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NOTES ON THE SEPARABILITY OF C*-ALGEBRAS

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Abstract. It is well-known that any abelian C*-algebra can be identified, by Gelfand transform, as $C_0(\Omega)$. Here, the spectrum Ω is locally compact Hausdorff in general, and compact exactly when the algebra is unital. In this survey article, we collect some results about the relation between the metrizability of Ω and the separability of $C_0(\Omega)$. A brief discussion for the topological structure of the spectrum of a general separable C*-algebra A is also given.

1. INTRODUCTION

Let E be a real or complex Banach space with Banach dual E^*. Then the closed unit ball U_{E^*} of E^* is a weak* compact convex set. The canonical linear map $E \hookrightarrow C(U_{E^*})$ gives an isometric embedding. In case E is separable, U_{E^*} is metrizable, and thus $C(U_{E^*})$ is also separable. Indeed, if $\{x_n\}$ is a dense sequence in E then the weak* topology of U_{E^*} arises from the metric

$$d(f,g)=\sum_n \frac{|f(x_n)-g(x_n)|}{2^n(1+|f(x_n)-g(x_n)|)}, \quad \forall f,g\in U_{E^*}.$$

So every (separable) Banach space can be considered as a closed subspace of continuous functions on a compact (metrizable) Hausdorff space. Therefore, it is always interesting to study (separable) spaces of continuous functions.

C*-algebras [4, p. 36] are involutive Banach algebras satisfying the C*-equation $\|a^*a\|=\|a\|^2$. In particular, any nonzero abelian C*-algebra A is an abelian Banach algebra with nonempty character space, or the spectrum, $\Omega(A)$, which is the set of nonzero homomorphisms from A into $\mathbb{C}$ [4, p. 40]. Here, $\Omega(A)$ is locally compact Hausdorff in general, and compact exactly when A is unital [4, p. 15]. Therefore, by Gelfand transform, A is *-isomorphic to $C_0(\Omega(A))$, the space of complex-valued continuous functions on $\Omega(A)$ vanishing at infinity.

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It is well-known that, if Ω is a locally compact Hausdorff space, and A is the space $C_0(\Omega)$, then the map $\omega \in \Omega \mapsto \tilde{\omega} \in \Omega(A)$ given by $\tilde{\omega}(x) = x(\omega), \forall x \in A$, is a homeomorphism between Ω and $\Omega(A)$ [9, p. 18]. Since *-isomorphic C*-algebras are isometric [4, p. 80], thus, for any two locally compact Hausdorff spaces X and Y, $C_0(X)$ is *-isomorphic to $C_0(Y)$ if and only if X is homeomorphic to Y. Therefore, the study of abelian C*-algebras is the study of the corresponding topological spaces.

In many important cases, e.g., when Ω is a compact metric space, we can recover Ω from $C_0(\Omega) = C(\Omega)$. In this view, some says Ω has "commutative geometry". For example, it is well-known that if Ω is a locally compact metric space, then Ω is separable if and only if $C_0(\Omega)$ is separable [3, p. 221]. (It is proved there for the compact case, but the argument carries to the locally compact case as follows. If $\{p_n | n \in \mathbb{N}\}$ is dense in Ω, then for all $n, m \in N$ such that $\overline{B_{1/m}(p_n)}$ is compact in Ω, there exists a continuous function $f_{n,m}$ from Ω into $[0,1]$ with support in $B_{1/m}(p_n)$ and $f_{n,m}(B_{1/2m}(p_n)) = \{1\}$. Since $\{p_n | n \in \mathbb{N}\}$ is dense in Ω, the collection of these $f_{n,m}$'s separate points in $\Omega \cup \{\infty\}$. Hence, by the Stone-Weierstrass theorem [7, p. 167], the set of all the finite complex-rational linear combinations of the finite products of these $f_{n,m}$'s is dense in $C_0(\Omega)$.) In general, if $C_0(\Omega)$ is separable then Ω is separable, but the converse might not hold when Ω is not metrizable. We shall provide some concrete counter examples in Section 2. Anyway, we shall see that if Ω is locally compact Hausdorff, $C_0(\Omega)$ is separable if and only if Ω is σ-compact and metrizable, if and only if Ω is second countable.

We also give a brief account of the topological structure of the spectrum of a separable general C*-algebra. Although we do not claim any result in this note is new, we think it would be interesting to present them in a neat way for a better picture of the role of the separability of a C*-algebra takes.

2. Separability of $C*$-Algebras

We first give three concrete examples that Ω is locally compact Hausdorff and separable, but $C_0(\Omega)$ is not separable. Two of them are even compact.

Example 2.1. Let Ω be the set of real numbers $\mathbb{R}$ with the rational sequence topology [8, p. 87]. That is, for every irrational number x, we fix a rational sequence $\{r_k^{[x]}\}_{k \in \mathbb{N}}$ approaching to x in the usual topology of $\mathbb{R}$. Let $\mathcal{B} = \{\{q\} | q \in \mathbb{Q}\} \cup \{U_n^{[x]} | x \in (\mathbb{R} \backslash \mathbb{Q})$ and $n \in \mathbb{N}\}$, where $U_n^{[x]} = \{r_k^{[x]} | k \geq n\} \cup \{x\}$. Let $\mathcal{T}$ be the topology generated by the basis $\mathcal{B}$. It is well-known that $(\Omega, \mathcal{T})$ is locally compact Hausdorff, separable, first countable, but not second countable or metrizable. For any $B \in \mathcal{B}$, since B is both open and compact, the characteristic function χ_B is continuous with compact support. For any two irrational $x_1 \neq x_2$ and for any $m, n \in$

$\mathbb{N}$, since $x_1 \in U_m^{[x_1]} \setminus U_n^{[x2]}$ and $x_2 \in U_n^{[x_2]} \setminus U_m^{[x_1]}$, we have $d(\chi_{U_m^{[x_1]}}, \chi_{U_n^{[x2]}}) = 1$. Hence any dense subset of $C_0(\Omega)$ must be uncountable. Therefore, $C_0(\Omega)$ is not separable.

Example 2.2. Let I denote the unit interval $[0,1]$ and Ω be the set I^I with product topology $\mathcal{T}$ [8, p. 125]. It is well-known that $(\Omega, \mathcal{T})$ is compact Hausdorff, separable, not first countable, hence not second countable and not metrizable. For any $x \in [0,1]$, the projection map $\pi_x : I^{[0,1]} \to I_x$ is continuous, and for any $x_1 \neq x_2$ in $[0,1]$, $d(\pi_{x_1}, \pi_{x_2}) = 1$. Hence any dense subset of $C(\Omega)$ must be uncountable. Therefore, $C(\Omega)$ is not separable.

Example 2.3. Let $(\Omega, \mathcal{T})$ be the Helly space [8, p. 127]. That is, first we identify I^I as the set of all functions from I into I, and then let Ω be the subspace of I^I consisting of nondecreasing functions. It is well-known that $(\Omega, \mathcal{T})$ is compact Hausdorff, separable, first countable, but not second countable, and hence not metrizable. For any $x \in [0,1]$, the restriction of the projection map $\pi_x|_\Omega : \Omega \to I_x$ is continuous, and for any $x_1 \neq x_2$ in $[0,1]$, $d(\pi_{x_1}|_\Omega, \pi_{x_2}|_\Omega) = 1$. Hence any dense subset of $C(\Omega)$ must be uncountable. Therefore, $C(\Omega)$ is not separable.

Theorem 2.4. *For a locally compact Hausdorff space Ω, the following are equivalent.*

(a) *The abelian C*-algebra $C_0(\Omega)$ is separable.*

(b) *Ω is σ-compact and metrizable.*

(c) *Ω is second countable.*

Proof. By the Uryshon metrizability theorem, a regular space is metrizable if it is second countable. Since locally compact Hausdorff spaces are completely regular, the condition (c) implies that Ω is metrizable. Also, (c) implies the separability of Ω. Thus $C_0(\Omega)$ is separable, as demonstrated in the Introduction. This gives the implication from (c) to (a).

Assume (a), and $\{f_n\}$ is a dense sequence in $C_0(\Omega)$. As a subset of the compact metrizable space $U_{C_0(\Omega)^*}$, we see that Ω is metrizable. On the other hand, the countable family of compact sets $K_{n,m} := \{x \in \Omega : |f_n(x)| \geq 1/m\}$, $n, m = 1, 2, \ldots$ covers Ω, and thus Ω is σ-compact. This gives the implication from (a) to (b).

Finally, assume that Ω is metrizable and $\Omega = \bigcup_n K_n$ as a countable union of compact subsets K_n. Since each compact metrizable space K_n is second countable, Ω is second countable. This gives the implication from (b) to (c). ■

Next, we give a brief discussion on the non-abelian case. In the following, let A be a general C*-algebra. A positive norm one linear functional of A is called a pure state if it is not a convex combination of another two distinct such functionals. Let $P(A)$ be the pure state space of A. In the abelian case, $P(C_0(\Omega)) \cong \Omega$. In general, we set $Q(A) = \{\varphi \in U_{A^*} : \varphi \geq 0\}$ to be the quasi-state space of A. Then $Q(A)$ is a weak* compact convex set with extreme boundary $P(A) \cup \{0\}$.

Recall that a (closed) ideal I of a C*-algebra A is primitive if it is the kernel of an irreducible representation of A. The primitive ideal space Prim(A) of A consists of all primitive ideals of A in the hull-kernel topology. In case $A = C_0(\Omega)$, primitive ideals are exactly maximal ideals and thus Prim$(C_0(\Omega)) \cong \Omega$. However, we know that Prim(A) is only a T_0-space in general. On the other hand, the map $\varphi \in P(A) \mapsto \ker \pi_\varphi \in$ Prim(A) is surjective, open and continuous (see, e.g., [5, Theorem 4.3.3]). Here, π_φ is the irreducible representation arising from the GNS construction through the pure state φ.

The *spectrum* of a C*-algebra is the set Spec(A) of (spatial) equivalence classes of irreducible representations. The topology of Spec(A) is the one induced from Prim(A) through the natural map $\pi \mapsto \ker \pi$. Again, in the abelian case we have Spec$(C_0(\Omega)) \cong \Omega$. However, although being locally compact, Spec(A) is even not a T_0-space in general.

If A is separable, then the closed dual ball U_{A^*} is weak* compact and metrizable, and thus second countable. Hence, $P(A)$ is second countable. Consequently, both Prim(A) and Spec(A) are second countable whenever A is separable. The converse does not hold, however. For a counter example, we can think of $A = K(H)$, the C*-algebra of compact operators on an inseparable Hilbert space H. In this case, Prim$(A) =$ Spec(A) consists of only one point. Anyway, we have

Theorem 2.5. *Let A be a separable C*-algebra. Then, A is a GCR (resp. CCR) if and only if Spec(A) is T_0 (resp. T_1). In general, a GCR C*-algebra is a CCR if and only if its spectrum is T_1.*

Proof. The first statement can be found in [1], and the second is [2, Theorem 4]. ∎

Remark that a GCR C*-algebra is also called a type I C*-algebra.

Corollary 2.6. *Let A be a separable C*-algebra A. Then its spectrum Spec(A) is metrizable if and only if Spec(A) is Hausdorff. In this case, A is a CCR.*

Proof. It follows from the local compactness of Spec(A) that Spec(A) is completely regular if and only if Spec(A) is Hausdorff. Since Spec(A) is second countable, these conditions are also equivalent to the metrizability of Spec(A) by the Uryshon metrizability theorem. The last assertion follows from Theorem 2.5 ∎

Corollary 2.6 says that every separable CCR C*-algebra with Hausdorff spectrum has metrizable spectrum. We note that every such C*-algebra can be represented as a continuous field of separable elementary C*-algebras over the spectrum [6, §5.1]. An example is $C_0(\Omega)\otimes K(\ell_2) \cong C_0(\Omega, K(\ell_2))$, the C*-algebra of continuous compact operator-valued functions on a locally compact and second countable space Ω vanishing at infinity.

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