$ for any $E \in \text{Lat}\mathcal{A}$. Then $\tau = (\tau_\sim)_\sim$ if and only if Condition A is satisfied.*

Proof. Sufficiency. Clearly, the rank-one submodule is not empty. For any $E \in \text{Lat}\mathcal{A}$ and $e \otimes f \in \mathcal{U}$, we first show

$$(e \otimes f)E \subseteq (\tau_\sim)_\sim(E) = \vee\{F \in \text{Lat}\mathcal{A} : \tau_\sim(F) \not\geq E\}.$$

By virtue of Lemma 2, there is an element $L \in \text{Lat}\mathcal{A}$ such that $e \in L$ and $f \in \tau_\sim(L)^\perp$. If $\tau_\sim(L) \geq E$ then

$$(e \otimes f)E = L(e \otimes f)\tau_\sim(L)^\perp E = (0) \subseteq (\tau_\sim)_\sim(E);$$

if $\tau_\sim(L) \not\geq E$ then $L \leq (\tau_\sim)_\sim(E)$. Thus

$$(e \otimes f)E = L(e \otimes f)\tau_\sim(L)^\perp E \subseteq L \subseteq (\tau_\sim)_\sim(E).$$

So each rank-one operator of $\mathcal{U}$ maps E to $(\tau_\sim)_\sim(E)$ for any $E \in \text{Lat}\mathcal{A}$. For $A \in \mathcal{U}$ and $x \in E(\in \text{Lat}\mathcal{A})$, Condition A shows that $Ax \in (\tau_\sim)_\sim(E)$ and $AE \subseteq (\tau_\sim)_\sim(E)$. Accordingly, $\tau(E) = [\mathcal{U}E] \subseteq (\tau_\sim)_\sim(E)$ and $\tau \leq (\tau_\sim)_\sim$. This together with the relation $(\tau_\sim)_\sim \leq \tau$ (see Theorem 6) implies the equality $\tau = (\tau_\sim)_\sim$.

Necessity. Suppose that $\tau = (\tau_\sim)_\sim$ and suppose that $A \in \mathcal{U}, x \in \mathcal{H}$ and $\epsilon > 0$ are given. It will be shown that there is a finite sum F of rank-one operators of $\mathcal{U}$ such that $\| Ax - Fx \| < \epsilon$.

Define E by

$$E = \wedge\{F \in \text{Lat}\mathcal{A} : x \in F\}.$$

Observe that the intersection is over a non-empty family of subspaces of Lat$\mathcal{A}$ since $x \in \mathcal{H}$. Clearly, $x \in E$ and $E \in$ Lat$\mathcal{A}$. By the hypothesis,

$$\tau(E) = [\mathcal{U}E] = \vee\{G \in \text{Lat}\mathcal{A} : \tau_\sim(G) \not\geq E\}$$

and hence the set of all $G \in$ Lat$\mathcal{A}$ with $\tau_\sim(G) \not\geq E$ has a dense linear span in $[\mathcal{U}E]$. Therefore there is a finite set $G_i(1 \leq i \leq n)$ of subspaces of Lat$\mathcal{A}$ with $\tau_\sim(G_i) \not\geq E$ and a set of vectors $x_i \in G_i(1 \leq i \leq n)$ with the property that

$$\| Ax - (x_1 + \cdots + x_n) \| < \epsilon.$$

The definition of E and the condition $\tau_\sim(G_i) \not\geq E(1 \leq i \leq n)$ imply that $x \notin \tau_\sim(G_i)(1 \leq i \leq n)$ and so there exists $y_i \in \tau_\sim(G_i)^\perp$ with

$$< x, y_i > \neq 0 \qquad \forall 1 \leq i \leq n.$$

By suitably scaling y_i if needed we may assume that $\langle x, y_i \rangle = 1$ and so $(x_i \otimes y_i)x = x_i$ for $1 \leq i \leq n$. Lemma 2 shows that $x_i \otimes y_i$ is a rank-one operator in $\mathcal{U}$. Writing $F = \sum_{i=1}^{n} x_i \otimes y_i$, we get

$$\begin{aligned} \| Ax - Fx \| &= \| Ax - (\sum_{i=1}^{n} x_i \otimes y_i)x \| \\ &= \| Ax - (x_1 + \cdots + x_n) \| < \epsilon, \end{aligned}$$

and this concludes the proof. ■

It is mentioned in Section 1 that complete distributivity is a condition somewhat weaker than strong density of the rank-one submodule. This was indeed thought true for some time, although no example was known. Later, however, such an example was constructed. In [1], there is an example of a complete distributive subspace lattice satisfying Condition A yet the identity cannot be approximated even at two points by operators of the corresponding rank one submodule.

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