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RELAXED EXTRAGRADIENT-LIKE METHOD FOR GENERAL SYSTEM OF GENERALIZED MIXED EQUILIBRIA AND FIXED POINT PROBLEM

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Abstract. In this paper, we introduce two iterative algorithms based on the relaxed extragradient-like method for finding a common element of the solution set of a general system of generalized mixed equilibria and the fixed point set of a strictly pseudocontractive mapping in a real Hilbert space. We will prove the weak convergence of the iterative algorithm under some mild conditions, while the strong convergence is obtained under some more restrictive conditions.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be ρ-Lipschitz mapping if $\rho \geq 0$ and

$$\|Tx - Ty\| \leq \rho\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $\rho \in [0, 1)$ then T is called a contraction on C, while if $\rho = 1$ then T is called a nonexpansive mapping on C. Let P_C be the metric projection of H onto C. A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

The variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

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The set of solutions of the variational inequality (1.1) is denoted by VI(A, C). Variational inequality theory has played an important role in the study of diverse disciplines, for instance, partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance. The variational inequality problem has been extensively studied in the literature; see [1-7] and the references therein.

Let $C \subset H$ be a nonempty set and $S : C \to C$ be a mapping. We denote by $\mathrm{Fix}(S) := \{x \in C \mid x = S(x)\}$ the set of fixed points of S. For finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$ under the assumptions that the set C is nonempty closed and convex, the mapping $S : C \to C$ is nonexpansive and the mapping $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0,$$

where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. It was shown in [8] that if $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$. Recently, motivated by the idea of Korpelevich [11], Nadezhkina-Takahashi [9] and Zeng-Yao [10] proposed some so-called extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. Further, these iterative methods were extended in [12] to develop a new iterative method for finding some elements of $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$.

On the other hand, let $F, G : C \times C \to \mathbf{R}$ be two bifunctions, $\phi, \psi : C \to \mathbf{R}$ be two functions and $A, B : C \to H$ be two nonlinear mappings. Consider the problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.2) \qquad \begin{cases} F(\bar{x}, x) + \phi(x) - \phi(\bar{x}) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\lambda}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ G(\bar{y}, y) + \psi(y) - \psi(\bar{y}) + \langle B\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

where $\lambda > 0$ and $\mu > 0$ are two constants, which is called a general system of generalized mixed equilibria. Next we present some special cases of problem (1.2) as follows:

If $\phi = \psi = 0$, then problem (1.2) reduces to the problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$(1.3) \qquad \begin{cases} F(\bar{x}, x) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\lambda}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \quad \forall x \in C, \\ G(\bar{y}, y) + \langle B\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \quad \forall y \in C, \end{cases}$$

This problem is called a general system of generalized equilibria and it was introduced and studied by Ceng and Yao [19]. The set of elements $\bar{x} \in C$ satisfying problem (1.3) is denoted by Ω.

If $F = G$ and $A = B$, then problem (1.3) reduces to the problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} F(\bar{x}, x) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\lambda}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, & \forall x \in C, \\ F(\bar{y}, y) + \langle A\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, & \forall y \in C, \end{cases}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. The above problem is called a new system of generalized equilibria and it was introduced in [19].

If $F = G,\ A = B$ and $\bar{x} = \bar{y}$, then problem (1.3) reduces to the following generalized equilibrium problem (introduced and studied by Takahashi and Takahashi in [20]): find $\bar{x} \in C$ such that

$$F(\bar{x}, y) + \langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C.$$

The solution set of the above problem is denoted by MEP.

Subsequently, Peng and Yao [23] introduced the following generalized mixed equilibrium problem: find $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle T\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

where $T : C \to H$ is a nonlinear mapping, $\varphi : C \to \mathbf{R}$ is a function and $\Theta : C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of problem (1.4) is denoted by $GMEP$. Out of question, the generalized mixed equilibrium problem covers the generalized equilibrium problem as a special case. It is assumed in [23] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (A1) or (A2), where

(H1) $\Theta(x, x) = 0,\ \forall x \in C$;

(H2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0,\ \forall x, y \in C$;

(H3) for each $y \in C,\ x \mapsto \Theta(x, y)$ is weakly upper semicontinuous;

(H4) for each $x \in C,\ y \mapsto \Theta(x, y)$ is convex and lower semicontinuous;

(A1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(A2) C is a bounded set.

If $F = G = 0$, then problem (1.3) reduces to the following general system of variational inequalities: find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} \langle \lambda A\bar{y} + \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, & \forall x \in C, \\ \langle \mu B\bar{x} + \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.5}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. The above problem was considered by Ceng, Wang and Yao [15]. The set of elements $\bar{x} \in C$ satisfying problem (1.5) is denoted by Γ.

If $A = B$ in (1.5), then problem (1.5) reduces to the following new system of variational inequalities (which was proposed by Verma [13], see also [14]): find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} \langle \lambda A\bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu A\bar{x} + \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, \quad \forall y \in C. \end{cases} \tag{1.6}$$

Finally, notice that if $\bar{x} = \bar{y}$ in (1.6), then problem (1.6) reduces to the classical variational inequality (1.1).

Very recently, Yao, Liou and Kang [21] proposed and analyzed an iterative algorithm based on the extragradient method for finding a common element of the solution set of the general system (1.5) of variational inequalities and the fixed point set of a strictly pseudocontractive mapping $S : C \to C$ in a real Hilbert space H.

Theorem 1.1 ([21, Theorem 3.2]). *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \mu Bx_n), \\ y_n = \alpha_n Qx_n + (1 - \alpha_n)P_C(z_n - \lambda Az_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda Az_n) + \delta_n Sy_n, \quad \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\lambda \in (0, 2\alpha),\ \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0, 1]$ such that

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \geq 0$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right) = 0.$

Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap\Gamma} \cdot Qx^*$ and (x^*, y^*) is a solution of the general system (1.5) of variational inequalities, where $y^* = P_C(x^* - \mu Bx^*)$.

On the other hand, Ceng and Yao [19] also considered a relaxed extragradient-like method for finding a common solution of problem (1.3), problem (1.4) and the fixed point problem of a strictly pseudocontractive mapping $S : C \to C$ in a real Hilbert space H. Notice that for the notations $T_{\lambda_n}^{(\Theta,\varphi)}$, T_λ^F and T_μ^G in the following result, we ask the readers to refer to Lemma 2.1 and Remark 2.1 in Section 2.

Theorem 1.2 ([19, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta, F, G : C\times C \to \mathbf{R}$ be three bifunctions which satisfy the assumptions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with restriction (A1) or (A2). Let the mappings $T, A, B : C \to H$ be η-inverse-strongly monotone, α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap GMEP \cap \Omega \neq \emptyset$. For fixed $u \in C$ and $x_0 \in C$ arbitrary, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n T x_n), \\ y_n = T_\lambda^F[T_\mu^G(z_n - \mu B z_n) - \lambda A T_\mu^G(z_n - \mu B z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$, and $\{\lambda_n\} \subset [0, 2\eta]$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$*;*

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\eta$ *and* $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0.$

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{\mathrm{Fix}(S)\cap GMEP\cap\Omega} u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.3), where $\bar{y} = T_\mu^G(\bar{x} - \mu B\bar{x})$.

Related iterative methods for solving fixed point problems, variational inequalities and optimization problems can be found in [28-45].

In this paper, let C be a nonempty bounded closed convex subset of a real Hilbert space H. Suppose $F, G : C \times C \to \mathbf{R}$ are two bifunctions satisfying conditions (H1)-(H4), $\phi, \psi : C \to \mathbf{R}$ are two lower semicontinuous and convex functions and the mappings $A, B : C \to H$ are α-inverse strongly monotone and β-inverse strongly monotone, respectively. Assume $S : C \to C$ is a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$, where for the notation Ξ we refer to Remark 2.2 in Section 2. Let $Q : C \to H$ be a ρ-Lipschitz mapping. We first propose the following iterative algorithm.

Algorithm I. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively via the scheme

$$(1.9) \qquad \begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu B x_n), \\ y_n = T_\lambda^{(F,\phi)}[\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)], \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha),\ \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$.

It is proven that under some mild conditions $\{x_n\}, \{y_n\}$ converge weakly to the same element $\hat{x} \in \mathrm{Fix}(S) \cap \Xi$, and $\{z_n\}$ converges weakly to $\hat{y} = T_\mu^{(G,\psi)}(\hat{x} - \mu B \hat{x})$ where $(\hat{x}, \hat{y})$ is a solution of the general system (1.2) of generalized mixed equilibria. On the other hand, whenever Q is a ρ-contraction of C into itself with $\rho \in [0, \frac{1}{2})$, we consider the following iterative algorithm.

Algorithm II. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively via the scheme

$$(1.10) \qquad \begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu B x_n), \\ y_n = \alpha_n Q x_n + (1-\alpha_n) T_\lambda^{(F,\phi)}(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha),\ \mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$.

It is proven that under some appropriate conditions, the sequence $\{x_n\}$ generated by (1.10) converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap\Xi} \cdot Q x^*$ and (x^*, y^*) is a solution of the general system (1.2) of generalized mixed equilibria, where $y^* = T_\mu^{(G,\psi)}(x^* - \mu B x^*)$.

In particular, whenever $F = G = 0$ and $\phi = \psi = 0$, our strong convergence result reduces to Yao, Liu and Kang's Theorem 3.2 in [21]. Thus, the results presented in the paper improve and extend the corresponding theorems in Yao, Liu and Kang [21].

2. Preliminaries

Let H be a real Hilbert space and denote by I the identity mapping of H. If C is a nonempty closed convex subset of H then, for every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C, i.e.,

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

It is also known that $P_C x$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H \text{ and } y \in C. \tag{2.1}$$

In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$; see Takahashi [27].

Recall that a mapping $S : C \to C$ is called a strictly pseudocontractive mapping if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

In this case, we say that S is a k-strict pseudocontraction. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any inverse strongly monotone mapping is Lipschitz. Furthermore, observe that (2.2) is equivalent to

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.3}$$

From [16] we know that if S is a k-strictly pseudocontractive mapping, then S is $\frac{1+k}{1-k}$-Lipschitz. Thus, it is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings.

In order to prove our main results in this paper, we need the following lemmas.

Lemma 2.1. ([22]). *Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)-(H4) and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Assume that either (A1) or (A2) holds. Then the following assertions hold:

(i) $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ *for each* $x \in H$ *and* $T_r^{(\Theta,\varphi)}$ *is single-valued;*
(ii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, i.e., for any* $x, y \in H$*,*

$$\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = MEP(\Theta,\varphi)$*;*
(iv) $MEP(\Theta,\varphi)$ *is closed and convex.*

Remark 2.1. If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ will be denoted by T_r^{Θ}.

Lemma 2.2. *Let* C *be a nonempty closed convex subset of* H*. Let* $F, G : C \times C \to \mathbf{R}$ *be two bifunctions satisfying conditions (H1)-(H4), let* $\phi, \psi : C \to \mathbf{R}$ *be two lower semicontinuous and convex functions with restriction (A1) or (A2) and let the mappings* $A, B : C \to H$ *be* α*-inverse strongly monotone and* β*-inverse strongly monotone, respectively. Let* $\lambda \in (0, 2\alpha)$ *and* $\mu \in (0, 2\beta)$*, respectively. Then, for given* $\bar{x}, \bar{y} \in C$*,* $(\bar{x}, \bar{y})$ *is a solution of problem (1.2) if and only if* $\bar{x}$ *is a fixed point of the mapping* $W : C \to C$ *defined by*

$$W(x) = T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(x - \mu Bx) - \lambda A T_\mu^{(G,\psi)}(x - \mu Bx)], \quad \forall x \in C,$$

where $\bar{y} = T_\mu^{(G,\psi)}(\bar{x} - \mu B\bar{x})$*.*

Proof. Observe that

$$\begin{cases} F(\bar{x}, x) + \phi(x) - \phi(\bar{x}) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\lambda}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, & \forall x \in C, \\ G(\bar{y}, y) + \psi(y) - \psi(\bar{y}) + \langle B\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, & \forall y \in C \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \bar{x} = T_\lambda^{(F,\phi)}(\bar{y} - \lambda A\bar{y}), \\ \bar{y} = T_\mu^{(G,\psi)}(\bar{x} - \mu B\bar{x}) \end{cases}$$

$$\Updownarrow$$

$$\bar{x} = T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(\bar{x} - \mu B\bar{x}) - \lambda A T_\mu^{(G,\psi)}(\bar{x} - \mu B\bar{x})].$$

Corollary 2.1. ([19, Lemma 2.2]). *Let* C *be a nonempty closed convex subset of* H*. Let* $F, G : C \times C \to \mathbf{R}$ *be two bifunctions satisfying conditions (H1)-(H4) and let the mappings* $A, B : C \to H$ *be* α*-inverse strongly monotone and* β*-inverse strongly monotone, respectively. Let* $\lambda \in (0, 2\alpha)$ *and* $\mu \in (0, 2\beta)$*, respectively. Then, for given* $\bar{x}, \bar{y} \in C$*,* $(\bar{x}, \bar{y})$ *is a solution of problem (1.3) if and only if* $\bar{x}$ *is a fixed point of the mapping* $\widetilde{W} : C \to C$ *defined by*

$$\widetilde{W}(x) = T_\lambda^F[T_\mu^G(x - \mu Bx) - \lambda A T_\mu^G(x - \mu Bx)], \quad \forall x \in C,$$

where $\bar{y} = T_\mu^G(\bar{x} - \mu B\bar{x})$.

Corollary 2.2. ([15, Lemma 2.1]). *For given* $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ *is a solution of problem (1.5) if and only if* $\bar{x}$ *is a fixed point of the mapping* $\widehat{W} : C \to C$ *defined by*

$$\widehat{W}(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \quad \forall x \in C,$$

where $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

Proof. Putting $F = G = 0$ and utilizing Lemma 2.1 we deduce that $T_\lambda^F = T_\mu^G = P_C$. Thus from Corollary 2.1 we obtain the desired result. ■

Remark 2.2. In the conditions of Lemma 2.2., the set of fixed points of the mapping W is denoted by Ξ.

Lemma 2.3. ([17]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4. (Demiclosedness Principle, see [16]). *Assume that* T *is a* k*-strictly pseudocontractive self-mapping on a nonempty closed convex subset* C *of a real Hilbert space* H. *Then, the mapping* $I - T$ *is demiclosed. That is, whenever* $\{x_n\}$ *is a sequence in* C *converging weakly to some* $x^* \in C$ *(for short,* $x_n \rightharpoonup x^* \in C$*), and the sequence* $\{(I - T)x_n\}$ *converges strongly to some* y *(for short,* $(I - T)x_n \to y$*), it follows that* $(I - T)x^* = y$.

Lemma 2.5. ([18]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(*i*) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space* H, *there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Notice that, if $\{x_n\}$ is a sequence and x is a point in a normed space X, then the symbols $x_n \to x$ and $x_n \rightharpoonup x$ denote, respectively, strong and weak convergence to x of the sequence $\{x_n\}$. In addition, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{\bar{x} \in X : x_{n_i} \rightharpoonup \bar{x} \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

3. Weak Convergence Theorem

We are now in position to state and prove the main result in this section. We first need a technical lemma whose proof is an immediate consequence of Qpial's property [24] of a Hilbert space. Recall that a Banach space X satisfies Opial's property [24] provided, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x.$$

It is known [24] that each l^p $(1 \leq p < \infty)$ enjoys this property, while L^p does not unless $p = 2$. It is known [25] that any separable Banach space can be equivalently renormed so that it satisfies Opial's property.

Lemma 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H satisfying the properties:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each $x \in K$;*

(ii) $\omega_w(x_n) \subset K$.

Then $\{x_n\}$ is weakly convergent to a point in K.

Proof. It is sufficient to show that $\omega_w(x_n)$ is a singleton. Indeed, let $\bar{x}$ and $\hat{x}$ be two elements in $\omega_w(x_n)$ with $\bar{x} \neq \hat{x}$. Then there are two subsequences $\{x_{n_i}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ and $x_{m_j} \rightharpoonup \hat{x}$. By Opial's property of H we reach the following contradiction:

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \bar{x}\| = \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| < \lim_{i\to\infty} \|x_{n_i} - \hat{x}\| = \lim_{j\to\infty} \|x_{m_j} - \hat{x}\| \\ < \lim_{n\to\infty} \|x_{m_j} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|.\end{aligned}$$

This implies that $\omega_w(x_n)$ is a singleton. Consequently, $\{x_n\}$ is weakly convergent to a point in K. ■

We also use the following elementary lemma.

Lemma 3.2. (See [26]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. Then $\lim_{n\to\infty} a_n$ exists.*

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $F, G : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)-(H4), let $\phi, \psi : C \to \mathbf{R}$ be two lower semicontinuous and convex functions and let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$. Let $Q : C \to H$ be a ρ-Lipschitz mapping with constant $\rho \geq 0$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$(3.1) \qquad \begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu B x_n), \\ y_n = T_\lambda^{(F,\phi)}[\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)], \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0.$

Then $\{x_n\}, \{y_n\}$ converge weakly to the same element $\hat{x} \in \mathrm{Fix}(S) \cap \Xi$, and $\{z_n\}$ converges weakly to $\hat{y} = T_\mu^{(G,\psi)}(\hat{x} - \mu B \hat{x})$ where $(\hat{x}, \hat{y})$ is a solution of the general system (1.2) of generalized mixed equilibria.

Proof. We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

First, (3.1) can be rewritten as

$$x_{n+1} = \beta_n x_n + (1-\beta_n) u_n, \quad \forall n \geq 0,$$

where $u_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. It follows that

$$(3.2) \qquad \begin{aligned} & u_{n+1} - u_n \\ &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1} T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1}) + \delta_{n+1} S y_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1}[T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1}) - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)] + \delta_{n+1}(S y_{n+1} - S y_n)}{1-\beta_{n+1}} \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\right) S y_n. \end{aligned}$$

Combining (2.2) with (2.3), we have

$$\begin{aligned}
&\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|^2\\
=\ &\gamma_{n+1}^2\|y_{n+1}-y_n\|^2+\delta_{n+1}^2\|Sy_{n+1}-Sy_n\|^2\\
&+2\gamma_{n+1}\delta_{n+1}\langle y_{n+1}-y_n, Sy_{n+1}-Sy_n\rangle\\
\le\ &\gamma_{n+1}^2\|y_{n+1}-y_n\|^2+\delta_{n+1}^2[\|y_{n+1}-y_n\|^2+k\|(I-S)y_{n+1}-(I-S)y_n\|^2]\\
&+2\gamma_{n+1}\delta_{n+1}[\|y_{n+1}-y_n\|^2-\frac{1-k}{2}\|(I-S)y_{n+1}-(I-S)y_n\|^2]\\
=\ &(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2+[\delta_{n+1}^2k\\
&-(1-k)\gamma_{n+1}\delta_{n+1}]\|(I-S)y_{n+1}-(I-S)y_n\|^2\\
=\ &(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2+\delta_{n+1}[(\gamma_{n+1}+\delta_{n+1})k\\
&-\gamma_{n+1}]\|(I-S)y_{n+1}-(I-S)y_n\|^2\\
\le\ &(\gamma_{n+1}+\delta_{n+1})^2\|y_{n+1}-y_n\|^2,
\end{aligned}$$

which implies that

$$\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|\le(\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|. \tag{3.3}$$

From (3.1) and (3.3) we get

$$\begin{aligned}
&\|\gamma_{n+1}[T_\lambda^{(F,\phi)}(z_{n+1}\\
&-\lambda Az_{n+1})-T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)]+\delta_{n+1}(Sy_{n+1}-Sy_n)\|\\
\le\ &\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|\\
&+\gamma_{n+1}\|[T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})-y_{n+1}]+[y_n-T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)]\|\\
\le\ &\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|\\
&+\gamma_{n+1}\|T_\lambda^{(F,\phi)}(z_{n+1}\\
-&\lambda Az_{n+1})-T_\lambda^{(F,\phi)}[\alpha_{n+1}Qx_{n+1}+(1-\alpha_{n+1})(z_{n+1}-\lambda Az_{n+1})]\|\\
&+\gamma_{n+1}\|T_\lambda^{(F,\phi)}[\alpha_nQx_n+(1-\alpha_n)(z_n-\lambda Az_n)]-T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)\|\\
\le\ &(\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|+\gamma_{n+1}\alpha_{n+1}\|Qx_{n+1}-(z_{n+1}-\lambda Az_{n+1})\|\\
&+\gamma_{n+1}\alpha_n\|Qx_n-(z_n-\lambda Az_n)\|.
\end{aligned} \tag{3.4}$$

Since A, B are α-inverse strongly monotone mapping and β-inverse strongly monotone mapping, respectively, we have

$$\begin{aligned}
&\|(I-\lambda A)x-(I-\lambda A)y\|^2\\
&=\|x-y\|^2-2\lambda\langle Ax-Ay, x-y\rangle+\lambda^2\|Ax-Ay\|^2\\
&\le\|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2,
\end{aligned} \tag{3.5}$$

and

$$\|(I-\mu B)x-(I-\mu B)y\|^2\le\|x-y\|^2+\mu(\mu-2\beta)\|Bx-By\|^2. \tag{3.6}$$

It is clear that if $0 \leq \lambda \leq 2\alpha$ and $0 \leq \mu \leq 2\beta$, then $(I - \lambda A)$ and $(I - \mu B)$ are nonexpansive. It follows that

$$\begin{aligned}
&\|z_{n+1} - \lambda A z_{n+1} - (z_n - \lambda A z_n)\| \\
\leq\ & \|z_{n+1} - z_n\| \\
=\ & \|T_\mu^{(G,\psi)}(x_{n+1} - \mu B x_{n+1}) - T_\mu^{(G,\psi)}(x_n - \mu B x_n)\| \\
\leq\ & \|(x_{n+1} - \mu B x_{n+1}) - (x_n - \mu B x_n)\| \\
\leq\ & \|x_{n+1} - x_n\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|T_\lambda^{(F,\phi)}[\alpha_{n+1} Q x_{n+1} + (1-\alpha_{n+1})(z_{n+1} - \lambda A z_{n+1})] \\
&\quad -T_\lambda^{(F,\phi)}[\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)]\| \\
&\leq \|[\alpha_{n+1} Q x_{n+1} + (1-\alpha_{n+1})(z_{n+1} \\
&\quad -\lambda A z_{n+1})] - [\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)]\| \\
&\leq \|z_{n+1} - \lambda A z_{n+1} - (z_n - \lambda A z_n)\| \\
&\quad +\alpha_{n+1}\|Q x_{n+1} - (z_{n+1} - \lambda A z_{n+1})\| \\
&\quad +\alpha_n \|Q x_n - (z_n - \lambda A z_n)\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|Q x_n - (z_n - \lambda A z_n)\| \\
&\quad +\alpha_{n+1}\|Q x_{n+1} - (z_{n+1} - \lambda A z_{n+1})\|.
\end{aligned} \tag{3.7}$$

Therefore, from (3.2), (3.4) and (3.7), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \Big(1 + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big)\alpha_n \|Q x_n - (z_n - \lambda A z_n)\| \\
&\quad +\Big(1 + \frac{\gamma_{n+1}}{1-\beta_{n+1}}\Big)\alpha_{n+1}\|Q x_{n+1} - (z_{n+1} - \lambda A z_{n+1})\| \\
&\quad +\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\Big|\big(\|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| + \|S y_n\|\big).
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.3 we get $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|u_n - x_n\| = 0. \tag{3.8}$$

Step 2. $\lim_{n\to\infty}\|x_n - x^*\|$ exists for each $x^* \in \mathrm{Fix}(S) \cap \Xi$; moreover, $\lim_{n\to\infty}\|A z_n - A y^*\| = 0$ and $\lim_{n\to\infty}\|B x_n - B x^*\| = 0$.

Indeed, take a fixed $x^* \in \mathrm{Fix}(S) \cap \Xi$ arbitrarily. Then by Lemma 2.2 we have $x^* = S x^*$ and

$$x^* = T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(x^* - \mu Bx^*) - \lambda A T_\mu^{(G,\psi)}(x^* - \mu Bx^*)].$$

Put $y^* = T_\mu^{(G,\psi)}(x^* - \mu Bx^*)$. Then $x^* = T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)$. From (3.5) and (3.6), we have

$$\begin{aligned}(3.9)\qquad &\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2\\ &\le \|(z_n - \lambda Az_n) - (y^* - \lambda Ay^*)\|^2\\ &\le \|z_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2,\end{aligned}$$

and

$$\begin{aligned}(3.10)\qquad \|z_n - y^*\|^2 &= \|T_\mu^{(G,\psi)}(x_n - \mu Bx_n) - T_\mu^{(G,\psi)}(x^* - \mu Bx^*)\|^2\\ &\le \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\|^2\\ &\le \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2.\end{aligned}$$

It follows from (3.1), (3.9) and (3.10) that

$$\begin{aligned}(3.11)\qquad &\|y_n - x^*\|^2\\ &= \|T_\lambda^{(F,\phi)}[\alpha_n Qx_n + (1-\alpha_n)(z_n - \lambda Az_n)] - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2\\ &\le \|[\alpha_n Qx_n + (1-\alpha_n)(z_n - \lambda Az_n)] - (y^* - \lambda Ay^*)\|^2\\ &\le \alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 + (1-\alpha_n)\|(z_n - \lambda Az_n) - (y^* - \lambda Ay^*)\|^2\\ &\le \alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 + \|z_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2\\ &\le \alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 + \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2\\ &\quad + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2.\end{aligned}$$

Utilizing the convexity of $\|\cdot\|$, we have

$$\begin{aligned}(3.12)\qquad &\|x_{n+1} - x^*\|^2\\ &= \|\beta_n(x_n - x^*) + (1-\beta_n)\frac{1}{1-\beta_n}[\gamma_n(T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*)\\ &\quad + \delta_n(Sy_n - x^*)]\|^2\\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n}{1-\beta_n}(T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*)\\ &\quad + \frac{\delta_n}{1-\beta_n}(Sy_n - x^*)\|^2\\ &= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1-\beta_n}\\ &\quad + \frac{\gamma_n}{1-\beta_n}(T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - y_n)\|^2\\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\frac{\|\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\|}{1-\beta_n}\\ &\quad + \frac{\gamma_n}{1-\beta_n}\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - T_\lambda^{(F,\phi)}[\alpha_n Qx_n\\ &\quad + (1-\alpha_n)(z_n - \lambda Az_n)]\|]^2\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\frac{(\gamma_n+\delta_n)\|y_n - x^*\|}{1-\beta_n} \\
&\quad + \frac{\gamma_n}{1-\beta_n}\|(z_n - \lambda Az_n) - [\alpha_n Qx_n + (1-\alpha_n)(z_n - \lambda Az_n)]\|]^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\|y_n - x^*\| + \frac{\alpha_n\gamma_n}{1-\beta_n}\|Qx_n \\
&\quad -(z_n - \lambda Az_n)\|]^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n,
\end{aligned}
$$

where $M > 0$ is some appropriate constant. So, from (3.11) and (3.12) we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 + \|x_n - x^*\|^2 \\
&\quad +\mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2] + M\alpha_n \\
&\le \|x_n - x^*\|^2 + \mu(\mu - 2\beta)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\
&\quad +\lambda(\lambda - 2\alpha)(1-\beta_n)\|Az_n - Ay^*\|^2 + (M + \|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n \\
&\le \|x_n - x^*\|^2 + (M + \|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n.
\end{aligned}
$$

Utilizing Lemma 3.2 we conclude from $\sum_{n=0}^{\infty}\alpha_n < \infty$ that $\lim_{n\to\infty}\|x_n - x^*\|$ exists for each $x^* \in \mathrm{Fix}(S) \cap \Xi$. In addition, observe that

$$
\begin{aligned}
&\lambda(2\alpha - \lambda)(1-\beta_n)\|Az_n - Ay^*\|^2 + \mu(2\beta - \mu)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (M + \|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n \\
&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + (M + \|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n.
\end{aligned}
$$

Since $\liminf_{n\to\infty}\lambda(2\alpha-\lambda)(1-\beta_n) > 0,\ \liminf_{n\to\infty}\mu(2\beta-\mu)(1-\beta_n) > 0,\ \|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we have

$$
\lim_{n\to\infty}\|Az_n - Ay^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Bx_n - Bx^*\| = 0.
$$

Step 3. $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

Indeed, set $v_n = T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)$. Noting that both $T_\lambda^{(F,\phi)}$ and $T_\mu^{(G,\psi)}$ are firmly nonexpansive, then we have

$$
\begin{aligned}
&\|z_n - y^*\|^2 = \|T_\mu^{(G,\psi)}(x_n - \mu Bx_n) - T_\mu^{(G,\psi)}(x^* - \mu Bx^*)\|^2 \\
&\le \langle (x_n - \mu Bx_n) - (x^* - \mu Bx^*), z_n - y^* \rangle \\
&= \frac{1}{2}(\|x_n - x^* - \mu(Bx_n - Bx^*)\|^2 + \|z_n - y^*\|^2 - \|(x_n - x^*) \\
&\quad -\mu(Bx_n - Bx^*) - (z_n - y^*)\|^2) \\
&\le \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|(x_n - z_n) - \mu(Bx_n - Bx^*) - (x^* - y^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 \\
&\quad +2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^* \rangle - \mu^2\|Bx_n - Bx^*\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|v_n - x^*\|^2 = \|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2 \\
&\le \langle (z_n - \lambda Az_n) - (y^* - \lambda Ay^*), v_n - x^* \rangle \\
&= \frac{1}{2}(\|z_n - \lambda Az_n - (y^* - \lambda Ay^*)\|^2 + \|v_n - x^*\|^2 \\
&\quad -\|z_n - \lambda Az_n - (y^* - \lambda Ay^*) - (v_n - x^*)\|^2) \\
&\le \frac{1}{2}(\|z_n - y^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*) \rangle - \lambda^2\|Az_n - Ay^*\|^2) \\
&\le \frac{1}{2}(\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*) \rangle)
\end{aligned}
$$

(due to (3.10)). Thus, we have

$$
\begin{aligned}
\|z_n - y^*\|^2 \le{}& \|x_n - x^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 \\
&+2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^* \rangle - \mu^2\|Bx_n - Bx^*\|^2,
\end{aligned}
\tag{3.13}
$$

and

$$
\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 + 2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\|.
$$

It follows that

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|y_n - [\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)] \\
&\quad +[\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)] - x^*\|^2 \\
&\le [\|y_n - [\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)]\| \\
&\quad +\|[\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)] - x^*\|]^2 \\
&\le [\alpha_n\|Qx_n - y_n\| + (1-\alpha_n)\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - y_n\| + \alpha_n\|Qx_n - x^*\| \\
&\quad +(1-\alpha_n)\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*\|]^2 \\
&\le [\alpha_n\|Qx_n - y_n\| + (1-\alpha_n)\alpha_n\|Qx_n - (z_n - \lambda Az_n)\| + \alpha_n\|Qx_n - x^*\| \\
&\quad +(1-\alpha_n)\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*\|]^2 \\
&= [\alpha_n\|Qx_n - y_n\| + (1-\alpha_n)\alpha_n\|Qx_n - (z_n - \lambda Az_n)\| + \alpha_n\|Qx_n - x^*\| \\
&\quad +(1-\alpha_n)\|v_n - x^*\|]^2 \\
&\le [\alpha_n\|Qx_n - y_n\| + \alpha_n\|Qx_n - (z_n - \lambda Az_n)\| + \alpha_n\|Qx_n - x^*\| + \|v_n - x^*\|]^2 \\
&\le \widetilde{M}\alpha_n + \|v_n - x^*\|^2 \\
&\le \widetilde{M}\alpha_n + \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\|,
\end{aligned}
\tag{3.14}
$$

where $\widetilde{M} > 0$ is some appropriate constant. From (3.11), (3.12) and (3.13), we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 \\
&\quad +(1-\beta_n)\|z_n - y^*\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|Qx_n - (y^* - \lambda Ay^*)\|^2 + (1-\beta_n)[\|x_n - x^*\|^2 \\
&\quad -\|x_n - z_n - (x^* - y^*)\|^2 + 2\mu\|x_n - z_n - (x^* - y^*)\|\|Bx_n - Bx^*\|] + M\alpha_n \\
&= \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - z_n - (x^* - y^*)\|^2 \\
&\quad +2(1-\beta_n)\mu\|x_n - z_n - (x^* - y^*)\|\|Bx_n - Bx^*\| \\
&\quad +(M + (1-\beta_n)\|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n.
\end{aligned}
$$

It follows that

$$\begin{aligned}
&(1-\beta_n)\|x_n - z_n - (x^* - y^*)\|^2 \\
\le\ & (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+(M + (1-\beta_n)\|Qx_n - (y^* - \lambda Ay^*)\|^2)\alpha_n \\
&+2(1-\beta_n)\mu\|x_n - z_n - (x^* - y^*)\|\|Bx_n - Bx^*\|.
\end{aligned}$$

Note that $\|x_{n+1} - x_n\| \to 0,\ \alpha_n \to 0$ and $\|Bx_n - Bx^*\| \to 0.$ Then we immediately get

$$\lim_{n\to\infty} \|x_n - z_n - (x^* - y^*)\| = 0. \tag{3.15}$$

Also, from (3.12) and (3.14), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n \\
\le\ & \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\widetilde{M}\alpha_n + \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&+2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\|] + M\alpha_n \\
=\ & \|x_n - x^*\|^2 - (1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2 \\
&+2\lambda(1-\beta_n)\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\| + (M + (1-\beta_n)\widetilde{M})\alpha_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1-\beta_n)\|z_n - v_n + (x^* - y^*)\|^2 \\
\le\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\lambda(1-\beta_n)\|Az_n \\
&-Ay^*\|\|z_n - v_n + (x^* - y^*)\| + (M + (1-\beta_n)\widetilde{M})\alpha_n \\
\le\ & (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&+2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\| + (M + \widetilde{M})\alpha_n.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|z_n - v_n + (x^* - y^*)\| = 0.$$

This together with $\|y_n - v_n\| \le \alpha_n\|Qx_n - (z_n - \lambda Az_n)\| \to 0$ implies that

$$\lim_{n\to\infty} \|z_n - y_n + (x^* - y^*)\| = 0. \tag{3.16}$$

Thus, from (3.15) and (3.16), we deduce that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since

$$
\begin{aligned}
\|\delta_n(Sy_n - x_n)\| &\le \|x_{n+1} - x_n\| + \gamma_n \|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) - x_n\| \\
&\le \|x_{n+1} - x_n\| + \gamma_n \|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) - y_n\| + \gamma_n \|y_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \gamma_n \alpha_n \|Q x_n - (z_n - \lambda A z_n)\| + \gamma_n \|y_n - x_n\|.
\end{aligned}
$$

Therefore,

$$
\lim_{n\to\infty} \|Sy_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Sy_n - y_n\| = 0.
$$

Step 4. $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Xi$.

Indeed, as $\{y_n\}$ is bounded, we take an element $v \in \omega_w(x_n)$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup v$. First, it is clear from Lemma 2.4 that $v \in \mathrm{Fix}(S)$. Next, we prove that $v \in \Xi$. We note that

$$
\begin{aligned}
\|y_n - v_n\| &= \|T_\lambda^{(F,\phi)}[\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)] - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
&\le \|[\alpha_n Q x_n + (1-\alpha_n)(z_n - \lambda A z_n)] - (z_n - \lambda A z_n)\| \\
&= \alpha_n \|Q x_n - (z_n - \lambda A z_n)\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\|y_n - W(y_n)\| \\
&= \|y_n - [\alpha_n Q x_n + (1-\alpha_n) v_n] + [\alpha_n Q x_n + (1-\alpha_n) v_n] - W(y_n)\| \\
&\le \alpha_n \|Q x_n - y_n\| + (1-\alpha_n)\|v_n - y_n\| + \alpha_n \|Q x_n - W(y_n)\| \\
&\quad + (1-\alpha_n)\|v_n - W(y_n)\| \\
&\le \alpha_n \|Q x_n - y_n\| + (1-\alpha_n)\alpha_n \|Q x_n - (z_n - \lambda A z_n)\| + \alpha_n \|Q x_n - W(y_n)\| \\
&\quad + (1-\alpha_n)\|T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(x_n - \mu B x_n) - \lambda A T_\mu^{(G,\psi)}(x_n - \mu B x_n)] - W(y_n)\| \\
&= \alpha_n [\|Q x_n - y_n\| + (1-\alpha_n)\|Q x_n - (z_n - \lambda A z_n)\| + \|Q x_n - W(y_n)\|] \\
&\quad + (1-\alpha_n)\|W(x_n) - W(y_n)\| \\
&\le \alpha_n [\|Q x_n - y_n\| + (1-\alpha_n)\|Q x_n - (z_n - \lambda A z_n)\| + \|Q x_n - W(y_n)\|] \\
&\quad + (1-\alpha_n)\|x_n - y_n\| \to 0.
\end{aligned}
$$

According to Lemma 2.4 we obtain $v \in \Xi$. Therefore, $v \in \mathrm{Fix}(S) \cap \Xi$. This shows that $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Xi$.

Step 5. $\{x_n\}, \{y_n\}$ converge weakly to the same element $\hat{x} \in \mathrm{Fix}(S) \cap \Xi$ and $\{z_n\}$ converges weakly to $\hat{y} = T_\mu^{(G,\psi)}(\hat{x} - \mu B \hat{x})$.

Indeed, since $\lim_{n\to\infty} \|x_n - x^*\|$ exists for each $x^* \in \text{Fix}(S) \cap \Xi$ (due to Step 2) and $\omega_w(x_n) \subset \text{Fix}(S) \cap \Xi$ (due to Step 4), utilizing Lemma 3.1 we deduce that $\{x_n\}$ converges weakly to an element $\hat{x} \in \text{Fix}(S) \cap \Xi$. Hence it follows from $\|x_n - y_n\| \to 0$ that $y_n \rightharpoonup \hat{x} \in \text{Fix}(S) \cap \Xi$. Note that $\hat{x} \in \Xi$ implies that

$$\hat{x} = T_\lambda^{(F,\phi)}(\hat{y} - \lambda A\hat{y})$$

where $\hat{y} = T_\mu^{(G,\psi)}(\hat{x} - \mu B\hat{x})$. Hence from (3.6) it follows that for each $f \in H$

$$\begin{aligned}|\langle z_n - \hat{y}, f\rangle| &= |\langle z_n - y_n + (\hat{x} - \hat{y}), f\rangle + \langle y_n - \hat{x}, f\rangle| \\ &\le \|z_n - y_n + (\hat{x} - \hat{y})\|\|f\| + |\langle y_n - \hat{x}, f\rangle| \to 0.\end{aligned}$$

This shows that $\{z_n\}$ converges weakly to $\hat{y}$. ■

4. Strong Convergence Theorem

In this section, Yao, Liou and Kang relaxed extragradient method [19] is extended to develop a new iterative algorithm for finding an element of $\text{Fix}(S) \cap \Xi$. Moreover, we derive a strong convergence theorem.

Theorem 4.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $F, G : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)-(H4), let $\phi, \psi : C \to \mathbf{R}$ be two lower semicontinuous and convex functions and let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\text{Fix}(S) \cap \Xi \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\tag{4.1} \begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu Bx_n), \\ y_n = \alpha_n Qx_n + (1 - \alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n), \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) + \delta_n Sy_n, \quad \forall n \ge 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0, 1]$ such that

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \ge 0$;

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(*iv*) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0.$

Then the sequence $\{x_n\}$ *generated by (4.1) converges strongly to* $x^* = P_{\mathrm{Fix}(S)\cap\Xi} \cdot Qx^*$ *and* (x^*, y^*) *is a solution of the general system (1.2) of generalized mixed equilibria, where* $y^* = T_\mu^{(G,\psi)}(x^* - \mu Bx^*)$.

Proof. We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

First, (4.1) can be rewritten as

$$x_{n+1} = \beta_n x_n + (1-\beta_n)u_n, \quad \forall n \geq 0,$$

where $u_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. It follows that

$$\begin{aligned}
&u_{n+1} - u_n \\
&= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})+\delta_{n+1}Sy_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)+\delta_n Sy_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}[T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})-T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)]+\delta_{n+1}(Sy_{n+1}-Sy_n)}{1-\beta_{n+1}} \\
&\quad +(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) + (\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n})Sy_n.
\end{aligned} \tag{4.2}$$

Combining (2.2) with (2.3) and repeating the computation in the proof of Theorem 3.1, we have

$$\|\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Sy_{n+1}-Sy_n)\| \leq (\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\|. \tag{4.3}$$

From (4.1) and (4.3) we get

$$\begin{aligned}
&\|\gamma_{n+1}[T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})-T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)]+\delta_{n+1}(Sy_{n+1}-Sy_n)\| \\
&\leq \|\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Sy_{n+1}-Sy_n)\| \\
&\quad +\gamma_{n+1}\|[T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1}) - y_{n+1}] + [y_n - T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)]\| \\
&\leq \|\gamma_{n+1}(y_{n+1}-y_n) + \delta_{n+1}(Sy_{n+1}-Sy_n)\| \\
&\quad +\gamma_{n+1}\|T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1}) - [\alpha_{n+1}Qx_{n+1} \\
&\quad +(1-\alpha_{n+1})T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})]\| \\
&\quad +\gamma_{n+1}\|[\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)] - T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)\| \\
&\leq (\gamma_{n+1}+\delta_{n+1})\|y_{n+1}-y_n\| + \gamma_{n+1}\alpha_{n+1}\|Qx_{n+1} - T_\lambda^{(F,\phi)}(z_{n+1}-\lambda Az_{n+1})\| \\
&\quad +\gamma_{n+1}\alpha_n\|Qx_n - T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)\|.
\end{aligned} \tag{4.4}$$

Since A, B are α-inverse strongly monotone mapping and β-inverse strongly monotone mapping, respectively, it is known that if $0 \leq \lambda \leq 2\alpha$ and $0 \leq \mu \leq 2\beta$, then $(I - \lambda A)$ and $(I - \mu B)$ are nonexpansive. It hence follows that

$$\begin{aligned}
&\|T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1}) - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
&\leq \|z_{n+1} - \lambda A z_{n+1} - (z_n - \lambda A z_n)\| \\
&\leq \|z_{n+1} - z_n\| \\
&= \|T_\mu^{(G,\psi)}(x_{n+1} - \mu B x_{n+1}) - T_\mu^{(G,\psi)}(x_n - \mu B x_n)\| \\
&\leq \|(x_{n+1} - \mu B x_{n+1}) - (x_n - \mu B x_n)\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned}$$

Then,

$$\begin{aligned}
&|y_{n+1} - y_n\| \\
&= \|[\alpha_{n+1} Q x_{n+1} + (1 - \alpha_{n+1}) T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1})] \\
&\quad -[\alpha_n Q x_n + (1 - \alpha_n) T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)]\| \\
&\leq \|T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1}) - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
&\quad +\alpha_{n+1}\|Q x_{n+1} - T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1})\| \\
&\quad +\alpha_n \|Q x_n - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|Q x_n - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
&\quad +\alpha_{n+1}\|Q x_{n+1} - T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1})\|.
\end{aligned} \tag{4.5}$$

Therefore, from (4.2), (4.4) and (4.5), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| \leq\ & \|x_{n+1} - x_n\| + (1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}})\alpha_n \|Q x_n - T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| \\
& +(1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}})\alpha_{n+1}\|Q x_{n+1} - T_\lambda^{(F,\phi)}(z_{n+1} - \lambda A z_{n+1})\| \\
& +|\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|(\|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)\| + \|S y_n\|).
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.3 we get $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|u_n - x_n\| = 0. \tag{4.6}$$

Step 2. $\lim_{n\to\infty} \|Az_n - Ay^*\| = 0$ and $\lim_{n\to\infty} \|Bx_n - Bx^*\| = 0$.

Indeed, take a fixed $x^* \in \mathrm{Fix}(S) \cap \Xi$ arbitrarily. Then by Lemma 2.2 we have $x^* = Sx^*$ and

$$x^* = T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(x^* - \mu Bx^*) - \lambda A T_\mu^{(G,\psi)}(x^* - \mu Bx^*)].$$

Put $y^* = T_\mu^{(G,\psi)}(x^* - \mu Bx^*)$. Then $x^* = T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)$. From (3.5) and (3.6), we have

$$\begin{aligned} &\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2 \\ &\le \|(z_n - \lambda Az_n) - (y^* - \lambda Ay^*)\|^2 \\ &\le \|z_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\|z_n - y^*\|^2 \\ &= \|T_\mu^{(G,\psi)}(x_n - \mu Bx_n) - T_\mu^{(G,\psi)}(x^* - \mu Bx^*)\|^2 \\ &\le \|(x_n - \mu Bx_n) - (x^* - \mu Bx^*)\|^2 \\ &\le \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2. \end{aligned} \tag{4.8}$$

It follows from (4.1), (4.7) and (4.8) that

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &= \|[\alpha_n Qx_n + (1-\alpha_n)T_\lambda^{(F,\phi)}(z_n - \lambda Az_n)] - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2 \\ &\le \alpha_n\|Qx_n - x^*\|^2 + (1-\alpha_n)\|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) \\ &\quad - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2 \\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|z_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2 \\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 \\ &\quad + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2. \end{aligned} \tag{4.9}$$

Utilizing the convexity of $\|\cdot\|$, we have

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &= \|\beta_n(x_n - x^*) + (1-\beta_n)\frac{1}{1-\beta_n}[\gamma_n(T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*) \\ &\quad + \delta_n(Sy_n - x^*)]\|^2 \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n}{1-\beta_n}(T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - x^*) \\ &\quad + \frac{\delta_n}{1-\beta_n}(Sy_n - x^*)\|^2 \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1-\beta_n} \\
&\quad + \frac{\gamma_n\alpha_n}{1-\beta_n}(T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) - Qx_n)\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1-\beta_n}\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n,
\end{aligned}
$$

where $M > 0$ is some appropriate constant. So, from (4.9) and (4.10) we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_n - Ay^*\|^2] + M\alpha_n \\
&\le \|x_n - x^*\|^2 + \mu(\mu - 2\beta)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\
&\quad + \lambda(\lambda - 2\alpha)(1-\beta_n)\|Az_n - Ay^*\|^2 + (M + \|Qx_n - x^*\|^2)\alpha_n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lambda(2\alpha - \lambda)(1-\beta_n)\|Az_n - Ay^*\|^2 + \mu(2\beta - \mu)(1-\beta_n)\|Bx_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (M + \|Qx_n - x^*\|^2)\alpha_n \\
&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + (M + \|Qx_n - x^*\|^2)\alpha_n.
\end{aligned}
$$

Since $\liminf_{n\to\infty} \lambda(2\alpha - \lambda)(1-\beta_n) > 0$, $\liminf_{n\to\infty} \mu(2\beta - \mu)(1-\beta_n) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we have

$$
\lim_{n\to\infty} \|Az_n - Ay^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Bx_n - Bx^*\| = 0.
$$

Step 3. $\lim_{n\to\infty} \|Sy_n - y_n\| = 0$.

Indeed, set $v_n = T_\lambda^{(F,\phi)}(z_n - \lambda A z_n)$. Noting that both $T_\lambda^{(F,\phi)}$ and $T_\mu^{(G,\psi)}$ are firmly nonexpansive, then we have

$$
\begin{aligned}
&\|z_n - y^*\|^2 \\
&= \|T_\mu^{(G,\psi)}(x_n - \mu B x_n) - T_\mu^{(G,\psi)}(x^* - \mu B x^*)\|^2 \\
&\le \langle (x_n - \mu B x_n) - (x^* - \mu B x^*), z_n - y^* \rangle \\
&= \frac{1}{2}(\|x_n - x^* - \mu(Bx_n - Bx^*)\|^2 \\
&\quad + \|z_n - y^*\|^2 - \|(x_n - x^*) - \mu(Bx_n - Bx^*) - (z_n - y^*)\|^2)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|(x_n - z_n) - \mu(Bx_n - Bx^*) - (x^* - y^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|z_n - y^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 \\
&\quad +2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^*\rangle - \mu^2\|Bx_n - Bx^*\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|v_n - x^*\|^2 \\
&= \|T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) - T_\lambda^{(F,\phi)}(y^* - \lambda Ay^*)\|^2 \\
&\le \langle (z_n - \lambda Az_n) - (y^* - \lambda Ay^*), v_n - x^*\rangle \\
&= \frac{1}{2}(\|z_n - \lambda Az_n - (y^* - \lambda Ay^*)\|^2 + \|v_n - x^*\|^2 - \|z_n \\
&-\lambda Az_n - (y^* - \lambda Ay^*) - (v_n - x^*)\|^2) \\
&\le \frac{1}{2}(\|z_n - y^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*)\rangle - \lambda^2\|Az_n - Ay^*\|^2) \\
&\le \frac{1}{2}(\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\langle Az_n - Ay^*, z_n - v_n + (x^* - y^*)\rangle)
\end{aligned}
$$

(due to (4.8)). Thus, we have

$$
\begin{aligned}
\|z_n - y^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - z_n - (x^* - y^*)\|^2 \\
&\quad +2\mu\langle x_n - z_n - (x^* - y^*), Bx_n - Bx^*\rangle - \mu^2\|Bx_n - Bx^*\|^2,
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
\|v_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&\le \alpha_n\|Qx_n - x^*\|^2 + (1 - \alpha_n)\|v_n - x^*\|^2 \\
&\le \alpha_n\|Qx_n - x^*\|^2 + \|v_n - x^*\|^2 \\
&\le \alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - v_n + (x^* - y^*)\|^2 \\
&\quad +2\lambda\|Az_n - Ay^*\|\|z_n - v_n + (x^* - y^*)\|.
\end{aligned}
\tag{4.12}
$$

From (4.9), (4.10) and (4.11), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
\le\ &\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+M\alpha_n\\
\le\ &\beta_n\|x_n-x^*\|^2+(1-\beta_n)\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\|z_n-y^*\|^2+M\alpha_n\\
\le\ &\beta_n\|x_n-x^*\|^2+(1-\beta_n)\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)[\|x_n-x^*\|^2\\
&-\|x_n-z_n-(x^*-y^*)\|^2+2\mu\|x_n-z_n-(x^*-y^*)\|\|Bx_n-Bx^*\|]+M\alpha_n\\
=\ &\|x_n-x^*\|^2-(1-\beta_n)\|x_n-z_n-(x^*-y^*)\|^2\\
&+2(1-\beta_n)\mu\|x_n-z_n-(x^*-y^*)\|\|Bx_n-Bx^*\|\\
&+(M+(1-\beta_n)\|Qx_n-x^*\|^2)\alpha_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\beta_n)\|x_n-z_n-(x^*-y^*)\|^2\\
\le\ &(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|+(M+(1-\beta_n)\|Qx_n-x^*\|^2)\alpha_n\\
&+2(1-\beta_n)\mu\|x_n-z_n-(x^*-y^*)\|\|Bx_n-Bx^*\|.
\end{aligned}$$

Note that $\|x_{n+1}-x_n\|\to 0,\ \alpha_n\to 0$ and $\|Bx_n-Bx^*\|\to 0.$ Then we immediately get

$$\lim_{n\to\infty}\|x_n-z_n-(x^*-y^*)\|=0. \tag{4.13}$$

Also, from (4.10) and (4.12), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
\le\ &\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+M\alpha_n\\
\le\ &\beta_n\|x_n-x^*\|^2+(1-\beta_n)[\alpha_n\|Qx_n-x^*\|^2+\|x_n-x^*\|^2\\
&-\|z_n-v_n+(x^*-y^*)\|^2+2\lambda\|Az_n-Ay^*\|\|z_n-v_n+(x^*-y^*)\|]+M\alpha_n\\
\le\ &\|x_n-x^*\|^2-(1-\beta_n)\|z_n-v_n+(x^*-y^*)\|^2\\
&+2\lambda(1-\beta_n)\|Az_n-Ay^*\|\|z_n-v_n+(x^*-y^*)\|+(M+\|Qx_n-x^*\|^2)\alpha_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1-\beta_n)\|z_n-v_n+(x^*-y^*)\|^2\\
\le\ &\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\lambda(1-\beta_n)\|Az_n-Ay^*\|\|z_n-v_n+(x^*-y^*)\|\\
&+(M+\|Qx_n-x^*\|^2)\alpha_n\\
\le\ &(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\
&+2\lambda\|Az_n-Ay^*\|\|z_n-v_n+(x^*-y^*)\|+(M+\|Qx_n-x^*\|^2)\alpha_n.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|z_n - v_n + (x^* - y^*)\| = 0.$$

This together with $\|y_n - v_n\| \le \alpha_n\|Qx_n - v_n\| \to 0$ implies that

$$\lim_{n\to\infty}\|z_n - y_n + (x^* - y^*)\| = 0. \tag{4.14}$$

Thus, from (4.13) and (4.14), we deduce that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Since

$$\begin{aligned}\|\delta_n(Sy_n - x_n)\| &\le \|x_{n+1} - x_n\| + \gamma_n\|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) - x_n\| \\ &\le \|x_{n+1} - x_n\| + \gamma_n\|T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) - y_n\| + \gamma_n\|y_n - x_n\| \\ &= \|x_{n+1} - x_n\| + \gamma_n\alpha_n\|Qx_n - v_n\| + \gamma_n\|y_n - x_n\|.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\|Sy_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|Sy_n - y_n\| = 0.$$

Step 4. $\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle \le 0$ where $x^* = P_{\mathrm{Fix}(S)\cap\Xi}\cdot Qx^*$.

Indeed, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle Qx^* - x^*, y_n - x^*\rangle = \lim_{i\to\infty}\langle Qx^* - x^*, y_{n_i} - x^*\rangle.$$

As $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_i} \rightharpoonup v$. First, it is clear from Lemma 2.4 that $v \in \mathrm{Fix}(S)$. Next, we prove that $v \in \Xi$. We note that

$$\begin{aligned}&\|y_n - W(y_n)\| \\ &\le \alpha_n\|Qx_n - W(y_n)\| + (1-\alpha_n)\|T_\lambda^{(F,\phi)}[T_\mu^{(G,\psi)}(x_n - \mu B x_n) \\ &\quad -\lambda A T_\mu^{(G,\psi)}(x_n - \mu B x_n)] - W(y_n)\| \\ &= \alpha_n\|Qx_n - W(y_n)\| + (1-\alpha_n)\|W(x_n) - W(y_n)\| \\ &\le \alpha_n\|Qx_n - W(y_n)\| + (1-\alpha_n)\|x_n - y_n\| \to 0.\end{aligned}$$

According to Lemma 2.4 we obtain $v \in \Xi$. Therefore, $v \in \mathrm{Fix}(S)\cap\Xi$. Hence, it follows from (2.1) that

$$\begin{aligned}\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle &= \limsup_{n\to\infty}\langle Qx^* - x^*, y_n - x^*\rangle \\ &= \lim_{i\to\infty}\langle Qx^* - x^*, y_{n_i} - x^*\rangle \\ &= \langle Qx^* - x^*, v - x^*\rangle \\ &\le 0.\end{aligned}$$

Step 5. $\lim_{n\to\infty}\|x_n - x^*\| = 0$ where $x^* = P_{\mathrm{Fix}(S)\cap\Xi}\cdot Qx^*$.
Indeed, utilizing Lemma 2.6, from (4.1) and the convexity of $\|\cdot\|$ we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&= \|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\\
&\quad +\gamma_n\alpha_n(T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)-Qx_n)\|^2\\
&\le \|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\|^2\\
&\quad +2\gamma_n\alpha_n\langle T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)-Qx_n, x_{n+1}-x^*\rangle\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|\frac{1}{1-\beta_n}[\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)]\|^2\\
&\quad +2\gamma_n\alpha_n\langle T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)-x^*, x_{n+1}-x^*\rangle\\
&\quad +2\gamma_n\alpha_n\langle x^*-Qx_n, x_{n+1}-x^*\rangle,
\end{aligned}\tag{4.15}$$

which hence implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2\\
&\quad +2\gamma_n\alpha_n\|T_\lambda^{(F,\phi)}(z_n-\lambda Az_n)-x^*\|\|x_{n+1}-x^*\|+2\gamma_n\alpha_n\langle x^*-Qx_n, x_{n+1}-x^*\rangle\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)[(1-\alpha_n)\|z_n-y^*\|^2+2\alpha_n\langle Qx_n-x^*, y_n-x^*\rangle]\\
&\quad +2\gamma_n\alpha_n\|z_n-y^*\|\|x_{n+1}-x^*\|+2\gamma_n\alpha_n\langle x^*-Qx_n, x_{n+1}-x^*\rangle.
\end{aligned}$$

From (4.8), we note that $\|z_n-y^*\|\le\|x_n-x^*\|$. Hence we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-x^*\|^2\\
&\quad +2\alpha_n(1-\beta_n)\langle Qx_n-x^*, y_n-x^*\rangle\\
&\quad +2\gamma_n\alpha_n\|x_n-x^*\|\|x_{n+1}-x^*\|+2\gamma_n\alpha_n\langle x^*-Qx_n, x_{n+1}-x^*\rangle.
\end{aligned}$$

Repeating the remainder of the proof in Yao, Liou and Kang [19, Theorem 3.2], in terms of Lemma 2.5 we can obtain the desired conclusion. ∎

Remark 4.1. It is easy to see that if $F = G = 0$ and $\phi = \psi = 0$, then our Theorem 4.1 reduces to Yao, Liou and Kang's Theorem 3.2 [21]. Hence our Theorem 4.1 covers Yao, Liou and Kang's Theorem 3.2 [21] as a special case.

Corollary 4.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $F, G : C\times C\to\mathbf{R}$ be two bifunctions satisfying conditions (H1)-(H4), let $\phi,\psi : C\to\mathbf{R}$ be two lower semicontinuous and convex functions and let the mappings $A, B : C\to H$ be α-inverse strongly monotone and β-inverse* *strongly monotone, respectively. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu B x_n), \\ y_n = \alpha_n u + (1-\alpha_n) T_\lambda^{(F,\phi)}(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$ *for all* $n \geq 0$*;*

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*iii*) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(*iv*) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0.$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap\Xi} u$ and (x^*, y^*) is a solution of the general system (1.2) of generalized mixed equilibria, where $y^* = T_\mu^{(G,\psi)}(x^* - \mu B x^*)$.

Corollary 4.2. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $F, G : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (H1)-(H4), let $\phi, \psi : C \to \mathbf{R}$ be two lower semicontinuous and convex functions and let the mappings $A, B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, \frac{1}{2})$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu B x_n), \\ y_n = \alpha_n Q x_n + (1-\alpha_n) T_\lambda^{(F,\phi)}(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda A z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that

(*i*) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1-2\rho)\delta_n$ *for all* $n \geq 0$*;*

(*ii*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$*;*

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0.$

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{\mathrm{Fix}(S)\cap\Xi} \cdot Qx^*$ *and* (x^*, y^*) *is a solution of the general system (1.2) of generalized mixed equilibria, where* $y^* = T_\mu^{(G,\psi)}(x^* - \mu Bx^*)$.

Corollary 4.3. *Let* C *be a nonempty bounded closed convex subset of a real Hilbert space* H. *Let* $F, G : C \times C \to \mathbf{R}$ *be two bifunctions satisfying conditions (H1)-(H4), let* $\phi, \psi : C \to \mathbf{R}$ *be two lower semicontinuous and convex functions and let the mappings* $A, B : C \to H$ *be* α*-inverse strongly monotone and* β*-inverse strongly monotone, respectively. Let* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Xi \neq \emptyset$*. For fixed* $u \in C$ *and given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{cases} z_n = T_\mu^{(G,\psi)}(x_n - \mu Bx_n), \\ y_n = \alpha_n u + (1-\alpha_n) T_\lambda^{(F,\phi)}(z_n - \lambda Az_n), \\ x_{n+1} = \beta_n x_n + \gamma_n T_\lambda^{(F,\phi)}(z_n - \lambda Az_n) + \delta_n Sy_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda \in (0, 2\alpha),\ \mu \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ *are four sequences in* $[0, 1]$ *such that*

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1-2\rho)\delta_n$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$*;*

(iv) $\lim_{n\to\infty} (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0.$

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_{\mathrm{Fix}(S)\cap\Xi} \cdot Qx^*$ *and* (x^*, y^*) *is a solution of the general system (1.2) of generalized mixed equilibria, where* $y^* = T_\mu^{(G,\psi)}(x^* - \mu Bx^*)$.

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