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# CONVERGENCE THEOREMS FOR THE $H_1$-INTEGRAL

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**Abstract.** We present two convergence theorems for the $H_1$-integral.

The Henstock integral is now relatively well-known. An attempt has been made by Garces, Lee, and Zhao [2] to define the Henstock integral as the Moore-Smith limit of Riemann sums. The resulting integral is the so-called $H_1$-integral. It has the property that a function $f$ is Henstock integrable on $[a, b]$ if and only if there is an $H_1$-integrable function $g$ such that $f(x) = g(x)$ almost everywhere in $[a, b]$. Every integral has a corresponding convergence theorem. For example, the Denjoy integral has the controlled convergence theorem, whereas the Perron integral has the generalized dominated convergence theorem. Corresponding to the Henstock integral, which is equivalent to both the integrals of Denjoy and Perron, is the equi-integrability theorem with the strong Lusin condition. It is the purpose of the current paper to present two (well-known) convergence theorems that hold for the $H_1$-integral. We assume that the reader is familiar with the definition of the Henstock integral [5].

A division $D$ of $[a, b]$ is a finite set of interval-point pairs $([u, v], \xi)$ such that the intervals $[u, v]$ are non-overlapping, $[a, b] = \cup[u, v]$, and $\xi \in [u, v]$. If $\delta(x) > 0$ for $x \in [a, b]$, then a division $D = \{([u, v], \xi)\}$ is said to be $\delta$-fine if $\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for each $([u, v], \xi) \in D$. A function $f$ is said to be Henstock integrable to a real number $A$ on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function $\delta$ on $[a, b]$ such that for every $\delta$-fine division $D$, we have

$$|(D) \sum f(\xi)(v - u) - A| < \epsilon.$$

Let $\mathcal{D}$ be the family of all $\delta$-fine divisions of $[a, b]$ for some given $\delta(x) > 0$, $x \in [a, b]$. For $D_1, D_2 \in \mathcal{D}$, we write $D_2 \geq D_1$ if for every $([s, t], \eta) \in D_2$ there exists $([u, v], \xi) \in D_1$ such that $[s, t] \subset [u, v]$, and $\{\xi : ([u, v], \xi) \in D_1\} \subset \{\eta :$

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$([s,t],\eta) \in D_2\}$. Then $(\mathcal{D}, \geq)$ is a directed set. A function $f$ is $H_1$-*integrable* to a real number $A$ on $[a,b]$ if $A$ is the Moore-Smith limit [1] of the Riemann sums using the directed set $(\mathcal{D}, \geq)$; that is, there exists a positive function $\delta$ on $[a,b]$ such that for every $\epsilon > 0$ there exists a $\delta$-fine division $D_0$ such that for every $\delta$-fine division $D \geq D_0$ of $[a,b]$, we have

$$|(D)\sum f(\xi)(v-u) - A| < \epsilon.$$

Here, $A$ is the $H_1$-*integral* of $f$ on $[a,b]$. Some examples of $H_1$-integrable functions were considered in [2]. It is easy to see that every $H_1$-integrable function on $[a,b]$ is Henstock integrable there and the two integrals are equal. Note that the Cauchy Criterion and the Saks-Henstock Lemma [5] also hold for the $H_1$-integral. For convenience, we say that $f$ is $H_1$-integrable on a set $X \subset [a,b]$ if $f\mathcal{X}_X$ is $H_1$-integrable on $[a,b]$, where $\mathcal{X}_X$ denotes the characteristic function of $X$ on $[a,b]$.

Let a function $F$ be defined on $[a,b]$ and $X \subset [a,b]$. Then $F$ is said to be $AC^*(X)$ if for every $\epsilon > 0$ there exists $\eta > 0$ such that for any partial division $D = \{([u,v],\xi)\}$ with $u$ or $v \in X$, we have

$$(D)\sum |v-u| < \eta \qquad \text{implies} \qquad (D)\sum |F(u,v)| < \epsilon,$$

where $F(u,v) = F(v) - F(u)$. On the other hand, a sequence $\{F_n\}$ of functions defined on $[a,b]$ is said to be $UAC^*(X)$ if, in the definition of $AC^*(X)$ above, $\eta > 0$ is independent of $n$. Further, $\{F_n\}$ is $UACG^*$ on $[a,b]$ if $[a,b] = \cup X_i$ such that $\{F_n\}$ is $UAC^*(X_i)$ for each $i$. We can assume that $X_i$ is closed for each $i$.

Our proof of the first convergence theorem we want to establish is based on the following three lemmas, in which Lemma 1 is easy.

**Lemma 1.** *Let $\{f_n\}$ be a sequence of $H_1$-integrable functions on $[a,b]$, with $\{F_n\}$ the sequence of primitives of $\{f_n\}$. If $\{f_n\}$ converges uniformly on $[a,b]$, then $\{F_n\}$ is $UACG^*$ on $[a,b]$.*

**Lemma 2.** *Let $X \subset [a,b]$ be closed. If $f$ is $H_1$-integrable on $[a,b]$ and its primitive $F$ is $AC^*(X)$, then $f$ is $H_1$-integrable on $X$.*

*Proof.* Since $f$ is $H_1$-integrable on $[a,b]$, by the Cauchy Criterion, there exists $\delta(x) > 0$ such that given $\epsilon > 0$ there exists a $\delta$-fine division $D_0$ of $[a,b]$ such that for any $\delta$-fine divisions $D, D' \geq D_0$ of $[a,b]$, we have

$$|(D)\sum f(\xi)(v-u) - (D')\sum f(\xi)(v-u)| < \epsilon.$$

Since $F$ is $AC^*(X)$, there exists $\eta > 0$ such that for any partial division $D = \{([u,v],\xi)\}$ of $[a,b]$ with $u$ or $v \in X$,

$$(D)\sum |v-u| < \eta \qquad \text{implies} \qquad (D)\sum |F(u,v)| < \epsilon.$$

Also, there exists a finite union $E$ of closed intervals such that $E \supset X$ and $|E-X| < \eta$. We can assume that a subset of $D_0$ forms a division of $E$, and we can modify $\delta(x) > 0$ such that if $\xi \in E - X$, then $(\xi - \delta(\xi), \xi + \delta(\xi)) \cap X = \emptyset$.

Now, let $D$ be any $\delta$-fine division of $E$. Then there are only two kinds of intervals in $D$: those that do not intersect $X$ and those that do. The latter form a finite cover of $X$, and the union of the former consists of intervals pairwise disjoint, each of which , denoted by $[u,v]$ again, can be expressed as a difference of two intervals, namely, $[w,v]-[w,u)$ or $[u,w]-(v,w]$ with $w \in X$ such that $(D)\sum_{[u,v]\cap X=\emptyset} |w-u| < \eta$ and $(D)\sum_{[u,v]\cap X=\emptyset} |w-v| < \eta$. Thus, $|(D)\sum_{\xi\in E-X} F(u,w)| < \epsilon$ and $|(D)\sum_{\xi\in E-X} F(w,v)| < \epsilon$. Consequently, $|(D)\sum_{\xi\in E-X} F(u,v)| < 2\epsilon$. Meanwhile, by the Saks-Henstock Lemma, for any partial division $D \geq D_0$ of $[a,b]$, we have

$$|(D)\sum_{\xi\in E-X}\{f(\xi)(v-u) - F(u,v)\}| < \epsilon.$$

Hence, $|(D)\sum_{\xi\in E-X}(v-u)| < \eta$ implies

$$\begin{aligned}|(D)\sum_{\xi\in E-X} f(\xi)(v-u)| &< |(D)\sum_{\xi\in E-X}\{f(\xi)(v-u)-F(u,v)\}| \\ &\quad + |(D)\sum_{\xi\in E-X} F(u,v)| \\ &< 3\epsilon.\end{aligned}$$

For any $\delta$-fine divisions $D_1, D_2 \geq D_0$, let $D_1^*$ and $D_2^*$ be the respective subsets of $D_1$ and $D_2$ which form divisions of $E$. Further, let $D_3 = D_1 - D_1^*$. Then $D = D_3 \cup D_1^*$ and $D' = D_3 \cup D_2^*$ are $\delta$-fine divisions of $[a,b]$ with $D, D' \geq D_0$. Therefore,

$$\begin{aligned}&|(D_1)\sum_{\xi\in X} f(\xi)(v-u) - (D_2)\sum_{\xi\in X} f(\xi)(v-u)| \\ \leq\ & |(D)\sum f(\xi)(v-u) - (D')\sum f(\xi)(v-u)| \\ &+ |(D_1^*)\sum_{\xi\in E-X} f(\xi)(v-u)| + |(D_2^*)\sum_{\xi\in E-X} f(\xi)(v-u)| \\ <\ & 7\epsilon.\end{aligned}$$

By the Cauchy Criterion again, the above sequence of inequalities implies that $f$ is $H_1$-integrable on $X$. ∎

**Lemma 3** [2]. *Let $f$ be $H_1$-integrable on a closed set $X_1 \subset [a,b]$ using $\delta_1$, and on another closed set $X_2 \subset [a,b]$, with $f(x) = 0$ for $x \notin X_1 \cup X_2$. If the primitive $F$ of $f$ on $[a,b]$ is absolutely continuous there, then $f$ is $H_1$-integrable on $X_1 \cup X_2$ using $\delta$, where $\delta(x) = \delta_1(x)$ when $x \in X_1$.*

We now present the uniform convergence theorem.

**Theorem 4** [Uniform Convergence Theorem]. *Let $\{f_n\}$ be a sequence of $H_1$-integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to some function $f$ on $[a,b]$, then $f$ is $H_1$-integrable on $[a,b]$ and $\int f = \lim \int f_n$.*

*Proof.* We may assume that $f$ is Henstock integrable on $[a,b]$. Let $\{F_n\}$ be the sequence of primitives of $\{f_n\}$. By Lemma 1, $\{F_n\}$ is $UACG^*$ on $[a,b]$, that is, there exists a sequence $\{X_i\}$ of closed subsets of $[a,b]$ such that $[a,b] = \cup X_i$ such that $\{F_n\}$ is $UAC^*(X_i)$ for each $i$. In particular, $F_n$ is $AC^*(X_i)$ for each $n$ and for each $i$. Hence, by Lemma 2, each $f_n$ is $H_1$-integrable on $X_i$ for all $i$.

It follows from the $UACG^*$ property (see [5, Theorem 9.8]) that for every $i$ there exists an integer $n(i) \geq i$ such that for any partial division $D$ of $[a,b]$ with $u$ or $v \in X_i$, we have

$$|(D) \sum \{F_{n(i)}(u,v) - F(u,v)\}| < \frac{1}{2^i},$$

where $F$ is the Henstock primitive of $f$ on $[a,b]$.

Since $f_{n(i)}$ is $H_1$-integrable on $X_i$ (and, therefore, Henstock integrable there), by the Saks-Henstock Lemma, there exists $\delta_{n(i)}(x) > 0$ such that for any $\delta_{n(i)}$-fine division $D$ of $[a,b]$, we have

$$(D) \sum_{\xi \in X_i} |f_{n(i)}(\xi)(v-u) - F_{n(i)}(u,v)| < \frac{1}{2^i}$$

for all $i$.

Let $Y_1 = X_1$ and $Y_i = X_i - (X_1 \cup X_2 \cup \cdots \cup X_{i-1})$ for $i = 2, 3, \ldots$. Put $\delta(x) = \delta_{n(i)}(x)$ if $x \in Y_i$. We may modify $\delta_{n(i)}$, if necessary, such that $(x - \delta_{n(i)}(x), x + \delta_{n(i)}(x)) \cap X_i = \emptyset$ for $x \notin X_i$.

Given $\epsilon > 0$, there exists a positive integer $N = n(i_0)$ such that

$$\sum_{i=i_0+1}^{\infty} \frac{1}{2^i} < \epsilon \qquad \text{and} \qquad |f_n(\xi) - f(\xi)| < \frac{\epsilon}{b-a}$$

for all $n \geq N$ and for all $\xi \in [a,b]$. Further, by Lemma 3, there exists a $\delta$-fine division $D_N$ of $[a,b]$ such that for any $\delta$-fine division $D \geq D_N$ of $[a,b]$, we have

$$|(D)\sum_{\xi \in X_{i_0}} \{f_N(\xi)(v-u) - F_N(u,v)\}| < \epsilon.$$

Write $n(\xi) = n(i_0)$ when $\xi \in X_{i_0}$ and $n(\xi) = n(i)$ when $n(i) > N = n(i_0)$. Thus,

$$\begin{aligned} |(D)\sum\{f(\xi)(v-u) - F(u,v)\}| \leq\ & |(D)\sum\{f(\xi)(v-u) - f_{n(\xi)}(\xi)(v-u)\}| \\ & +|(D)\sum\{F_{n(\xi)}(u,v) - F(u,v)\}| \\ & +|(D)\sum\{f_{n(\xi)}(\xi)(v-u) - F_{n(\xi)}(u,v)\}| \\ <\ & 4\epsilon. \end{aligned}$$

Hence, $f$ is $H_1$-integrable on $[a,b]$. ∎

We now consider the $H_1$-integral version of equi-integrability [4, 6] or uniformly Henstock integrable [3].

Let $\{f_n\}$ be a sequence of $H_1$-integrable functions on $[a,b]$. We say that $\{f_n\}$ is *equi-$H_1$-integrable* on $[a,b]$ if there exists $\delta(x) > 0$ such that for each $\epsilon > 0$ there exists a $\delta$-fine division $D_0$ of $[a,b]$ such that for any $\delta$-fine division $D \geq D_0$ of $[a,b]$, we have

$$|(D)\sum f_n(\xi)(v-u) - F_n(a,b)| < \epsilon$$

for all $n$, where $F_n$ is the primitive of $f_n$ on $[a,b]$.

**Lemma 5.** *Let $\{f_n\}$ be a sequence of $H_1$-integrable functions on $[a,b]$ with $F_n$ as the primitive of $f_n$ on $[a,b]$ such that $\{f_n\}$ converges pointwise to a function $f$ on $[a,b]$. If $\{f_n\}$ is equi-$H_1$-integrable on $[a,b]$, then $\{F_n(a,b)\}$ is a Cauchy sequence.*

*Proof.* By definition, there exist $\delta(x) > 0$ and a $\delta$-fine division $D_m$ of $[a,b]$ such that for all $\delta$-fine divisions $D \geq D_m$ of $[a,b]$, we have

$$|(D)\sum f_n(\xi)(v-u) - F_n(a,b)| < \frac{1}{2^m}$$

for all $n$. Given an $\epsilon > 0$, choose an integer $M > 0$ such that $1/2^M < \epsilon$ and

$$|(D_M)\sum f_n(\xi)(v-u) - (D_M)\sum f_m(\xi)(v-u)| < \epsilon$$

for all $n, m \geq M$, where $D_M$ is a $\delta$-fine division of $[a,b]$. Then, for $n, m \geq M$,

$$\begin{aligned}|F_n(a,b) - F_m(a,b)| &\leq |F_n(a,b) - (D_M)\sum f_n(\xi)(v-u)| \\ &\quad + |(D_M)\sum f_n(\xi)(v-u) - (D_M)\sum f_m(\xi)(v-u)| \\ &\quad + |(D_M)\sum f_m(\xi)(v-u) - F_m(a,b)| \\ &< 3\epsilon.\end{aligned}$$

Hence, $\{F_n(a,b)\}$ is a Cauchy sequence. ■

**Theorem 6.** *If the conditions of Lemma 5 are satisfied, then $f$ is $H_1$-integrable on $[a,b]$ and $\int f = \lim \int f_n$.*

*Proof.* By definition, there exists $\delta(x) > 0$ such that for every $\epsilon > 0$ there exists a $\delta$-fine division $D_0$ on $[a,b]$ such that for any $\delta$-fine division $D \geq D_0$ of $[a,b]$, we have

$$|(D)\sum f_n(\xi)(v-u) - F_n(u,v)| < \epsilon$$

for all $n$. By Lemma 5, there exists an integer $N > 0$ such that

$$|F_n(a,b) - F(a,b)| < \epsilon$$

for all $n \geq N$, where $F(a,b)$ is the limit of $\{F_n(a,b)\}$. Let $D$ be any $\delta$-fine division of $[a,b]$ with $D \geq D_0$. Then there exists an integer $k \geq N$ such that

$$|(D)\sum f(\xi)(v-u) - (D)\sum f_k(\xi)(v-u)| < \epsilon$$

since $D$ is finite and $f_n \to f$ pointwise. Hence,

$$\begin{aligned}|(D)\sum f(\xi)(v-u) - F(a,b)| &\leq |(D)\sum f(\xi)(v-u) - (D)\sum f_k(\xi)(v-u)| \\ &\quad + |(D)\sum f_k(\xi)(v-u) - F_k(a,b)| \\ &\quad + |F_k(a,b) - F(a,b)| \\ &< 3\epsilon.\end{aligned}$$

Thus, $f$ is $H_1$-integrable to $F(a,b)$ on $[a,b]$. ■

It should be noted that uniform convergence (Theorem 4) will also follow as a consequence of equi-$H_1$-integrability (Theorem 6), but the proof is as lengthy as the direct one given for Theorem 4.

So far, no other convergence theorems have been established for this relatively new integral. After a quite long battle for a proof of the uniform convergence theorem, the authors are still optimistic that the other known convergence theorems for other integrals will also hold for the $H_1$-integral.

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